An intuitionistic version of Cantor’s theorem

Dario Maguolo, Silvio Valentini
Dipartimento di Matematica Pura ed Applicata
Università di Padova
via G. Belzoni n.7, I–35131 Padova, Italy
valentini@pdmat1.math.unipd.it

September 24, 1996

Abstract

An intuitionistic version of Cantor’s theorem, which shows that there is no surjective function from the type of the natural numbers $\mathbb{N}$ into the type $\mathbb{N} \to \mathbb{N}$ of the functions from $\mathbb{N}$ into $\mathbb{N}$, is proved within Martin-Löf’s Intuitionistic Type Theory with the universe of the small types.

Mathematics Subject Classification: 03B15, 03B20.

Keywords: Intuitionistic type theory, Cartesian closed categories.

1 The intuitionistic Cantor’s theorem

In this work we show that within Martin-Löf’s Intuitionistic Type Theory with the universe of the small types [ML84, NPS90] (ITT for short in the following) a version of Cantor’s theorem holds, which shows that there is no surjective function from the type of the natural numbers $\mathbb{N}$ into the type $\mathbb{N} \to \mathbb{N}$ of the functions from $\mathbb{N}$ into $\mathbb{N}$. As the matter of fact a similar result can be stated for any not-empty type $A$ such that there exists a function from $A$ into $A$ which has no fixed point, as is the case of the successor function for the type $\mathbb{N}$. In order to express Cantor’s theorem within ITT we need the Equality proposition: let $A$ is a type and $a, c \in A$, then by $a =_A c$ we mean the Equality proposition for elements of type $A$ [NPS90].

Theorem 1.1 (ITT Cantor’s theorem) Let $\mathbb{N}$ be the type of the natural numbers; then

$$\neg(\exists f \in \mathbb{N} \to (\mathbb{N} \to \mathbb{N}))(\forall y \in \mathbb{N} \to \mathbb{N})(\exists x \in \mathbb{N}) f(x) =_{\mathbb{N} \to \mathbb{N}} y$$

To prove this theorem some lemmas are useful. Indeed we need to obtain a contradiction from the assumption

$$(\exists f \in \mathbb{N} \to (\mathbb{N} \to \mathbb{N}))(\forall y \in \mathbb{N} \to \mathbb{N})(\exists x \in \mathbb{N}) f(x) =_{\mathbb{N} \to \mathbb{N}} y$$

1
i.e. from the two assumptions

\[ f \in \mathcal{N} \rightarrow (\mathcal{N} \rightarrow \mathcal{N}) \]

and

\[ (\forall y \in \mathcal{N} \rightarrow \mathcal{N})(\exists x \in \mathcal{N}) \ f(x) =_{\mathcal{N} \rightarrow \mathcal{N}} y \]

By using the basic idea of the classic proof of Cantor’s theorem, from the first assumption we can prove \( \lambda x. s(f(x)(x)) \in \mathcal{N} \rightarrow \mathcal{N} \), where \( s : (x : \mathcal{N})\mathcal{N} \) is the successor function, by the following deduction:

\[
\begin{array}{c}
[x : \mathcal{N}]_1 f \in \mathcal{N} \rightarrow (\mathcal{N} \rightarrow \mathcal{N}) \\
\hline \\
\frac{f(x)(x) \in \mathcal{N}}{\lambda x. s(f(x)(x)) \in \mathcal{N} \rightarrow \mathcal{N}}_1
\end{array}
\]

We can now use this function in the second assumption in order to obtain \( (\exists x \in \mathcal{N}) f(x) =_{\mathcal{N} \rightarrow \mathcal{N}} \lambda x. s(f(x)(x)) \). So our problem becomes to obtain a contradiction from the two assumptions \( x : \mathcal{N} \) and \( f(x) =_{\mathcal{N} \rightarrow \mathcal{N}} \lambda x. s(f(x)(x)) \). We can use these assumptions to prove, by transitivity of the equality proposition, that \( f(x)(x) =_{\mathcal{N}} s(f(x)(x)) \) is true since in general if \( A \) and \( B \) are types and \( a =_A c \) and \( f =_{A \rightarrow B} g \) then \( f(a) =_B g(c) \) and obviously \( (\lambda x. s(f(x)(x)))(x) =_{\mathcal{N}} s(f(x)(x)) \) is true.

We can thus re-state our aim by saying that we have to prove that ITT is not consistent with the assumption that the successor function has a fixed point. To prove this result we can transpose a well known categorical arguments within ITT [L69, HP90]. Let us recall that we can solve the usual recursive definition of the sum between two natural numbers

\[
\begin{cases}
  n + 0 = n : \mathcal{N} \\
  n + s(x) = s(n + x) : \mathcal{N}
\end{cases}
\]

by putting \( n + x \equiv \text{Rec}(x, n, (u, v) \ s(v)) \). Then the following lemma can be proved by induction.

**Lemma 1.2** For any \( n, x \in \mathcal{N} \), \( n + s(x) =_\mathcal{N} s(n) + x \).

As for the sum, we can solve the recursive equation for the predecessor function

\[
\begin{cases}
  p(0) = 0 : \mathcal{N} \\
  p(s(x)) = x : \mathcal{N}
\end{cases}
\]

by putting \( p(x) \equiv \text{Rec}(x, 0, (u, v) \ u) \), and then that for the subtraction

\[
\begin{cases}
  n - 0 = n : \mathcal{N} \\
  n - s(x) = p(n - x) : \mathcal{N}
\end{cases}
\]

by putting \( n - x \equiv \text{Rec}(x, n, (u, v) \ p(v)) \).
Lemma 1.3 For any \( x \in \mathcal{N} \), \( (\forall n \in \mathcal{N}) (n + x) - x =_\mathcal{N} n \).

Proof. By induction on \( x \). If \( x = 0 \) then \( (n + 0) - 0 =_\mathcal{N} n + 0 =_\mathcal{N} n \); let us now suppose that \( (\forall n \in \mathcal{N}) (n + x) - x =_\mathcal{N} n \), then \( (n + s(x)) - s(x) =_\mathcal{N} p((n + s(x)) - x) =_\mathcal{N} p((s(n) + x) - x) =_\mathcal{N} p(s(n)) =_\mathcal{N} n \). □

We can apply this lemma to the case \( n = 0 \) and obtain the following corollary.

Corollary 1.4 For any \( x \in \mathcal{N} \), \( x - x =_\mathcal{N} 0 \).

Proof. Immediate, since \( 0 + x =_\mathcal{N} x \) holds for each \( x \in \mathcal{N} \). □

Now we conclude our proof. Let us write \( \omega \) to mean the fixed point of the successor function, i.e. \( \omega =_\mathcal{N} s(\omega) \); then the following lemma holds.

Lemma 1.5 For any \( x \in \mathcal{N} \), \( \omega - x =_\mathcal{N} \omega \).

Proof. Again a proof by induction on \( x \). If \( x = 0 \) then \( \omega - 0 =_\mathcal{N} \omega \) and, supposing \( \omega - x =_\mathcal{N} \omega \), we obtain \( \omega - s(x) =_\mathcal{N} p(\omega - x) =_\mathcal{N} p(\omega) =_\mathcal{N} p(s(\omega)) =_\mathcal{N} \omega \). □

So we proved that \( \omega - \omega =_\mathcal{N} 0 \) by corollary 1.4 and also that \( \omega - \omega =_\mathcal{N} \omega \) by lemma 1.5; hence \( 0 =_\mathcal{N} \omega =_\mathcal{N} s(\omega) \). Finally we reach a contradiction.

Theorem 1.6 For any \( x \in \mathcal{N} \), \( \neg(0 =_\mathcal{N} s(x)) \)

Proof. By an elimination rule for the type \( \mathcal{N} \), from the assumption \( y : \mathcal{N} \), we obtain \( \text{Rec}(y, \bot, (u, v) \top) \in U_0 \), where \( U_0 \) is the universe of the small types, \( \bot \) is the empty type and \( \top \) is the one-element type. Now let us assume that \( x \in \mathcal{N} \) and that \( 0 =_\mathcal{N} s(x) \) is true, then \( \text{Rec}(0, \bot, (u, v) \top) =_{U_0} \text{Rec}(s(x), \bot, (u, v) \top) \) since in general if \( A \) and \( B \) are types and \( a =_A c \) is true and \( b(x) \in B \) \( x : A \) then \( b(a) =_B b(c) \) is true. Hence, by transitivity of the equality proposition, \( \bot =_{U_0} \top \) since \( \bot =_{U_0} \text{Rec}(0, \bot, (u, v) \top) \) and \( \text{Rec}(s(x), \bot, (u, v) \top) =_{U_0} \top \). Then, because of one of the properties of the equality proposition for the elements of the type \( U_0 \), \( \bot \) is inhabited since \( \top \) is and hence, by discharging the assumption \( 0 =_\mathcal{N} s(x) \), we obtain that \( \neg(0 =_\mathcal{N} s(x)) \) is true. □

Thus the proof of theorem 1.1 is finished since we have obtained the contradiction we were looking for. Anyhow we stress on the fact that a similar result holds for any type \( A \) such that there exists a function from \( A \rightarrow A \) with no fixed point. In fact, in this hypothesis, we can prove that there exists a function \( g \) from \( A \rightarrow (A \rightarrow A) \) into \( A \rightarrow A \) which supplies, for any function \( h \) from \( A \) into \( A \rightarrow A \), a function \( g(h) \in A \rightarrow A \) which is not in the image of \( h \).

Theorem 1.7 Let \( A \) be a type; then

\[
(\exists f : A \rightarrow A)(\forall x : A) \neg (f(x) =_A x) \\
\rightarrow (\exists g : (A \rightarrow (A \rightarrow A)) \rightarrow (A \rightarrow A)) \\
(\forall h : A \rightarrow (A \rightarrow A)) \\
(\forall x : A) \neg (g(h) =_{A \rightarrow A} h(x))
\]
The proof of this theorem is similar to the first part of the proof of theorem 1.1. In fact we only have to use the function \( f \in A \to A \), instead of the successor function, to construct the function \( g \equiv \lambda k.\lambda y. f(k(y)(y)) \in (A \to (A \to A)) \to (A \to A) \) such that, for any \( h \in A \to (A \to A) \) and any \( x \in A \), allows to prove \( h(x)(x) =_A f(h(x)(x)) \), which is contrary to the assumption that the function \( f \) has no fixed point.

References


