A complete formalization of constructive topology

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Abstract

Looking for a complete formalization of constructive topology we analyzed the structure of the subsets of a Heyting algebra which correspond to the concrete closed and open sets of a topological space over its formal points. After this has been done, the rules for a formalization of constructive topology, which is both predicative and complete, are unveiled.

1 Introduction

The aim of formal topology is to develop topology in a constructive framework where the adjective "constructive" is meant to imply both intuitionistic and predicative.

One of the main problems with the original definition of formal topology (see for instance [GS99] or the introductory section in [CSSV]) is that it misses a complete formalization of topological spaces since only *some* of the valid conditions on the cover and the positivity relations are considered. This is confirmed by the fact that, while in the pointset setting basic opens are sufficient to determine both the open and the closed subsets of a topological space this in general does not happen for a formal topology (see [Val00] for a discussion on the problem of finding a complete formalization).

In this paper we propose new rules for a predicative inductive generation of formal topologies and justify their completeness. It is interesting to observe that this completeness result relies on using a predicative set theory, like Martin-Löf's type theory [ML84]; thus predicativity is essential not only from a philosophical point of view, that is, in order to be able to give a constructive meaning to quantification, but also because this assumption has mathematical consequences which cannot be proved without it.

In order to make the paper self contained, we will give in the next sections an introduction to formal topology.

1.1 Concrete topological spaces

The classical definition of topological space reads as follows: $(X, \Omega(X))$ is a topological space if X is a set and $\Omega(X)$ is a subset of $\mathcal{P}(X)$ which satisfies:

(Ω_1) $\emptyset, X \in \Omega(X)$;

(Ω_2) $\Omega(X)$ is closed under finite intersection;

(Ω_3) $\Omega(X)$ is closed under arbitrary union.

Usually, elements of X are called *points* and elements of $\Omega(X)$ are called *opens*.

The quantification implicitly used in (Ω_3) is of the third order, since it says

$$(\forall F \in \mathcal{P}(\mathcal{P}(X)))\ (F \subseteq \Omega(X) \rightarrow \bigcup F \in \Omega(X))$$

We can "go down" one step by thinking of $\Omega(X)$ as a family of subsets indexed by a set S through a map $\mathsf{ext} : S \rightarrow \mathcal{P}(X)$, that is, a binary relation between S and X. In fact, we can now quantify on S rather than on $\Omega(X)$. But we still have to say

$$(\forall U \in \mathcal{P}(S))(\exists c \in S)\ (\cup_{a \varepsilon U} \mathsf{ext}(a) = \mathsf{ext}(c))$$

which is still impredicative[1].

We can "go down" another step by defining opens to be of the form

$$\mathsf{Ext}(U) \equiv \bigcup_{a \varepsilon U} \mathsf{ext}(a)$$

for an arbitrary subset U of S. In this way $\emptyset$ is open, because $\mathsf{Ext}(\emptyset) = \emptyset$, and closure under union is automatic, because obviously $\cup_{i \in I} \mathsf{Ext}(U_i) = \mathsf{Ext}(\cup_{i \in I} U_i)$. So, all we have to do is to require that $\mathsf{Ext}(S)$ be the whole X, that is,

$$(B_1) \quad X = \mathsf{Ext}(S)$$

and that closure under finite intersections holds, that is,

$$\begin{aligned}(B_2) \quad & (\forall a, b \in S)(\forall x \in X)\ ((x\ \varepsilon\ \mathsf{ext}(a) \cap \mathsf{ext}(b)) \rightarrow \\ & \quad (\exists c \in S)\ (x\ \varepsilon\ \mathsf{ext}(c)\ \&\ \mathsf{ext}(c) \subseteq \mathsf{ext}(a)\ \&\ \mathsf{ext}(c) \subseteq \mathsf{ext}(b)))\end{aligned}$$

It is not difficult to realize that this amounts to the standard definition saying that $\{\mathsf{ext}(a) \subseteq X |\ a \in S\}$ is a base (see for instance [Eng77]).

We can make (B_2) a little shorter by introducing an abbreviation, that is

$$a \downarrow b \equiv \{c : S |\ \mathsf{ext}(c) \subseteq \mathsf{ext}(a)\ \&\ \mathsf{ext}(c) \subseteq \mathsf{ext}(b)\}$$

so that it becomes $(\forall a, b \in S)\ \mathsf{ext}(a) \cap \mathsf{ext}(b) \subseteq \mathsf{Ext}(a \downarrow b)$. But note that $c \varepsilon a \downarrow b$ implies that $\mathsf{ext}(c) \subseteq \mathsf{ext}(a) \cap \mathsf{ext}(b)$, and hence $\mathsf{Ext}(a \downarrow b) \subseteq \mathsf{ext}(a) \cap \mathsf{ext}(b)$. Thus we arrived at the following definition.

[1] All the set-theoretical notions that we use are conform to the subset theory for Martin-Löf's type theory as presented in [SV98]. In particular, we will use the symbol $\in$ for the membership relation between an element and a set or a collection and ε for the membership relation between an element and a subset, which is never a set but a propositional function so that $a \varepsilon U$ means $U(a)$.

Definition 1.1 *A* concrete topological space *is a triple* $\mathcal{X} \equiv (X, S, \mathsf{ext})$ *where* X *and* S *are sets and* ext *is a binary relation from* S *to* X *satisfying:*

$$(B_1) \quad X = \mathsf{Ext}(S)$$

$$(B_2) \quad (\forall a, b \in S)\ \mathsf{ext}(a) \cap \mathsf{ext}(b) = \mathsf{Ext}(a \downarrow b)$$

1.2 Towards formal topologies

The notion of formal topology arises by describing the structure induced by a concrete topological space on the *formal side*, that is, the side of the set S of the names. The reason for such a move is that the definition of concrete topological space is too restrictive, given that in the most interesting cases of topological space we do not have, from a constructive point of view, a *set* of points to start with. One way to obtain such a structure is to introduce two operators which link the *concrete side*, that is, the side of the set X of the concrete points, with the formal side. The intention is to allow dealing with concrete open and closed subsets of the topological space X by using only the names of the basic opens, that is, the elements of the set S.

The problem to identify the open sets is easily solved. Since the elements in S are names for basic opens of the topology on X, we can obtain their *extension*, that is, the concrete basic open, by using the operator ext. Now, by definition, any open set is the union of basic opens and hence it can be specified in the formal side by using the subset of all the (names of the) basic opens which are used to form it.

It is not difficult to check that, provided the conditions (B_1) and (B_2) are satisfied, in this way we obtain a topology on the set X. Moreover, it is the correct topology, namely, any open subset of the topology whose base is the family $(\mathsf{ext}(a))_{a \in S}$ is extensionally equal to the extension of some subset of S. In fact, from a topological point of view, an open subset A of X is characterized by the property of being the union of all the basic opens that it contains or, equivalently, to coincide with its interior $\mathsf{Int}(A)$, where

$$\mathsf{Int}(A) \equiv \{x \in X |\ (\exists a \in S)\ x \varepsilon \mathsf{ext}(a)\ \&\ \mathsf{ext}(a) \subseteq A\}$$

Of course, for any $A \subseteq X$, $\mathsf{Int}(A) \subseteq A$ and thus a subset A is open if and only if $A \subseteq \mathsf{Int}(A)$.

Theorem 1.2 *Let* $U \subseteq S$. *Then* $\mathsf{Ext}(U)$ *is an open subset of* X.

Proof. Let us suppose that $x \varepsilon \mathsf{Ext}(U)$ then there exists $a \in S$ such that $x \varepsilon \mathsf{ext}(a)$ and $a \varepsilon U$; but the latter yields $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$ and hence $x \varepsilon \mathsf{Int}(\mathsf{Ext}(U))$, that is, $\mathsf{Ext}(U)$ is open. ∎

We can now prove the following theorem which characterizes the open subsets of X.

Theorem 1.3 *Let* $A \subseteq X$. *Then* A *is an open subset if and only if there exists a subset* U *of* S *such that* $A = \mathsf{Ext}(U)$.

Proof. In the previous theorem we proved that, for any subset U of S, $\mathsf{Ext}(U)$ is an open subset of X. Now, let A be an open subset of X and consider the subset $U_A \equiv \{a \in S |\ \mathsf{ext}(a) \subseteq A\}$. Then $A = \mathsf{Ext}(U_A)$. In fact $\bigcup_{a \varepsilon U_A} \mathsf{ext}(a) \subseteq A$ is obvious and if $x \varepsilon A$ then there exists $a \in S$ such that $x \varepsilon \mathsf{ext}(a)$ and $\mathsf{ext}(a) \subseteq A$, since A is open; hence $a \varepsilon U_A$ and so $x \varepsilon \mathsf{Ext}(U_A)$. ■

The proof of the previous theorem shows how to find, for any given open subset A of X, a suitable subset U_A of S such that A and $\mathsf{Ext}(U_A)$ are extensionally equal; we chose the *biggest* among the possible subsets, that is, the one which contains *all* of the suitable basic opens. It is clear that in general this is not the only choice and that it is well possible that two different subsets of S have the same extension. Thus, we don't have a bijective correspondence between concrete opens and subsets of S and we need to introduce an equivalence relation on the formal side if we want to obtain such a correspondence. What we need is a relation which identifies the subsets U and V when $\mathsf{Ext}(U) = \mathsf{Ext}(V)$. We can simplify a little the search for such a relation if we realize that $\mathsf{Ext}(U) = \mathsf{Ext}(V)$ holds if and only if, for any $a \in S$, $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$ if and only if $\mathsf{ext}(a) \subseteq \mathsf{Ext}(V)$.

Theorem 1.4 *Let U and V be subsets of S. Then $\mathsf{Ext}(U) = \mathsf{Ext}(V)$ if and only if, for all $a \in S$, $(\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)) \leftrightarrow (\mathsf{ext}(a) \subseteq \mathsf{Ext}(V))$.*

Proof. From left to right the statement is obvious. On the other hand, let us suppose that $x \varepsilon \mathsf{Ext}(U)$, then $x \varepsilon \mathsf{ext}(u)$ for some $u \varepsilon U$; but $u \varepsilon U$ yields $\mathsf{ext}(u) \subseteq \mathsf{Ext}(U)$ and hence the assumption yields $\mathsf{ext}(u) \subseteq \mathsf{Ext}(V)$ and thus $x \varepsilon \mathsf{Ext}(V)$ follows from $x \varepsilon \mathsf{ext}(u)$. The proof of the other inclusion is completely similar. ■

Thus we need to introduce, in the formal side, an infinitary binary relation $\lhd$, that we will call *cover*, between elements and subsets of S, whose intended meaning is that, for $a \in S$ and $U \subseteq S$,

$$a \lhd U \text{ if and only if } \mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$$

The problem has now became to find a complete characterization for the cover relation, which can be expressed completely within the formal side, that is, making no reference to points. For the solution of this problem let us wait for the next section.

Meanwhile, let us turn our attention to closed subsets. Here we have to face the problem that from an intuitionistic point of view we cannot simply identify the closed subsets with the complements of the open subsets. Thus our plan is to follow for closed subsets an approach similar to the one that we used for the open subsets and hence we need a primitive definition for them too. Of course the problem is that we want to identify a closed subset by using only the basic opens, which are the only subsets of X that can be named in the formal side. But note that a subset $A \subseteq X$ is closed if and only if any point which cannot be separated from A by mean of a basic open is inside A, or, equivalently, if A is equal to its closure $\mathsf{Cl}(A)$, where

$$\mathsf{Cl}(A) \equiv \{x \in X |\ (\forall a \in S)\ x \varepsilon \mathsf{ext}(a) \rightarrow \mathsf{ext}(a) \between A\}$$

where $\mathsf{ext}(a) \between A$ is a shorthand for $(\exists y \in X)\ y \varepsilon \mathsf{ext}(a)\ \&\ y \varepsilon A$ that we will read $\mathsf{ext}(a)$ *meets* A. It is straightforward to verify that, for any subset A of X, $A \subseteq \mathsf{Cl}(A)$ and hence a subset is closed if and only if it contains its closure.

The key observation to find a formal characterization of the concrete closed subsets is that a closed subset is completely determined by the collection of the basic opens which meet it.

Theorem 1.5 *Let A and B be two closed subsets of the concrete topological space on the set X whose base is the family $(\mathsf{ext}(a))_{a \in S}$. Then A and B are equal if and only if, for any $a \in S$, $(\mathsf{ext}(a) \between A) \leftrightarrow (\mathsf{ext}(a) \between B)$.*

Proof. If $A = B$ then obviously $(\mathsf{ext}(a) \between A) \leftrightarrow (\mathsf{ext}(a) \between B)$ holds for any $a \in S$. On the other hand, if we assume that, for any $a \in S$, $(\mathsf{ext}(a) \between A) \leftrightarrow (\mathsf{ext}(a) \between B)$, then, by using the fact that A and B are closed subsets, for any $x \in X$, we obtain that $x \varepsilon A$ iff $x \varepsilon \mathsf{Cl}(A)$ iff $(\forall a \in S)\ x \varepsilon \mathsf{ext}(a) \rightarrow \mathsf{ext}(a) \between A$ iff $(\forall a \in S)\ x \varepsilon \mathsf{ext}(a) \rightarrow \mathsf{ext}(a) \between B$ iff $x \varepsilon \mathsf{Cl}(B)$ iff $x \varepsilon B$, that is, A and B are equal. ■

So, in order to have a complete information on a concrete closed subset we can simply collect, in the formal side, all the basic opens which meet it. It is then necessary to introduce a new operator, besides Ext, which links the formal side with the concrete one and which allows to obtain back the closed subset when we are given with the collection of the basic opens which meet it. To this aim we are going to characterize the set of all the points in X which are of adherence for a given family of basic opens, that is, supposing F is a subset of basic opens, we want to consider the subset of X of the points whose basic neighborhoods are all in F. Thus, for any $F \subseteq S$, we put

$$\mathsf{Rest}(F) \equiv \{x \in X |\ (\forall a \in S)\ x \varepsilon \mathsf{ext}(a)\ \rightarrow\ a \varepsilon F\}$$

Let us state immediately some obvious facts on the operator $\mathsf{Rest}(-)$.

Lemma 1.6 *Let $a \in S$ and $F \subseteq S$. Then, if $\mathsf{ext}(a) \between \mathsf{Rest}(F)$ then $a \varepsilon F$.*

Proof. Suppose that $\mathsf{ext}(a) \between \mathsf{Rest}(F)$ holds, then there exists a point $y \in X$ such that $y \varepsilon \mathsf{ext}(a)$ and $y \varepsilon \mathsf{Rest}(F)$; the latter means that, for all $b \in S$, if $y \varepsilon \mathsf{ext}(b)$ then $b \varepsilon F$ and hence $a \varepsilon F$ follows since $y \varepsilon \mathsf{ext}(a)$. ■

Theorem 1.7 *Let $F \subseteq S$. Then $\mathsf{Rest}(F)$ is a closed subset of X.*

Proof. Suppose $x \varepsilon \mathsf{Cl}(\mathsf{Rest}(F))$. Then, for any $a \in S$, $x \varepsilon \mathsf{ext}(a)$ yields that $\mathsf{ext}(a) \between \mathsf{Rest}(F)$ and hence, by the previous lemma, if $x \varepsilon \mathsf{ext}(a)$ then $a \varepsilon F$ that is, $x \varepsilon \mathsf{Rest}(F)$. ■

We can finally prove the following theorem.

Theorem 1.8 *Let A be a subset of X. Then A is a closed subset if and only if there exists a subset F of S such that $A = \mathsf{Rest}(F)$.*

Proof. We already proved that, for any subset F of S, $\mathsf{Rest}(F)$ is a closed subset of X. So, let A be a closed subset of X and consider the subset $F_A \equiv \{a \in S |\ \mathsf{ext}(a) \between A\}$. Then $A = \mathsf{Rest}(F_A)$. In fact, supposing $x \varepsilon A$, for any $a \in S$, if $x \varepsilon \mathsf{ext}(a)$ then $\mathsf{ext}(a) \between A$ and hence $a \varepsilon F_A$ and thus $x \varepsilon \mathsf{Rest}(F_A)$; let us now suppose that $x \varepsilon \mathsf{Rest}(F_A)$, thus, for all $a \in S$, if $x \varepsilon \mathsf{ext}(a)$ then $a \varepsilon F_A$, that is, $\mathsf{ext}(a) \between A$, thus $x \varepsilon \mathsf{Cl}(A)$ which yields $x \varepsilon A$ since A is closed. ■

Thus, we have solved the problem of dealing with concrete closed subsets by using only subsets of the set S of names of basic opens, but, as in the previous case with open subsets, in the proof of the theorem above we chose a suitable subset which corresponds to a given concrete closed subset and there may well be other subsets which correspond to the same closed subset. So, also in this case we need to define an equivalence relation between subsets of S such that two subsets F and G are equal if and only if $\mathsf{Rest}(F) = \mathsf{Rest}(G)$.

Theorem 1.9 *Let F and G be two subsets of S. Then $\mathsf{Rest}(F) = \mathsf{Rest}(G)$ if and only if, for all $a \in S$, $(\mathsf{ext}(a) \between \mathsf{Rest}(F)) \leftrightarrow (\mathsf{ext}(a) \between \mathsf{Rest}(G))$.*

Proof. We have already proved that, for any $F, G \subseteq S$, $\mathsf{Rest}(F)$ and $\mathsf{Rest}(G)$ are closed subsets. Then the result is an immediate consequence of theorem 1.5. ■

Then, in order to have a completely formal counterpart of a closed subset, we need to find the formal conditions which state that $\mathsf{ext}(a) \between \mathsf{Rest}(F)$. To this aim we introduce an infinitary relation $\ltimes$ between elements and subsets of S, that we call *positivity predicate*, whose intended meaning is that, for any $a \in S$ and $F \subseteq S$,

$$a \ltimes F \text{ if and only if } \mathsf{ext}(a) \between \mathsf{Rest}(F)$$

As for the cover relation above, here also the problem is to find the correct conditions on the positivity predicate. The rest of the paper is devoted to this topic.

2 Heyting algebras and formal topologies

To find the correct conditions on the cover relation and the positivity predicate we will work with a typical case study, that is, the case the basic opens are the elements of a Heyting algebra H. Since we are just looking for the correct conditions and we are not actually developing constructive topology, we will not feel stuck to intuitionistic logic and we will use sometimes classical reasoning. We are confident that this approach is not too dangerous and we hope to be forgiven by the most strict believers.

Our strategy will be to look for conditions which let us construct a bijective correspondence between the collection of the concrete opens and the one of the formal opens and between the collection of the concrete closed subsets and the one of the formal closed subsets. In this way we will be able to deal with open and closed subsets of a concrete topological space by using only the corresponding subsets of the set of the basic opens. The problem is that at present there is no concrete side. In order to recognize the usual topological situation we have to fill the basic opens with points.

Definition 2.1 (Formal point) *A subset α of H is a* formal point *if, for all $a, b \in H$ and all subset U of H such that $\bigvee U$ exists,*

$$\textit{(positivity)} \quad \neg(0_H \varepsilon \alpha) \qquad \textit{(non-emptyness)} \quad 1_H \varepsilon \alpha$$

$$\textit{(down-closure)} \quad \frac{a\varepsilon\alpha \qquad b\varepsilon\alpha}{a \wedge b\varepsilon\alpha} \qquad \textit{(up-closure)} \quad \frac{a\varepsilon\alpha \qquad a \leq \bigvee U}{U \between \alpha}$$

where $U \between \alpha$ is a short-hand for $(\exists b \in H)\ b\varepsilon U\ \&\ b\varepsilon\alpha$.

From a technical point of view a formal point is a completely prime proper filter. However, its geometric interpretation should be clear: a formal point α is "contained" in a basic open a if a is an element of the filter α. Thus, we can obtain a topological space on the collection Pt_H of all the formal points by setting, for any $a \in H$,

$$\mathsf{ext}(a) \equiv \{\alpha \in \mathsf{Pt}_H |\ a\varepsilon\alpha\}$$

and using the collection $\{\mathsf{ext}(a)|\ a \in H\}$ as a base. In fact, the conditions B_1 and B_2 of the previous section 1.1 are obviously satisfied since for any formal point α there is an element in H, namely 1_H, which "contains" α, and if α is "contained" both in a and b, then there exists an element c in H, namely $a \wedge b$, such that α is "contained" in c and $\mathsf{ext}(c) \subseteq \mathsf{ext}(a) \cap \mathsf{ext}(b)$ since formal points are filters.

Our aims are now, first, to find two suitable collections of subsets of H such that a bijective correspondence can be defined with the collection of the open subsets and the collection of the closed subsets of this topology and, second, to characterize the collections determined in this way without any reference to the formal points.

Let us begin with the open sets. In analogy with what we did in the proof of theorem 1.3, we can associate to any open subset A of Pt_H the subset $\mathsf{Id}(A)$ of H by using the following definition.

$$\mathsf{Id}(A) \equiv \{a \in H |\ \mathsf{ext}(a) \subseteq A\}$$

Consider now any *complete ideal* of H, that is, any subset I of H such that, for any $a \in H$ and any $U \subseteq H$ such that $\bigvee U$ exists, satisfies the following condition:

$$\frac{a \leq \bigvee U \qquad U \subseteq I}{a\varepsilon I}$$

It is easy to check that the single condition above is sufficient to define an ideal. In fact, $0_H \varepsilon I$ because $0_H \leq \bigvee \emptyset$; moreover, provided $a, b\varepsilon I$, we obtain that $a \vee b\varepsilon I$, because $a \vee b \leq \bigvee\{a, b\}$; finally, if $a\varepsilon I$ and $b \leq a$ then obviously $b\varepsilon I$ since $\{a\} \subseteq I$ if and only if $a\varepsilon I$.

It is not difficult to prove that, for any open subset A of Pt_H, $\mathsf{Id}(A)$ is a complete ideal of H.

Theorem 2.2 *Let A be any open subset of the topological space on Pt_H. Then $\mathsf{Id}(A)$ is a complete ideal of H.*

Proof. Let us suppose both that U is a subset of $\mathsf{Id}(A)$ such that $\bigvee U$ exists and that $a \leq \bigvee U$. Assume now that β is a formal point such that $\beta \in \mathsf{ext}(a)$; then $a \varepsilon \beta$ and hence there exists $u \varepsilon U$ such that $u \varepsilon \beta$ since β is a formal point and $a \leq \bigvee U$. Thus $u \varepsilon \mathsf{Id}(A)$, since $U \subseteq \mathsf{Id}(A)$, and hence $\mathsf{ext}(u) \subseteq A$ which yields $\beta \in A$, since $\beta \in \mathsf{ext}(u)$. We thus proved that, for any formal point β, $\beta \in \mathsf{ext}(a)$ yields $\beta \in A$, that is, $a \varepsilon \mathsf{Id}(A)$. ■

Thus, "being a complete ideal" is a property that, for any open set A, is satisfied by $\mathsf{Id}(A)$. We will prove now that this is a characteristic property, that is, we will prove that complete ideals correspond to open subsets.

Let us set, for any subset I of H,

$$\mathsf{Ext}(I) \equiv \{\alpha \in \mathsf{Pt}_H \mid \alpha \between I\}$$

This definition is an instance of the analogous definition in section 1.1 and it agrees with the definition of ext above; indeed $\mathsf{ext}(a) = \mathsf{Ext}(\{a\})$ because, for any formal point α, $a \varepsilon \alpha$ if and only if $\alpha \between \{a\}$.

Then we can prove the following theorem which is completely analogous to theorem 1.2.

Theorem 2.3 *Let I be any subset of the Heyting algebra H. Then $\mathsf{Ext}(I)$ is an open subset of the topology on Pt_H.*

Proof. We have to prove that $\mathsf{Ext}(I) \subseteq \mathsf{Int}(\mathsf{Ext}(I))$. To this aim, let us suppose that $\alpha \in \mathsf{Ext}(I)$, that is, $(\exists a \in H)\ a \varepsilon \alpha\ \&\ a \varepsilon I$. Thus we know that there exists an element $a \in H$ such that $a \varepsilon \alpha$ and in order to conclude that $\alpha \in \mathsf{Int}(\mathsf{Ext}(I))$ we have to prove only that, supposing $\beta \in \mathsf{Pt}_H$ and $\beta \in \mathsf{ext}(a)$, that is, $a \varepsilon \beta$, we can conclude that $\beta \in \mathsf{Ext}(I)$, that is, $\beta \between I$. But this is immediate since $a \varepsilon I$ and hence β and I meets in a. ■

It is interesting to note that the previous proof works for any subset of H and not only for complete ideals. When A is an open subset we can "close the circle".

Theorem 2.4 *Let A be any open subset of the topological space on Pt_H. Then $\mathsf{Ext}(\mathsf{Id}(A)) = A$.*

Proof. The proof consists just in expanding the definitions. In fact, $\alpha \in \mathsf{Ext}(\mathsf{Id}(A))$ iff $\alpha \between \mathsf{Id}(A)$ iff there exists $a \in H$ such that $a \varepsilon \alpha$ and $a \varepsilon \mathsf{Id}(A)$ iff there exists $a \in H$ such that $a \varepsilon \alpha$ and $\mathsf{ext}(a) \subseteq A$ iff $\alpha \in \mathsf{Int}(A)$ iff $\alpha \in A$ where the last step holds because A is an open set. ■

In order to prove that the correspondence between complete ideals and open subsets is bijective we still have to prove that, for any complete ideal I of H, $\mathsf{Id}(\mathsf{Ext}(I)) = I$. One inclusion is straightforward: if $a \varepsilon I$ then, for any formal point β, if $a \varepsilon \beta$ then $\beta \between I$, that is, if $a \varepsilon \beta$ then $\beta \in \mathsf{Ext}(I)$, which means that

$a\varepsilon\mathsf{Id}(\mathsf{Ext}(I))$. The proof of the other inclusion is much more complex. In fact, we should prove that if, for any formal point β, $a\varepsilon\beta$ yields $\beta \between I$, then $a\varepsilon I$. It is well known that this result can not be proved in the most general setting since there are Heyting algebras with no formal points[2] and hence, in one of such a Heyting algebra the complete ideal $\{0_H\}$ can not be equal to $\mathsf{Id}(\mathsf{Ext}(\{0_H\}))$; in fact in this case $\mathsf{Id}(\mathsf{Ext}(\{0_H\}))$ is equal to the whole algebra H since the universal quantification on the collection of the formal points is always true. However, this statement can be proved if we stay strictly within a predicative setting, namely, if we assume that the collection $\mathcal{U}$ of the subsets of H which have a supremum is *set-indexed*, that is, there exists an inductive set[3] and a map from such a set onto $\mathcal{U}$. Indeed, in this case we can prove the following theorem.

Theorem 2.5 *Let H be a Heyting algebra such that the collection of the subsets which have a supremum is set-indexed. Then, if, for all formal points β, $a\varepsilon\beta$ yields $b\varepsilon\beta$, then $a \leq b$.*

Proof. In order to simplify the notation let us assume that the set used to index the collection of the subsets of H which have a supremum is the set of the natural numbers, so that induction is the usual one. The proof consists in building a formal point α such that $a\varepsilon\alpha$ and $(b\varepsilon\alpha) \to (a \leq b)$. Of course, if $a = 0_H$ it is not possible to build the formal point α such that $a\varepsilon\alpha$ since, for any formal point α, $0_H \not\varepsilon\alpha$ must hold. Thus, let us assume that $a \neq 0_H$.

To construct the formal point α we build, in a countable number of steps, a sequence $c_0, \ldots, c_n, \ldots$ of elements of H such that, for any $n \geq 0$, $c_n \neq 0_H$ and $c_n \leq b$ yields $a \leq b$.

Let $U_0, \ldots, U_n, \ldots$ be a list of the subsets of H which have a supremum and construct a new countable list $W_0, \ldots, W_m, \ldots$ of subsets of H out of the list $U_0, \ldots, U_n, \ldots$ in such a way that any subset U_i appears a countable number of times among $W_0, \ldots, W_m, \ldots$. Now, set $c_0 = a$; hence clearly $c_0 \neq 0_H$, since we assumed that $a \neq 0_H$, and $c_0 \leq b$ yields $a \leq b$. Let us suppose now that we have defined the element c_n such that $c_n \neq 0_H$ and $c_n \leq b$ yields $a \leq b$ and define c_{n+1} by cases as follows:

$$c_{n+1} = \begin{cases} c_n \wedge w_n & \text{if } c_n \leq \bigvee W_n \\ c_n & \text{otherwise} \end{cases}$$

where w_n is an element of W_n such that $c_n \wedge w_n \neq 0_H$ and $c_n \wedge w_n \leq b$ yields $a \leq b$. Indeed, we can prove that such an element w_n exists by using classical logic as follows. Let us set $T_n \equiv \{w\varepsilon W_n |\ c_n \wedge w \neq 0_H\}$; then we can prove that if $c_n \leq \bigvee W_n$ then the subset T_n is not empty; indeed if $c_n \leq \bigvee W_n$ we have that $c_n = c_n \wedge \bigvee W_n = \bigvee_{w\varepsilon W_n} c_n \wedge w = \bigvee_{w\varepsilon T_n} c_n \wedge w$ and hence if T_n would be empty then c_n would be equal to 0_H which is contrary to the inductive

[2] One can consider for instance the Heyting algebra of the Cantor space which has no completely prime filter.

[3] By inductive set we mean a set according to Martin-Löf's type theory where any set is defined by using inductively some introduction rules.

hypothesis[4]; then, according to classical logic, we can prove that the required element w_n exists by showing that if $c_n \wedge w \leq b$, for all $w \varepsilon T_n$, then $a \leq b$; now, provided $c_n \leq \bigvee W_n$, if we assume that, for all $w \varepsilon T_n$, $c_n \wedge w \leq b$ then we obtain that $c_n = c_n \wedge \bigvee W_n = \bigvee_{w \varepsilon W_n} c_n \wedge w = \bigvee_{w \varepsilon T_n} c_n \wedge w \leq b$, and hence $c_n \leq b$ so that $a \leq b$ follows by the inductive hypothesis.

We can now define the formal point α by setting $\alpha \equiv \bigcup_{n \in \omega} \uparrow c_n$, where $\uparrow c_n \equiv \{c \in H |\ c_n \leq c\}$. In fact, we can immediately prove that $\neg(0_H \varepsilon \alpha)$, otherwise there would be an index $n \in \omega$ such that $0_H \varepsilon \uparrow c_n$, that is, $c_n = 0_H$ and we already showed that this is not possible. Moreover α is a filter since it is the union of a chain of filters because, for all $n \geq 0$, $c_{n+1} \leq c_n$ and hence the filter $\uparrow c_n$ is contained in the filter $\uparrow c_{n+1}$. But it is also a completely prime filter since if $\bigvee U \varepsilon \alpha$ then there is an index $n \in \omega$ such that $\bigvee U \varepsilon \uparrow c_n$; hence, since any U appears a countable number of times in the list $W_0, \ldots, W_m, \ldots$, for some $h \geq n$ it happens that $W_h = U$ and thus $\bigvee W_h = \bigvee U \varepsilon \uparrow c_n \subseteq \uparrow c_h$; so, there exists $w_h \varepsilon U$ such that $w_h \varepsilon \uparrow c_{h+1} \subseteq \alpha$.

Finally, it is clear that $a \varepsilon \alpha$ since $a \varepsilon \uparrow c_0 \subseteq \alpha$ and that $b \varepsilon \alpha$ yields $a \leq b$ because $b \varepsilon \alpha$ means that there exists an index $n \geq 0$ such that $b \varepsilon \uparrow c_n$, that is, $c_n \leq b$, and we already proved that this yields $a \leq b$.

We can now conclude our proof. In fact, under the assumption that $a \neq 0_H$, we showed that $a \varepsilon \alpha$. Then the hypothesis in the statement of the theorem yields that $b \varepsilon \alpha$ and hence $a \leq b$ follows since we proved that $b \varepsilon \alpha$ yields $a \leq b$. Thus we obtain that $(a \neq 0_H) \to a \leq b$, but, $(a = 0_H) \to (a \leq b)$ holds for any a and b in H and hence, by using classical logic, we can conclude that $a \leq b$ holds with no assumption. ■

It is important to note that in the proof of the theorem above we used classical logic. When we will arrive at the definition of formal topology we will need to translate the classical principles that we used here into suitable conditions on a formal topology if we want to preserve the completeness of our formalization.

Let us recall now our original problem. Assume that $a \in H$ and I is a complete ideal of H; then we want to show that if, for any formal point $\beta \in \mathsf{Pt}_H$, $a \varepsilon \beta$ yields $\beta \between I$, then $a \varepsilon I$. To this aim let us consider the set

$$V_a \equiv \bigcup_{a \varepsilon \beta \in \mathsf{Pt}_H} (\beta \cap I)$$

Note that, provided a is contained in some formal point, V_a is not empty since, by assumption, $a \varepsilon \beta$ yields $\beta \between I$. Consider now the subset

$$U_a \equiv a \wedge V_a \equiv \{a \wedge v |\ v \varepsilon V_a\}$$

[4]It can be useful to recall that, given an element $a \in H$ and a subset W of H such that $\bigvee W$ exists, also $\bigvee_{w \varepsilon W} a \wedge w$ exists and it is equal to $a \wedge \bigvee W$. In fact, for any $h \varepsilon \{a \wedge w |\ w \varepsilon W\}$, $h \leq a \wedge \bigvee W$ and, for any $k \in H$ such that, for any $h \varepsilon \{a \wedge w |\ w \varepsilon W\}$, $h \leq k$ we can prove that $a \wedge \bigvee W \leq k$ as follows: by assumption, for any $w \varepsilon W$, $a \wedge w \leq k$ and hence $w \leq a \to k$, thus $\bigvee W \leq a \to k$ and so $a \wedge \bigvee W \leq k$.

We can immediately see that $U_a \subseteq I$. In fact, $u \varepsilon U_a$ means that there is $v \varepsilon V_a$ such that $u = a \wedge v$; but $V_a \subseteq I$, hence $v \varepsilon I$ and thus $u \varepsilon I$ because $u \leq v$.

A bit more complex is to prove that $\bigvee U_a$ exists. In fact, the supremum of U_a is a. Indeed, for any $u \varepsilon U_a$, there exists $v \varepsilon V_a$ such that $u = a \wedge v$ and hence $u \leq a$. On the other hand, assume that, for some $w \in H$, $u \leq w$ holds for any $u \varepsilon U_a$; then we can prove that $a \leq w$ by using theorem 2.5 as follows. Let $\beta \in \mathsf{Pt}_H$ and assume that $a \varepsilon \beta$. Then V_a is not empty and so U_a contains elements in β, namely, all those elements $a \wedge b$ of H such that $b \varepsilon \beta \cap I$. Then $w \varepsilon \beta$ since β is a filter and $a \wedge b \leq w$ since $a \wedge b \varepsilon U$ and all the elements of U are smaller then w. Thus we proved that, for any formal point β, if $a \varepsilon \beta$ then $w \varepsilon \beta$ which yields $a \leq w$.

Then both $U_a \subseteq I$ and $a \leq \bigvee U_a$ hold and hence $a \varepsilon I$ since I is a complete ideal.

It is interesting to note that the theorem

$$(*) \qquad \frac{(\forall \beta \in \mathsf{Pt}_H)\ a \varepsilon \beta \rightarrow \beta \between I}{a \varepsilon I}$$

that we proved now for any $a \in H$ and any complete ideal I, is indeed equivalent to the statement in theorem 2.5. In fact, we proved $(*)$ above as a consequence of theorem 2.5. On the other hand, suppose $b \in H$ and consider the subset $\downarrow b \equiv \{c \in H \mid c \leq b\}$. It is trivial to see that $\downarrow b$ is a complete ideal since, supposing U is a subset of H which have a supremum, $U \subseteq \downarrow b$ yields immediately that $\bigvee U \leq b$ and hence $a \leq \bigvee U$ yields $a \leq b$, that is, $a \varepsilon \downarrow b$. Now, for any formal point β, $\beta \between \downarrow b$ if and only if $b \varepsilon \beta$; moreover, $a \varepsilon \downarrow b$ if and only if $a \leq b$ and hence $(*)$ above yields $((\forall \beta \in \mathsf{Pt}_H)\ a \varepsilon \beta \rightarrow b \varepsilon \beta) \rightarrow (a \leq b)$, that is, the statement in theorem 2.5.

In a similar way we can prove that the fact that $a \neq 0_H \rightarrow a \leq b$ yields $a \leq b$ is also a consequence of $(*)$. In fact, let $\beta \in \mathsf{Pt}_H$ and $a \varepsilon \beta$, then $a \neq 0_H$ and hence $a \neq 0_H \rightarrow a \leq b$ yields $a \leq b$, that is, $a \varepsilon \downarrow b$; now, by using again the assumption that $a \varepsilon \beta$, we obtain that $\beta \between \downarrow b$; hence , by discharging all the assumptions, we get that $(\forall \beta \in \mathsf{Pt}_H)\ a \varepsilon \beta \rightarrow \beta \between \downarrow b$ holds which, by $(*)$, yields $a \varepsilon \downarrow b$, that is, $a \leq b$.

Let us sumerize the last three pages in one sentence. We proved that, under the predicative constraint that the collection of the subsets of H which have a supremum is set-indexed, it can be built a bijective correspondence between the collection of the open subsets of the topological space on Pt_H and the collection of the complete ideals of the Heyting algebra H.

The next question is how to provide a constructive presentation of complete ideals. We can solve this problem if we will be able to solve the following one. Let U be any subset of H; how to build the minimal complete ideal which contains U? A solution is a map $\mathsf{sat} : \mathcal{P}(H) \rightarrow \mathcal{P}(H)$ which satisfies the following conditions:

$$\begin{array}{ll}
\text{(reflexivity)} & U \subseteq \mathsf{sat}(U) \\[1ex]
\text{(infinity)} & \dfrac{a \leq \bigvee W \qquad W \subseteq \mathsf{sat}(U)}{a \varepsilon \mathsf{sat}(U)} \\[1ex]
\text{(minimality)} & \dfrac{U \subseteq V \qquad a\varepsilon V\ [a \leq \bigvee W,\ W \subseteq V]}{\mathsf{sat}(U) \subseteq V}
\end{array}$$

where *infinity* applies only for the subsets W such that $\bigvee W$ exists.

Indeed, *infinity* states that $\mathsf{sat}(U)$ is a complete ideal and *reflexivity* states that it contains U. Finally, *minimality* states that $\mathsf{stat}(U)$ is the minimal complete ideal which contains U.

After *minimality* has been stated explicitly, it should be clear the need for a set which indexes the collection of the subsets which have a supremum, at least from a predicative point of view; in fact, only in this case the impredicative universal quantification on subsets which appears in the minor premise of *minimality* can be transformed into a predicative quantification (see [CSSV] for a more detailed discussion of this topic). On the other hand, this is the only case we are interested in since only in this case we have a bijective correspondence between formal and concrete opens.

Till now we considered *reflexivity*, *infinity* and *minimality* like conditions that $\mathsf{sat}(-)$ has to satisfy, but it is obvious that *minimality* holds if *reflexivity* and *infinity* are considered like the only rules for an inductive generation of the subset $\mathsf{sat}(U)$.

We can prove now that *reflexivity* and *infinity* are the only rules that are necessary to have an inductive generation of all complete ideals. In fact, given any subset U of H we already proved that the inductively generated subset $\mathsf{sat}(U)$ is a complete ideal. On the other hand, any complete ideal I is the image by $\mathsf{sat}(-)$ of some subset of H because $I = \mathsf{sat}(I)$. In fact, *reflexivity* shows immediately that the inclusion $I \subseteq \mathsf{sat}(I)$ always holds while the quickest way to prove the other inclusion is to use *minimality* as follows:

$$\dfrac{I \subseteq I \qquad \dfrac{[a \leq \bigvee W]_1 \quad [W \subseteq I]_1}{a\varepsilon I}}{\mathsf{sat}(I) \subseteq I}\,1$$

where the minor premise holds because I is a complete ideal.

It is interesting to observe that what we did leads immediately to the solution of the problem of a complete and predicative formalization of the cover relation. Indeed, we can first prove that, for any subset U of H, $\mathsf{Ext}(U) = \mathsf{Ext}(\mathsf{sat}(U))$. In fact, *reflexivity*, that is, $U \subseteq \mathsf{sat}(U)$, yields $\mathsf{Ext}(U) \subseteq \mathsf{Ext}(\mathsf{sat}(U))$. On the other hand, suppose that $\alpha \in \mathsf{Ext}(\mathsf{sat}(U))$ holds for a formal point α; then there exists an element a in H such that $a\varepsilon\mathsf{sat}(U)$ and $a\varepsilon\alpha$. Now, since we are supposing that the subset $\mathsf{sat}(U)$ is inductively generated, two possibilities have to be considered for $a\varepsilon\mathsf{sat}(U)$ to hold. Either it holds because $a\varepsilon U$, and in this case we are done since this yields $\alpha \in \mathsf{Ext}(U)$, or there exists a subset W such that $W \subseteq \mathsf{sat}(U)$ and $a \leq \bigvee W$ and hence $a\varepsilon\alpha$ yields $W \between \alpha$, that is, there exists $w\varepsilon W$ such that $w\varepsilon\alpha$ and $w\varepsilon\mathsf{sat}(U)$ and we can obtain also in this case that $\alpha \in \mathsf{Ext}(U)$ by inductive hypothesis.

Recall now that $\mathsf{sat}(U)$ is a complete ideal and hence $\mathsf{Id}(\mathsf{Ext}(\mathsf{sat}(U))) = \mathsf{sat}(U)$. So we obtain

$$\begin{array}{rll} a\varepsilon\mathsf{sat}(U) & \text{iff} & a\varepsilon\mathsf{Id}(\mathsf{Ext}(\mathsf{sat}(U))) \\ & \text{iff} & \mathsf{ext}(a) \subseteq \mathsf{Ext}(\mathsf{sat}(U)) \\ & \text{iff} & \mathsf{ext}(a) \subseteq \mathsf{Ext}(U) \end{array}$$

Thus, if we put

$$a \lhd U \equiv a\varepsilon\mathsf{sat}(U)$$

we have found the rules on the cover relation that we were looking for, at least in the case there is a set-indexed family of subsets of H which have a supremum:

$$\text{(reflexivity)} \quad \frac{a\varepsilon U}{a \lhd U}$$

$$\text{(infinity)} \quad \frac{a \leq \bigvee W \qquad W \lhd U}{a \lhd U}$$

where $W \lhd U$ is a shorthand for $(\forall w\varepsilon W)\ w \lhd U$ and *infinity* applies only for the subsets W such that $\bigvee W$ exists.

A similar approach can be exploited also for the closed subsets of the topological space on Pt_H. As we saw in section 1.2, a subset of H can be associated to the closed subset C by setting

$$\mathsf{Up}(C) \equiv \{a \in H |\ \mathsf{ext}(a) \between C\}$$

Consider now any subset F of H such that, for any $a \in H$ and any $U \subseteq H$ such that $\bigvee U$ exists,

$$\frac{a \leq \bigvee U \qquad a\varepsilon F}{U \between F}$$

We will call a subset which satisfies such a condition an *up-complete set.* Note that any up-complete subset F is up-closed, that is, $a \leq b$ and $a\varepsilon F$ yield $b\varepsilon F$, and that an up-complete set F can never contain 0_H since $0_H \leq \bigvee \emptyset$ and hence $0_H\varepsilon F$ would yield $\emptyset \between F$ which is absurd.

It is possible to prove that, for any closed subset C of the topological space on Pt_H, $\mathsf{Up}(C)$ is an up-complete set .

Theorem 2.6 *Let C be a closed subset of the topological space on Pt_H. Then $\mathsf{Up}(C)$ is an up-complete set.*

Proof. Let us suppose that both U is a subset of H such that $\bigvee U$ exists, $a \leq \bigvee U$ and $a\varepsilon\mathsf{Up}(C)$. Then $\mathsf{ext}(a) \between C$, that is, there is a formal point α such that $a\varepsilon\alpha$ and $\alpha \in C$. Then $\alpha \between U$, since $a \leq \bigvee U$, and hence there exists $u \in U$ such that $u\varepsilon\alpha$ and $\alpha \in C$, that is, $u\varepsilon\mathsf{Up}(C)$, and hence $U \between \mathsf{Up}(C)$. ■

In order to prove that we gave the correct definition we have to be able to associate a closed subset to any up-complete subset of H. To this aim we can specialize to our present setting what we did in section 1.2 and set

$$\mathsf{Rest}(F) \equiv \{\alpha \in \mathsf{Pt}_H \mid \alpha \subseteq F\}$$

Then we can prove the following theorem, which is the analogous of theorem 1.7.

Theorem 2.7 *Let F be any subset of H. Then $\mathsf{Rest}(F)$ is a closed subset of the topological space on Pt_H.*

Proof. In order to prove that $\mathsf{Cl}(\mathsf{Rest}(F)) \subseteq \mathsf{Rest}(F)$, let us suppose that $\alpha \in \mathsf{Cl}(\mathsf{Rest}(F))$, that is, $(\forall a \in H)\ a\varepsilon\alpha \rightarrow \mathsf{ext}(a)\between\mathsf{Rest}(F)$, and assume that $a \in H$ and $a\varepsilon\alpha$. Then $\mathsf{ext}(a)\between\mathsf{Rest}(F)$, that is, there exists $\beta \in \mathsf{Pt}_H$ such that $a\varepsilon\beta$ and $\beta \subseteq F$, which yield that $a\varepsilon F$. Thus, for any $a \in H$ such that $a\varepsilon\alpha$ we showed that $a\varepsilon F$, that is, $\alpha \subseteq F$, i.e. $\alpha \in \mathsf{Rest}(F)$. ■

We are now ready to "close the circle" also for closed subsets.

Theorem 2.8 *Let C be any closed subset of the topological space on Pt_H. Then $C = \mathsf{Rest}(\mathsf{Up}(C))$.*

Proof. The proof is obtained also in this case by expanding the definitions: $\alpha \in \mathsf{Rest}(\mathsf{Up}(C))$ iff $\alpha \subseteq \mathsf{Up}(C)$ iff $(\forall a \in H)\ a\varepsilon\alpha \rightarrow a\varepsilon\mathsf{Up}(C)$ iff $(\forall a \in H)\ a\varepsilon\alpha \rightarrow \mathsf{ext}(a)\between C$ iff $\alpha \in \mathsf{Cl}(C)$ iff $\alpha \in C$ where the last step is a consequence of the fact that C is a closed subset. ■

In order to prove that the correspondence between closed subsets of Pt_H and up-complete subsets of H is bijective we have to show that, for any up-complete subset F of H, $\mathsf{Up}(\mathsf{Rest}(F)) = F$. One inclusion is straightforward: $a\varepsilon\mathsf{Up}(\mathsf{Rest}(F))$ if and only if $\mathsf{ext}(a)\between\mathsf{Rest}(F)$ if and only if there exists a formal point α such that $a\varepsilon\alpha$ and $\alpha \subseteq F$ which yields that $a\varepsilon F$. In order to prove the other inclusion all we need to do is to reverse the last implication above, that is, we need the following theorem.

Theorem 2.9 *Let H be a Heyting algebra such that the collection of the subsets which have a supremum is set-indexed, let F be an up-complete subset of H and suppose that $a\varepsilon F$. Then there exists a formal point α such that $a\varepsilon\alpha$ and $\alpha \subseteq F$.*

Proof. In order to simplify the notation let us assume also in this case that the set used to index the collection of the subsets of H which have a supremum is the set of the natural numbers. Let us build a list $U_0, U_1, \ldots$ of all the subsets which have a supremum such that each subset appears a countable number of times and let us consider the following inductive definition of a sequence $c_0, \ldots, c_n, \ldots$ of elements of H.

$$\begin{array}{lcl} c_0 & = & a \\ c_{n+1} & = & \left\{ \begin{array}{ll} \mathsf{i}(W_n \between F) & \text{if } c_n \leq \bigvee U_n \\ c_n & \text{otherwise} \end{array} \right. \end{array}$$

where $W_n \equiv \{c_n \wedge u|\ u\varepsilon U_n\}$ and $\mathsf{i}(W \between F)$ is an element in the intersection between W and F[5].

It is not difficult to prove by induction that, for any $n \geq 0$, $c_n \varepsilon F$ and that, provided $c_n \leq \bigvee U_n$, $W_n \between F$ holds. In fact, $c_0 = a\varepsilon F$ holds by assumption and $c_0 \leq \bigvee U_0$ yields $c_0 \leq c_0 \wedge \bigvee U_0 = \bigvee W_0$ and hence $W_0 \between F$ follows since F is up-complete. Moreover, if we assume, by inductive hypothesis, both that $c_n \varepsilon F$ and that $c_n \leq \bigvee U_n$ yields $W_n \between F$ then $c_{n+1}\varepsilon F$ holds; in fact, provided $c_n \leq \bigvee U_n$ we obtain $W_n \between F$ and thus c_{n+1} is an element both in W_n and F; otherwise c_{n+1} is equal to c_n and thus it belongs to F by hypothesis. Finally, $c_n \leq \bigvee U_n$ yields $c_n \leq c_n \wedge \bigvee U_n = \bigvee W_n$ and hence also in this case $W_n \between F$ follows since F is up-complete.

Moreover, it is immediate to see that, for any $n \geq 0$, $c_{n+1} \leq c_n$. Consider now, for any $n \geq 0$, the filter $\alpha_n \equiv \uparrow c_n$ generated by c_n, and put

$$\alpha \equiv \bigcup_{n\in\omega} \alpha_n$$

Then α is the formal point that we are looking for. In fact, $\alpha \subseteq F$ since, for each $i \geq 0$, $\alpha_i \subseteq F$ because $c_i \varepsilon F$ and F is up-complete. Moreover $a\varepsilon\alpha$ since $a\varepsilon\alpha_0 \subseteq \alpha$. We have to show now that α is a formal point. First of all $\neg(0_H \varepsilon \alpha)$ since $\alpha \subseteq F$ and $\neg(0_H \varepsilon F)$ because F is up-complete. Moreover $1_H \varepsilon \alpha$ since $1_H \varepsilon \ \uparrow a = \alpha_0 \subseteq \alpha$. Now, note that, for any $i \geq 0$, $\alpha_i \subseteq \alpha_{i+1}$, since $c_{i+1} \leq c_i$, and hence, supposing $b, d\varepsilon\alpha$, we obtain that there exists k such that $b, d\varepsilon\alpha_k$, that is, $c_k \leq b$ and $c_k \leq d$ and hence $c_k \leq b \wedge d$; thus $b \wedge d\varepsilon\alpha_k$ and hence $b \wedge d\varepsilon\alpha$. Finally, if $b\varepsilon\alpha$ and $b \leq \bigvee U$, then, for some natural number k, $b\varepsilon\alpha_k$, that is, $c_k \leq b$, and hence $c_k \leq \bigvee U$, and there is a natural number $h \geq k$ such that $U \equiv U_h$. Then $c_h \leq c_k \leq \bigvee U_h$ and hence $c_{h+1} \equiv \mathsf{i}(U_h \wedge c_h \between F)$ is an element such that $c_{h+1} \equiv u \wedge c_h$ for some $u\varepsilon U$; thus $u\varepsilon\alpha_{h+1}$ and hence $u\varepsilon\alpha$. ■

Thus we proved that, for any Heyting algebra such that the collection of the subsets which have a supremum is set-indexed, there is a bijective correspondence between closed subsets of Pt_H and up-complete subsets of H. The next problem is to find a constructive way to present them. We will follow an approach similar to the one that we used to present complete ideals, namely, we will generate by co-induction a map $\mathsf{pos} : \mathcal{P}(H) \to \mathcal{P}(H)$ which, given any subset F of H, gives the biggest among the up-complete subset which are contained in F. The necessary conditions are

$$\text{(anti-reflexivity)} \quad \mathsf{pos}(F) \subseteq F$$

$$\text{(compatibility)} \quad \frac{a \leq \bigvee W \qquad a\varepsilon\mathsf{pos}(F)}{W \between \mathsf{pos}(F)}$$

$$\text{(maximality)} \quad \frac{G \subseteq F \qquad W \between G\ [a \leq \bigvee W,\ a\varepsilon G]}{G \subseteq \mathsf{pos}(F)}$$

where *compatibility* applies only for the subsets W such that $\bigvee W$ exists.

[5] Provided $W \between F$ holds, the map $\mathsf{i}(-)$ can be defined by using the axiom of choice.

Now, *compatibility* states that $\mathsf{pos}(F)$ is an up-complete subset and *anti-reflexivity* states that it is contained in F. Finally, *maximality* states that $\mathsf{pos}(F)$ is the maximal up-complete subset which is contained in F.

It should be clear that in order to give a predicative meaning to *maximality* it is necessary that the collection of the subsets of H which have a supremum is indexed by a set.

Similarly to what we did with the map $\mathsf{sat}(-)$, we are going to consider here *anti-reflexivity* and *compatibility* like the only rules for a co-inductive generation of the subset $\mathsf{pos}(F)$; in this way *maximality* holds automatically and we do not need to require it. Then, we can prove that they are the only rules that are necessary to generate by co-induction all the up-complete subsets. In fact, given any subset F of H we already proved that $\mathsf{pos}(F)$ is an up-complete subset. On the other hand, any up-complete subset F is the image by $\mathsf{pos}(-)$ of some subset of H since $F = \mathsf{pos}(F)$. Indeed, *anti-reflexivity* yields trivially that $\mathsf{pos}(F) \subseteq F$. To prove that $F \subseteq \mathsf{pos}(F)$ we can use *maximality* as follows:

$$\dfrac{F \subseteq F \qquad \dfrac{[a \leq \bigvee W]_1 \quad [a \varepsilon F]_1}{W \between F}}{F \subseteq \mathsf{pos}(F)}\ 1$$

where the minor premise holds because F is up-complete.

Similarly to the case of the cover relation, we can use the map $\mathsf{pos}(-)$ to define the positivity predicate. Indeed, for any subset F of H, $\mathsf{Rest}(\mathsf{pos}(F)) = \mathsf{Rest}(F)$. In fact, *anti-reflexivity*, that is, $\mathsf{pos}(F) \subseteq F$, yields $\mathsf{Rest}(\mathsf{pos}(F)) \subseteq \mathsf{Rest}(F)$. On the other hand, suppose that, for a formal point α, $\alpha \in \mathsf{Rest}(F)$; then $\alpha \subseteq F$. But a formal point α is an up-complete subset, since $a \varepsilon \alpha$ and $a \leq \bigvee W$ yield $W \between \alpha$, and hence $\alpha \subseteq F$ yields $\alpha \subseteq \mathsf{pos}(F)$ since $\mathsf{pos}(F)$ is the biggest among the up-complete subsets which are contained in F. Thus $\alpha \in \mathsf{Rest}(\mathsf{pos}(F))$.

Recall now that $\mathsf{pos}(F)$ is up-complete and hence $\mathsf{Up}(\mathsf{Rest}(\mathsf{pos}(F))) = \mathsf{pos}(F)$ holds. So we obtain

$$\begin{array}{rll} a \varepsilon \mathsf{pos}(F) & \text{iff} & a \varepsilon \mathsf{Up}(\mathsf{Rest}(\mathsf{pos}(F))) \\ & \text{iff} & \mathsf{ext}(a) \between \mathsf{Rest}(\mathsf{pos}(F)) \\ & \text{iff} & \mathsf{ext}(a) \between \mathsf{Rest}(F) \end{array}$$

Thus, if we put

$$a \ltimes F \equiv a \varepsilon \mathsf{pos}(F)$$

we have found the rules on the positivity predicate that we were looking for, with the usual proviso that there must be a set-indexed family of subsets of H which have a supremum:

$$\text{(anti-reflexivity)} \quad \dfrac{a \ltimes F}{a \varepsilon F}$$

$$\text{(compatibility)} \quad \dfrac{a \leq \bigvee W \qquad a \ltimes F}{W \ltimes F}$$

where *compatibility* applies only for the subsets W such that $\bigvee W$ exists and $W \ltimes F$ is a shorthand for $(\exists w \varepsilon W)\ w \ltimes F$.[6]

3 Where formal topologies come from

In the previous section we showed that, given any Heyting algebra H, in order to be able to prove that there is a bijective correspondence between the open subsets of the topological space Pt_H and the complete ideals of H and the closed subsets of Pt_H and the up-complete subsets of H we have to require that the collection of subsets which has supremum is set-indexed. This condition was also essential to be able to provide a constructive presentation of these subsets of H by means of the maps $\mathsf{sat}(-)$ and $\mathsf{pos}(-)$ and the rules

$$\text{(reflexivity)} \quad \frac{a\varepsilon U}{a\varepsilon \mathsf{sat}(U)} \qquad \text{(infinity)} \quad \frac{a \le \bigvee W \qquad W \subseteq \mathsf{sat}(U)}{a\varepsilon \mathsf{sat}(U)}$$

$$\text{(anti-reflexivity)} \quad \frac{a\varepsilon \mathsf{pos}(F)}{a\varepsilon F} \qquad \text{(compatibility)} \quad \frac{a \le \bigvee W \qquad a\varepsilon \mathsf{pos}(F)}{W \between \mathsf{pos}(F)}$$

Indeed the proofs that we provided were by induction and in order to be able to argue by induction it is necessary that $\mathsf{sat}(-)$ and $\mathsf{pos}(-)$ are inductively generated which, from a predicative point of view, is possible only if the collection of the subsets of H which have a supremum is set-indexed.

Inductively generated formal topologies are the result of a generalization of this condition and the rules above from the case of a Heyting algebra and a set-indexed collection of axioms, like $a \le \bigvee U$, to the case of a set S of basic opens and an *axiom-set*, that is, a family of sets $I(a)$, for $a \in S$, and a family $C(a,i)$ of subsets of S, for $a \in S$ and $i \in I(a)$, whose purpose is to state that, for any $i \in I(a)$, a is covered by $C(a,i)$. Thus we arrive at the following definition.

Definition 3.1 *Let $I(a)$ be a set for any $a \in S$ and $C(a,i)$ be a subset of S for any $a \in S$ and $i \in I(a)$. Then, an* inductively generated formal topology *is a structure $(S, \lhd, \ltimes)$ where $\lhd$ is an infinitary relation inductively generated and $\ltimes$ is an infinitary relation co-inductively generated by using the following rules:*

$$\textit{(reflexivity)} \quad \frac{a\varepsilon U}{a \lhd U} \qquad \textit{(infinity)} \quad \frac{C(a,i) \lhd U}{a \lhd U}$$

$$\textit{(anti-reflexivity)} \quad \frac{a \ltimes F}{a\varepsilon F} \qquad \textit{(compatibility)} \quad \frac{a \ltimes F}{C(a,i) \ltimes F}$$

Then, *minimality* and *maximality* have to be adapted to the new framework:

$$(\textit{minimality}) \quad \frac{U \subseteq V \quad a\varepsilon V\ [C(a,i) \subseteq V]}{\lhd(U) \subseteq V}$$

$$(\textit{maximality}) \quad \frac{G \subseteq F \quad C(a,i) \between G\ [a\varepsilon G]}{G \subseteq \ltimes(F)}$$

[6] The reader should be aware of the notation we are using. Indeed, despite there similarity, $W \lhd U$ is a shorthand for $(\forall w \varepsilon W)\ w \lhd U$ while $W \ltimes F$ is a shorthand for $(\exists w \varepsilon W)\ w \ltimes F$.

where $\lhd(U) \equiv \{a \in S|\ a \lhd U\}$ and $\ltimes(F) \equiv \{a \in S|\ a \ltimes F\}$, that is, $\lhd(U) \equiv \mathsf{sat}(U)$ and $\ltimes(F) \equiv \mathsf{pos}(F)$.

Of course, we should add here also all the conditions on $\lhd$ and $\ltimes$ which are the formal counterpart of the properties of a Heyting algebra whose proof required classical logic and which are nevertheless valid in any concrete situation. We recall for instance that, given any a and b in a Heyting algebra H we needed to deduce that $a \leq b$ holds from the fact that $(a \neq 0_H) \to (a \leq b)$. We can express this property here in the language that we developed. In fact, $a \neq 0_H$ can be expressed by stating that $\mathsf{ext}(a)\between\mathsf{Rest}(H)$. Indeed $\mathsf{Rest}(H)$ is equal to the collection Pt_H of all the formal points of H and the proof of theorem 2.5 shows that, if $a \neq 0_H$ then it is contained in some formal point; on the other hand, if $\mathsf{ext}(a)\between\mathsf{Rest}(H)$ holds then there is a formal point which contains a and this yields that $a \neq 0_H$. Then the condition that we are interested in can be expressed in the language of a formal topology $(S, \lhd, \ltimes)$ by stating

$$(positivity) \qquad \frac{(a \ltimes S) \to (a \lhd U)}{a \lhd U}$$

where we have generalized from the order relation $\leq$ to the cover relation $\lhd$. It is now possible to prove that *positivity* is valid in any concrete topological space and hence it should be part of the definition of formal topology since it was used in the proof of completeness of our characterization of the open subsets. Here it follows a proof of its validity. Suppose that (X, S, ext) is a concrete topological space and assume that $(a \ltimes S) \to (a \lhd U)$ holds. This means that $(\exists x \in X)\ x\varepsilon\mathsf{ext}(a)\ \&\ x\varepsilon\mathsf{Rest}(S)$ yields $(\forall x \in X)\ x\varepsilon\mathsf{ext}(a) \to x\varepsilon\mathsf{Ext}(U)$. Since $\mathsf{Rest}(S) = X$ the antecedent of this implication can be simplified into $(\exists x \in X)\ x\varepsilon\mathsf{ext}(a)$. In order to show that $a \lhd U$ holds, that is, $(\forall x \in X)\ x\varepsilon\mathsf{ext}(a) \to x\varepsilon\mathsf{Ext}(U)$, let us assume that $x \in X$ and $x\varepsilon\mathsf{ext}(a)$. Then we immediately obtain that $(\exists x \in X)\ x\varepsilon\mathsf{ext}(a)$ holds and hence, under the same assumptions, also $(\forall x \in X)\ x\varepsilon\mathsf{ext}(a) \to x\varepsilon\mathsf{Ext}(U)$ holds. Let us now use the same assumptions again on this last conclusion and obtain that $x\varepsilon\mathsf{Ext}(U)$ holds. Thus, by discharging the assumptions, we finally obtain that $(\forall x \in X)\ x\varepsilon\mathsf{ext}(a) \to x\varepsilon\mathsf{Ext}(U)$ holds.

In the proof of theorems 2.5, we used the fact that, given any Heyting algebra H, any non-empty subset W of H, and any elements $a, b, c \in H$, in classical logic, $((\forall w\varepsilon W)\ c \wedge w \leq b) \to (a \leq b)$ yields $(\exists w\varepsilon W)\ (c \wedge w \leq b) \to (a \leq b)$. It is not difficult to transform this condition into a condition on the cover relation, but the problem is to find a condition which is classically equivalent to the one above and which can be proved constructively to be valid in any concrete topological space. This problem is still open.

Finally, in the proof of theorems 2.5 and 2.9, we used a definition by cases which rely on the decidability of the order relation in the considered Heyting algebra. Clearly, no valid condition can be proposed in formal topology for this choice.

The second step to arrive at the definition of formal topology consists in forgetting the condition which makes it possible to characterize the closed and the open subsets of a topological space by using only suitable subsets of basic opens and jump into a completely axiomatic definition which just considers *some* of the consequences of the previous rules and looses its character of constructive, namely, inductive, definition. On the other hand, in the definition of formal topology we require no special condition, apart for the stated ones which concern only the cover relation and the positivity predicate, and hence more mathematical structures fall into its realm.

Definition 3.2 *Let S be a set and $R_{\lhd}(a,U)$ and $R_{\ltimes}(a,F)$, for $a \in S$ and $U,F \subseteq S$, be propositions. Then a* formal topology *is a structure $(S,\lhd,\ltimes)$ such that $\lhd$ satisfies the following conditions:*

$$\textit{(reflexivity)} \qquad \frac{a \varepsilon U}{a \lhd U}$$

$$\textit{($\lhd$-axioms)} \qquad \frac{R_{\lhd}(a,U)}{a \lhd U}$$

$$\textit{($\lhd$-transitivity)} \qquad \frac{a \lhd V \qquad V \subseteq \lhd(U)}{a \lhd U}$$

$$\textit{($\lhd$-right)} \qquad \frac{a \lhd V \qquad\qquad a \lhd U}{a \lhd \{c|\ (\exists u \varepsilon U)\ c \lhd \{u\}\ \&\ (\exists v \varepsilon V)\ c \lhd \{v\}\}}$$

$\ltimes$ satisfies the following conditions:

$$\textit{(anti-reflexivity)} \qquad \frac{a \ltimes F}{a \varepsilon F}$$

$$\textit{($\ltimes$-axioms)} \qquad \frac{a \ltimes F}{R_{\ltimes}(a,F)}$$

$$\textit{($\ltimes$-transitivity)} \qquad \frac{a \ltimes F \qquad \ltimes(F) \subseteq G}{a \ltimes G}$$

and $\lhd$ and $\ltimes$ are linked by the following condition:

$$\textit{(compatibility)} \qquad \frac{a \lhd U \qquad a \ltimes F}{U \ltimes F}$$

From a constructive point of view, the problem with this definition of formal topology is to provide examples. Indeed, it is possible to show that the conditions above *cannot* be used to generate by induction a cover relation and by co-induction a positivity predicate (see [Val00]), at least if we want to rest within a predicative approach to topology, which seems to be the only one which guarantees a bijective correspondence between the concrete and the formal side. Anyhow, we will show here that all inductively generated formal topologies are indeed formal topologies.

A detailed discussion on the proof that, in the case of an inductively generated formal topology, $\lhd$-*axiom*, $\lhd$-*transitivity*, $\lhd$-*right* are consequences of *infinity* can be found in [CSSV]. Here we will show a quick proof of the validity of the first two of them and give only a short sketch of the proof of the validity of $\lhd$-*right* since to obtain such a proof one has to introduce some technicalities which are not relevant to the topic of this paper.

- ($\lhd$-transitivity) The quickest way to prove that this rule is valid is to prove its set-theoretic equivalent, that is, $(V \subseteq \lhd(U)) \Rightarrow (\lhd(V) \subseteq \lhd(U))$, by using *minimality* as follows:

$$\dfrac{V \subseteq \lhd(U) \qquad \dfrac{[C(a,i) \subseteq \lhd(U)]_1}{a\varepsilon \lhd(U)}}{\lhd(V) \subseteq \lhd(U)}\,1$$

- ($\lhd$-axiom) In the case of an inductively generated formal topology the only considered axioms have the following shape, for any $a \in S$ and $U \subseteq S$,

$$R_{\lhd}(a,U) \text{ if and only if } (\exists i \in I(a))\ C(a,i) \subseteq U$$

  To prove the validity of $\lhd$-*axiom* it is sufficient to show that, for any $a \in S$ and $i \in I(a)$, $a\varepsilon \lhd(C(a,i))$ holds. In fact, $a\varepsilon \lhd(U)$ can then be derived from $C(a,i) \subseteq \lhd(U)$, which is an immediate consequence of $C(a,i) \subseteq U$, by using $\lhd$-*transitivity* that we already proved to be valid. But $a\varepsilon \lhd(C(a,i))$ is immediate by *infinity* since $C(a,i) \subseteq \lhd(C(a,i))$ holds by *reflexivity*.

- ($\lhd$-right) In order to prove the validity of this rule the easiest way is to prove the validity of one of its equivalent in a structure where also an infimum operation $\wedge$ is present, namely,

$$(\textit{localization}) \qquad \lhd(U) \wedge b \subseteq \lhd(U \wedge b)$$

  To this am let us substitute *infinity* with its *localized* form

$$(\textit{localized-infinity}) \qquad \dfrac{C(a,i) \wedge c \subseteq \lhd(U)}{a \wedge c\varepsilon \lhd(U)}$$

  and modify *minimality* in the obvious way, that is,

$$(\textit{localized-minimality}) \qquad \dfrac{U \subseteq V \quad a \wedge c\varepsilon V\ [C(a,i) \wedge c \subseteq V]}{\lhd(U) \subseteq V}$$

  It is straightforward to see that *localized-infinity* is valid in the case of a Heyting algebra. Let us also introduce the following abbreviation which will make the next proof more readable

$$V \to W \equiv \{x \in S|\ x \wedge V \subseteq \lhd(W)\}$$

  It is immediate to see that $U \wedge V \subseteq \lhd(W)$ if and only if $U \subseteq V \to W$.

  Then *localization* can be proved by using *localized-minimality* as follows:

$$\dfrac{\dfrac{\dfrac{U \wedge b \subseteq \lhd(U \wedge b)}{U \subseteq \{b\} \to (U \wedge b)} \qquad \dfrac{\dfrac{\dfrac{[C(a,i) \wedge c \subseteq \{b\} \to (U \wedge b)]_1}{C(a,i) \wedge c \wedge b \subseteq \lhd(U \wedge b)}}{a \wedge c \wedge b\varepsilon \lhd(U \wedge b)}}{a \wedge c\varepsilon\{b\} \to (U \wedge b)}}{\lhd(U) \subseteq \{b\} \to (U \wedge b)}\,1}{\lhd(U) \wedge b \subseteq \lhd(U \wedge b)}$$

Let us now turn our attention on the conditions on $\ltimes$.

- ($\ltimes$-axiom) In the case of an inductively generated formal topology no axiom for the positivity predicate exists and hence nothing needs to be proved for $\ltimes$*-axiom*.
- ($\ltimes$-transitivity) The quickest way to prove the validity of this rule is to prove its set-theoretic equivalent, that is, $(\ltimes(F) \subseteq G) \Rightarrow (\ltimes(F) \subseteq \ltimes(G))$, by using *maximality* as follows:

$$\dfrac{\ltimes(F) \subseteq G \quad \dfrac{[a\varepsilon \ltimes(F)]_1}{C(a,i) \between \ltimes(F)}}{\ltimes(F) \subseteq \ltimes(G)}\ 1$$

- (compatibility) Validity of *compatibility* can be proved by induction on the length of the proof of $a\varepsilon \lhd(U)$. In fact, if $a\varepsilon \lhd(U)$ was derived from $a\varepsilon U$ then the result is immediate since U and $\ltimes(F)$ meet in a. On the other hand, if $a\varepsilon \lhd(U)$ is a consequence of $C(a,i) \subseteq \lhd(U)$, then, since $a\varepsilon \ltimes(F)$ yields $C(a,i) \between \ltimes(F)$, $U \between \ltimes(F)$ follows by induction.

4 Conclusion

No conclusion at present, but only a first attempt to describe a complete formalization of the concrete situation.

It is not clear if the move from inductively generated formal topologies to formal topologies is convenient: one has to balance between completeness of the formalization and wider number of interesting structures.

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