An intuitionistic version of Cantor's theorem

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Abstract

An intuitionistic version of Cantor's theorem, which shows that there is no surjective function from the type of the natural numbers $\mathcal{N}$ into the type $\mathcal{N} \to \mathcal{N}$ of the functions from $\mathcal{N}$ into $\mathcal{N}$, is proved within Martin-Löf's Intuitionistic Type Theory with the universe of the small types.

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1 The intuitionistic Cantor's theorem

In this work we show that within Martin-Löf's Intuitionistic Type Theory with the universe of the small types [ML84, NPS90] (ITT for short in the following) a version of Cantor's theorem holds, which shows that there is no surjective function from the type of the natural numbers $\mathcal{N}$ into the type $\mathcal{N} \to \mathcal{N}$ of the functions from $\mathcal{N}$ into $\mathcal{N}$. As the matter of fact a similar result can be stated for any not-empty type A such that there exists a function from A into A which has no fixed point, as is the case of the successor function for the type $\mathcal{N}$. In order to express Cantor's theorem within ITT we need the *Equality proposition*: let A is a type and $a, c \in A$, then by $a =_A c$ we mean the Equality proposition for elements of type A [NPS90].

Theorem 1.1 (ITT Cantor's theorem) *Let $\mathcal{N}$ be the type of the natural numbers; then*

$$\neg(\exists f \in \mathcal{N} \to (\mathcal{N} \to \mathcal{N}))(\forall y \in \mathcal{N} \to \mathcal{N})(\exists x \in \mathcal{N})\ f(x) =_{\mathcal{N}\to\mathcal{N}} y$$

To prove this theorem some lemmas are useful. Indeed we need to obtain a contradiction from the assumption

$$(\exists f \in \mathcal{N} \to (\mathcal{N} \to \mathcal{N}))(\forall y \in \mathcal{N} \to \mathcal{N})(\exists x \in \mathcal{N})\ f(x) =_{\mathcal{N}\to\mathcal{N}} y$$

i.e. from the two assumptions

$$f \in \mathcal{N} \rightarrow (\mathcal{N} \rightarrow \mathcal{N})$$

and

$$(\forall y \in \mathcal{N} \rightarrow \mathcal{N})(\exists x \in \mathcal{N})\ f(x) =_{\mathcal{N}\rightarrow\mathcal{N}} y$$

By using the basic idea of the classic proof of Cantor's theorem, from the first assumption we can prove $\lambda x.s(f(x)(x)) \in \mathcal{N} \rightarrow \mathcal{N}$, where $s : (x : \mathcal{N})\mathcal{N}$ is the successor function, by the following deduction:

$$\cfrac{\cfrac{\cfrac{[x : \mathcal{N}]_1 \qquad \cfrac{[x : \mathcal{N}]_1 \quad f \in \mathcal{N} \rightarrow (\mathcal{N} \rightarrow \mathcal{N})}{f(x) : \mathcal{N} \rightarrow \mathcal{N}}}{f(x)(x) \in \mathcal{N}}}{s(f(x)(x)) \in \mathcal{N}}}{\lambda x.s(f(x)(x)) \in \mathcal{N} \rightarrow \mathcal{N}}\ 1$$

We can now use this function in the second assumption in order to obtain $(\exists x \in \mathcal{N})\ f(x) =_{\mathcal{N}\rightarrow\mathcal{N}} \lambda x.s(f(x)(x))$. So our problem becomes to obtain a contradiction from the two assumptions $x : \mathcal{N}$ and $f(x) =_{\mathcal{N}\rightarrow\mathcal{N}} \lambda x.s(f(x)(x))$. We can use these assumptions to prove, by transitivity of the equality proposition, that $f(x)(x) =_{\mathcal{N}} s(f(x)(x))$ is true since in general if A and B are types and $a =_A c$ and $f =_{A\rightarrow B} g$ then $f(a) =_B g(c)$ and obviously $(\lambda x.s(f(x)(x)))(x) =_{\mathcal{N}} s(f(x)(x))$ is true.

We can thus re-state our aim by saying that we have to prove that ITT is not consistent with the assumption that the successor function has a fixed point. To prove this result we can transpose a well known categorical arguments within ITT [L69, HP90]. Let us recall that we can solve the usual recursive definition of the sum between two natural numbers

$$\begin{cases} n + 0 = n : \mathcal{N} \\ n + s(x) = s(n + x) : \mathcal{N} \end{cases}$$

by putting $n + x \equiv Rec(x, n, (u, v)\ s(v))$. Then the following lemma can be proved by induction.

Lemma 1.2 *For any $n, x \in \mathcal{N}, n + s(x) =_{\mathcal{N}} s(n) + x$.*

As for the sum, we can solve the recursive equation for the predecessor function

$$\begin{cases} p(0) = 0 : \mathcal{N} \\ p(s(x)) = x : \mathcal{N} \end{cases}$$

by putting $p(x) \equiv Rec(x, 0, (u, v)\ u)$, and then that for the subtraction

$$\begin{cases} n - 0 = n : \mathcal{N} \\ n - s(x) = p(n - x) : \mathcal{N} \end{cases}$$

by putting $n - x \equiv Rec(x, n, (u, v)\ p(v))$.

Lemma 1.3 *For any $x \in \mathcal{N}$, $(\forall n \in \mathcal{N})\ (n + x) - x =_{\mathcal{N}} n$.*

Proof. By induction on x. If $x = 0$ then $(n + 0) - 0 =_{\mathcal{N}} n + 0 =_{\mathcal{N}} n$; let us now suppose that $(\forall n \in \mathcal{N})\ (n + x) - x =_{\mathcal{N}} n$, then $(n + s(x)) - s(x) =_{\mathcal{N}} p((n + s(x)) - x) =_{\mathcal{N}} p((s(n) + x) - x) =_{\mathcal{N}} p(s(n)) =_{\mathcal{N}} n$. □

We can apply this lemma to the case $n = 0$ and obtain the following corollary.

Corollary 1.4 *For any $x \in \mathcal{N}$, $x - x =_{\mathcal{N}} 0$.*

Proof. Immediate, since $0 + x =_{\mathcal{N}} x$ holds for each $x \in \mathcal{N}$. □

Now we conclude our proof. Let us write ω to mean the fixed point of the successor function, i.e. $\omega =_{\mathcal{N}} s(\omega)$; then the following lemma holds.

Lemma 1.5 *For any $x \in \mathcal{N}$, $\omega - x =_{\mathcal{N}} \omega$.*

Proof. Again a proof by induction on x. If $x = 0$ then $\omega - 0 =_{\mathcal{N}} \omega$ and, supposing $\omega - x =_{\mathcal{N}} \omega$, we obtain $\omega - s(x) =_{\mathcal{N}} p(\omega - x) =_{\mathcal{N}} p(\omega) =_{\mathcal{N}} p(s(\omega)) =_{\mathcal{N}} \omega$. □

So we proved that $\omega - \omega =_{\mathcal{N}} 0$ by corollary 1.4 and also that $\omega - \omega =_{\mathcal{N}} \omega$ by lemma 1.5; hence $0 =_{\mathcal{N}} \omega =_{\mathcal{N}} s(\omega)$. Finally we reach a contradiction.

Theorem 1.6 *For any $x \in \mathcal{N}$, $\neg(0 =_{\mathcal{N}} s(x))$*

Proof. By an elimination rule for the type $\mathcal{N}$, from the assumption $y : \mathcal{N}$, we obtain $Rec(y, \bot, (u, v)\ \top) \in U_0$, where U_0 is the universe of the small types, $\bot$ is the empty type and $\top$ is the one-element type. Now let us assume that $x \in \mathcal{N}$ and that $0 =_{\mathcal{N}} s(x)$ is true, then $Rec(0, \bot, (u, v)\ \top) =_{U_0} Rec(s(x), \bot, (u, v)\ \top)$ since in general if A and B are types and $a =_A c$ is true and $b(x) \in B\ [x : A]$ then $b(a) =_B b(c)$ is true. Hence, by transitivity of the equality proposition, $\bot =_{U_0} \top$ since $\bot =_{U_0} Rec(0, \bot, (u, v)\ \top)$ and $Rec(s(x), \bot, (u, v)\ \top) =_{U_0} \top$. Then, because of one of the properties of the equality proposition for the elements of the type U_0, $\bot$ is inhabited since $\top$ is and hence, by discharging the assumption $0 =_{\mathcal{N}} s(x)$, we obtain that $\neg(0 =_{\mathcal{N}} s(x))$ is true. □

Thus the proof of theorem 1.1 is finished since we have obtained the contradiction we were looking for. Anyhow we stress on the fact that a similar result holds for any type A such that there exists a function from A into A with no fixed point. In fact, in this hypothesis, we can prove that there exists a function g from $A \to (A \to A)$ into $A \to A$ which supplies, for any function h from A into $A \to A$, a function $g(h) \in A \to A$ which is not in the image of h.

Theorem 1.7 *Let A be a type; then*

$$\begin{array}{l} (\exists f \in A \to A)(\forall x \in A)\ \neg(f(x) =_A x) \\ \to (\exists g \in (A \to (A \to A)) \to (A \to A)) \\ \qquad (\forall h \in A \to (A \to A)) \\ \qquad (\forall x \in A)\ \neg(g(h) =_{A \to A} h(x)) \end{array}$$

The proof of this theorem is similar to the first part of the proof of theorem 1.1. In fact we only have to use the function $f \in A \to A$, instead of the successor function, to construct the function $g \equiv \lambda k.\lambda y.f(k(y)(y)) \in (A \to (A \to A)) \to (A \to A)$ such that, for any $h \in A \to (A \to A)$ and any $x \in A$, allows to prove $h(x)(x) =_A f(h(x)(x))$, which is contrary to the assumption that the function f has no fixed point.

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