Fixed-points of continuous function between formal spaces

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Abstract

Formal topology is today an established topic in the development of constructive mathematics and constructive proofs for many classical results of general topology has been obtained by using this approach. In this work, after an introduction to the formal approach to constructive topology and its morphisms, we will show sufficient conditions for a continuous function between formal points to admit a fixed-point and we will illustrate an example to the problem of finding the fixed-point of a monotone operator which maps recursive subsets of the natural numbers into recursive subsets.

1 Formal topologies and their functions

In this section the basic definitions of formal topology will be quickly recalled. Anyhow, the reader interested to have more details on formal topology and a deeper analysis of the foundational motivations for the formal development of topology within Martin-Löf's constructive type theory [ML84] is invited to look for instance at [CSSV] or to wait for a full monograph on formal topology which is in preparation [Sam00].

1.1 Concrete topological spaces

The classical definition of topological space reads: $(X, \Omega(X))$ is a topological space if X is a set and $\Omega(X)$ is a subset of $\mathcal{P}(X)$ which satisfies:

(Ω_1) $\emptyset, X \in \Omega(X)$;

(Ω_2) $\Omega(X)$ is closed under finite intersection;

(Ω_3) $\Omega(X)$ is closed under arbitrary union.

Usually, elements of X are called points and elements of $\Omega(X)$ are called opens.

The quantification implicitly used in (Ω_3) is of the third order, since it says $(\forall F \subseteq \Omega(X)) \bigcup F \in \Omega(X)$, i.e. $(\forall F \in \mathcal{P}(\mathcal{P}(X))) (F \subseteq \Omega(X) \to \bigcup F \in \Omega(X))$. We can "go down" one step by thinking of $\Omega(X)$ as a family of subsets indexed by a set S through a map $\mathsf{n} : S \to \mathcal{P}(X)$, since we can now quantify on S rather than on $\Omega(X)$. But we still have to say $(\forall U \in \mathcal{P}(S))(\exists c \in S) (\cup_{a \varepsilon U} \mathsf{n}(a) = \mathsf{n}(c))$, which is still impredicative[1].

We can "go down" another step by defining opens to be of the form $\mathsf{N}(U) \equiv \cup_{a \varepsilon U} \mathsf{n}(a)$ for an arbitrary subset U of S. In this way $\emptyset$ is open, because $\mathsf{N}(\emptyset) = \emptyset$, and closure under union is automatic, because obviously $\cup_{i \in I} \mathsf{N}(U_i) = \mathsf{N}(\cup_{i \in I} U_i)$. So, all we have to do is to require $\mathsf{N}(S)$ to be the whole X and closure under finite intersections. It is not difficult to realize that this amounts to the standard definition saying that $\{\mathsf{n}(a) \subseteq X | \ a \in S\}$ is a base (see for instance [Eng77]). So, we reach the following definition:

Definition 1.1 *A* concrete topological space *is a triple* $\mathcal{X} \equiv (X, S, \mathsf{n})$ *where* X *is a set of* concrete points*,* S *is a set of* names for basic open subsets*,* n *is a map from* S *into subsets of* X*, called the* neighborhood *map, which associates the names with the basic open subsets and satisfies*

$$(B_1) \quad X = \cup_{a \in S} \mathsf{n}(a)$$

$$(B_2) \quad (\forall a, b \in S)(\forall x \in X) \ (x \varepsilon \mathsf{n}(a) \cap \mathsf{n}(b) \to (\exists c \in S) \ (x \varepsilon \mathsf{n}(c) \ \& \ \mathsf{n}(c) \subseteq \mathsf{n}(a) \cap \mathsf{n}(b)))$$

Now, a map $\mathsf{n} : S \to \mathcal{P}(X)$ is a propositional function with two arguments, that is $\mathsf{n}(a)(x) \ \mathsf{prop} \ [a : S, x : X]$, that is a binary relation. Then we can write it as

$$x \Vdash a \ \mathsf{prop} \ [x : X, a : S]$$

and read it "x lies in a" or "x forces a".

It is convenient to use the following two abbreviations:

$$\begin{array}{rcl} \mathsf{ext}(a) & \equiv & \{x : X | \ x \Vdash a\} \\ \mathsf{Ext}(U) & \equiv & \cup_{a \varepsilon U} \mathsf{ext}(a) \end{array}$$

Hence $x \Vdash a$ is the same as $x \varepsilon \mathsf{ext}(a)$ and thus the map n coincides with ext.

Then (B_1) and (B_2) can be rewritten as

$$(B_1) \quad X = \mathsf{Ext}(S)$$

$$(B_2) \quad (\forall a, b \in S)(\forall x \in X) \ ((x \varepsilon \mathsf{ext}(a) \cap \mathsf{ext}(b)) \to (\exists c \in S) \ (x \varepsilon \mathsf{ext}(c) \ \& \ \mathsf{ext}(c) \subseteq \mathsf{ext}(a) \ \& \ \mathsf{ext}(c) \subseteq \mathsf{ext}(b)))\}$$

[1] All the set-theoretical notions that we use conform to the subset theory for Martin-Löf's type theory as presented in [SV98]. In particular, we will use the symbol $\in$ for the membership relation between an element and a set or a collection and ε for the membership relation between an element and a subset, which is never a set but a propositional function.

We can make (B_2) a bit shorter by introducing another abbreviation, that is

$$a \downarrow b \equiv \{c : S \mid \mathsf{ext}(c) \subseteq \mathsf{ext}(a) \ \& \ \mathsf{ext}(c) \subseteq \mathsf{ext}(b)\}$$

so that it becomes

$$(B_2) \quad (\forall a, b \in S)\ \mathsf{ext}(a) \cap \mathsf{ext}(b) \subseteq \mathsf{Ext}(a \downarrow b)$$

which looks much better.

Note that $c \varepsilon a \downarrow b$ implies that $\mathsf{ext}(c) \subseteq \mathsf{ext}(a) \cap \mathsf{ext}(b)$, so that $\mathsf{Ext}(a \downarrow b) \equiv \cup_{c \varepsilon a \downarrow b} \mathsf{ext}(c) \subseteq \mathsf{ext}(a) \cap \mathsf{ext}(b)$. Then the definition of concrete topological space can be rewritten as follows:

Definition 1.2 *A* concrete topological space *is a triple* $\mathcal{X} \equiv (X, S, \Vdash)$ *where* X *and* S *are sets and* $\Vdash$ *is a binary relation from* X *to* S *satisfying:*

$$(B_1) \quad X = \mathsf{Ext}(S)$$

$$(B_2) \quad (\forall a, b \in S)\ \mathsf{ext}(a) \cap \mathsf{ext}(b) = \mathsf{Ext}(a \downarrow b)$$

1.2 Formal topologies

The notion of formal topology arises by describing as well as possible the structure induced by a concrete topological space on the *formal side*, that is the side of the set S of the names, and then by taking the result as an axiomatic definition. The reason for such a move is that the definition of concrete topological space is too restrictive, given that in the most interesting cases of topological space we do not have, from a constructive point of view[2], a *set* of points to start with and in the definition of concrete topological space we have to require that X and S are sets; in fact, we need to quantify over their elements in order to state the conditions (B_1) and (B_2) and this quantifications have a constructive meaning only if they are sets.

The problem to identify the open sets at the formal side can be solved as follows. Since the elements in S are names for basic opens of the topology on X, then we can obtain their *extension*, that is the concrete basic open, by using the operator ext. Now, by definition, any open set is the union of basic opens and hence it can be specified in the formal side by using the subset of all the (names of the) basic opens which are used to form it.

It is possible to check that, provided the conditions (B_1) and (B_2) are satisfied, in this way we really obtain a topology on the set X. Indeed, the whole space X is an open set, since $X = \mathsf{Ext}(S)$ because of (B_1). And $\emptyset = \mathsf{Ext}(\emptyset)$ is an open set as well; moreover an arbitrary union of open sets is an open set since $\bigcup_{i \in I} \mathsf{Ext}(U_i) = \mathsf{Ext}(\bigcup_{i \in I} U_i)$ can be proved by using a bit of intuitionistic logic; finally, also finite intersection of open sets is an open set since, as a

[2] Here we commit ourselves to Martin-Löf's constructive set theory; hence we distinguish between sets, which can be inductively generated, and collections.

consequence of (B_2), we can prove that $\mathsf{Ext}(U) \cap \mathsf{Ext}(V) = \mathsf{Ext}(U \downarrow V)$ where $U \downarrow V \equiv \{c \in S |\ (\exists u \varepsilon U)\ \mathsf{ext}(c) \subseteq \mathsf{ext}(u)\ \&\ (\exists v \varepsilon V)\ \mathsf{ext}(c) \subseteq \mathsf{ext}(v)\}$. In fact

$$\mathsf{Ext}(U) \cap \mathsf{Ext}(V) \equiv \cup_{a\varepsilon U} \mathsf{ext}(a) \cap \cup_{b\varepsilon V} \mathsf{ext}(b) = \cup_{a\varepsilon U} \cup_{b\varepsilon V}\ \mathsf{ext}(a) \cap \mathsf{ext}(b)$$

So, by (B_2), $\mathsf{Ext}(U) \cap \mathsf{Ext}(V) \subseteq \cup_{a\varepsilon U} \cup_{b\varepsilon V} \mathsf{Ext}(a \downarrow b)$ and hence $\mathsf{Ext}(U) \cap \mathsf{Ext}(V) \subseteq \mathsf{Ext}(U \downarrow V)$ follows since Ext distributes unions. The other inclusion is trivial.

From a topological point of view an open subset of X is characterized by the property of being the union of all the basic opens that it contains or, equivalently, to coincide with its interior $\mathsf{Int}(A)$, where, for any $A \subseteq X$,

$$\mathsf{Int}(A) \equiv \{x \in X |\ (\exists a \in S)\ x \Vdash a\ \&\ \mathsf{ext}(a) \subseteq A\}$$

Of course, for any $A \subseteq X$, $\mathsf{Int}(A) \subseteq A$ and thus a subset A is open if and only if $A \subseteq \mathsf{Int}(A)$.

Theorem 1.3 *Let $A \subset X$. Then A is an open subset if and only if there exists a subset U of S such that $A = \mathsf{Ext}(U)$*

Proof. Let A be an open subset of X and consider the subset $U \equiv \{a \in S |\ \mathsf{ext}(a) \subseteq A\}$. Then $A = \mathsf{Ext}(U)$; in fact $\bigcup_{a\varepsilon U} \mathsf{ext}(a) \subseteq A$ is obvious and if $x \varepsilon A$ then there exists $a \in S$ such that $x \Vdash a$ and $\mathsf{ext}(a) \subseteq A$, since A is open, and hence $a \varepsilon U$ and so $x \varepsilon \mathsf{Ext}(U)$.

On the other hand, let U be any subset of S and suppose that $x \varepsilon \mathsf{Ext}(U)$; then there exists $a \in S$ such that $x \Vdash a$ and $a \varepsilon U$; but the latter yields $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$ and hence $x \varepsilon \mathsf{Int}(\mathsf{Ext}(U))$, i.e. $\mathsf{Ext}(U)$ is open.

The proof of the previous theorem shows how to find, for any given open subset A of X, a suitable subset U of S such that A and $\mathsf{Ext}(U)$ are extensionally equal; we chose the *biggest* among the possible subsets, that is the one which contains *all* of the suitable basic opens. It is clear that in general this is not the only choice and that it is well possible that two different subsets of S have the same extension. Thus we don't have a bijective correspondence between concrete opens and subsets of S and we need to introduce an equivalence relation on the formal side if we want to obtain such a correspondence. What we need is a relation which identifies the subsets U and V when $\mathsf{Ext}(U) = \mathsf{Ext}(V)$. Of course, within a constructive set theory, we cannot introduce such a relation among subsets since the collection of the subsets of a set is not a set, but we can simplify a bit the problem if we realize that $\mathsf{Ext}(U) = \mathsf{Ext}(V)$ holds if and only if, for any $a \in S$, $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$ if and only if $\mathsf{ext}(a) \subseteq \mathsf{Ext}(V)$.

Theorem 1.4 *Let U and V be subsets of S. Then $\mathsf{Ext}(U) = \mathsf{Ext}(V)$ if and only if $(\forall a \in S)\ \mathsf{ext}(a) \subseteq \mathsf{Ext}(U) \leftrightarrow \mathsf{ext}(a) \subseteq \mathsf{Ext}(V)$.*

Proof. From left to right the statement is obvious. To prove the other implication, let us suppose that $x \varepsilon \mathsf{Ext}(U)$; then $x \varepsilon \mathsf{ext}(u)$ for some $u \varepsilon U$; but $u \varepsilon U$ yields $\mathsf{ext}(u) \subseteq \mathsf{Ext}(U)$ and hence the assumption yields $\mathsf{ext}(u) \subseteq \mathsf{Ext}(V)$ and thus $x \varepsilon \mathsf{Ext}(V)$ follows from $x \varepsilon \mathsf{ext}(u)$.

Thus, in order to define the equivalence relation among subsets that we are looking for, we need just to introduce, in the formal side, a new relation

$$\lhd : (a : S)(U : (x : S)\ \mathsf{prop})\ \mathsf{prop}$$

whose intended meaning is that

$$a \lhd U \text{ if and only if } \mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$$

In fact, after the previous theorem, we can define an equivalence relation over the subsets of S by putting

$$U =_{\lhd} V \equiv (\forall a \in S)\ a \lhd U \leftrightarrow a \lhd V$$

and this is the equivalence relation we were looking for since it is immediate to prove the following theorem.

Theorem 1.5 *Let U and V be two subsets of S. Then $U =_{\lhd} V$ if and only if* $\mathsf{Ext}(U) = \mathsf{Ext}(V)$.

Now, in order to obtain a bijective correspondence between formal and concrete open subsets, we could simply state that a formal open is an equivalence class of the relation $=_{\lhd}$. But, since we prefer to avoid to deal with collections of collections of subsets, we simply choose the "fullest" among the representative of an equivalence class by putting

$$\lhd(U) \equiv \{a \in S |\ a \lhd U\}$$

and say that a *formal open* is any subset $\lhd(U)$ for some subset U. Now, $\mathsf{Ext}(U) = \mathsf{Ext}(V)$ if and only if $\lhd(U) = \lhd(V)$ because $U =_{\lhd} V$ if and only if $\lhd(U) = \lhd(V)$. Moreover it is possible to prove that $\lhd(U)$ is a good representative of the equivalence class of the subset U because $\lhd(U) =_{\lhd} U$, i.e. $\lhd(\lhd(U)) = \lhd(U)$. In fact, it is easy to check that the following two conditions on $\lhd$ are valid and hence we can assume them like axiomatic conditions on the formal side.

$$\text{(reflexivity)} \qquad \frac{a \varepsilon U}{a \lhd U}$$

which holds since if $a\varepsilon U$ then $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$, and

$$\text{(transitivity)} \qquad \frac{a \lhd U \qquad U \lhd V}{a \lhd V}$$

where $U \lhd V$ is a short-hand for a derivation of $u \lhd V$ under the assumption that $u\varepsilon U$. The validity of *transitivity* is straightforward because the first assumption means that $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$ and the second yields that $\mathsf{Ext}(U) \subseteq \mathsf{Ext}(V)$.

We can re-write *reflexivity* and *transitivity* by using a set-theoretical notation

$$\text{(reflexivity)} \quad U \subseteq \lhd(U) \qquad \text{(transitivity)} \quad \frac{U \subseteq \lhd(V)}{\lhd(U) \subseteq \lhd(V)}$$

and hence we obtain $\lhd(U) \subseteq \lhd(\lhd(U))$ by *reflexivity* while $\lhd(\lhd(U)) \subseteq \lhd(U)$ is an immediate consequence of *transitivity*.

Thus, we found a relation, that is $\lhd$, and two conditions over it, that is *reflexivity* and *transitivity*, which allow to deal on the formal sides with concrete open subsets. But these conditions are not sufficient to describe completely the concrete situation; for instance there is no condition which describe formally the conditions (B_1) and (B_2).

To formulate (B_2) within the formal side, we can use the fact that we already proved that $\mathsf{Ext}(U) \cap \mathsf{Ext}(V) \subseteq \mathsf{Ext}(U \downarrow V)$. In fact, supposing $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$ and $\mathsf{ext}(a) \subseteq \mathsf{Ext}(V)$, we immediately obtain $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U) \cap \mathsf{Ext}(V)$ and hence $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U \downarrow V)$. Its formal counterpart is

$$(\downarrow\text{-right}) \qquad \frac{a \lhd U \qquad a \lhd V}{a \lhd U \downarrow V}$$

where $U \downarrow V$ must be read at the formal side as

$$U \downarrow V \equiv \{c \in S |\ (\exists u \varepsilon U)\ c \lhd \{u\}\ \&\ (\exists v \varepsilon V)\ c \lhd \{v\}\}$$

To express constructively in a formal way the fact that a basic open subset is inhabited it is convenient to introduce also a second primitive predicate Pos on the element of S whose intended meaning is that

$$\mathsf{Pos}(a) \text{ if and only if } (\exists x \in X)\ x \Vdash a$$

We will require two conditions on this predicate.

$$(\text{monotonicity}) \qquad \frac{\mathsf{Pos}(a) \qquad a \lhd U}{(\exists u \varepsilon U)\ \mathsf{Pos}(u)}$$

$$(\text{positivity}) \qquad \frac{\mathsf{Pos}(a) \to a \lhd U}{a \lhd U}$$

While the meaning of *monotonicity* is obvious and the proof of its validity in any concrete topological space is immediate, *positivity* can be not completely clear. It states two things in one rule: first the fact that a not-inhabited basic open subset is covered by any subset but also it allows proof by cases on the positivity of a when the conclusion is $a \lhd U$ (see [SVV96]). Also the proof of validity of *positivity* is straightforward and it uses only intuitionistic logic.

Note that from an intuitionistic and predicative point of view we cannot use the classically equivalent formulation $\neg(a \lhd \emptyset)$, which just yields to $\neg\neg\mathsf{Pos}(a)$, nor the impredicative formulation $(\forall U \subseteq S)\ a \lhd U \to U \neq \emptyset$, which require a quantification on the collection of all the subsets.

We thus arrived at the main definition.

Definition 1.6 *A* formal topology *is a triple $\mathcal{A} \equiv (S, \lhd, \mathsf{Pos})$ where S is a set, $\lhd$ is an infinitary relation, called* cover *relation, between elements and subsets* *of S, that is* $a \lhd U$ prop $[a : S, U \subseteq S]$*, satisfying the following conditions:*

$$\text{(reflexivity)} \quad \frac{a \varepsilon U}{a \lhd U}$$

$$\text{(transitivity)} \quad \frac{a \lhd U \qquad U \lhd V}{a \lhd V}$$

$$\text{(}\downarrow\text{-right)} \quad \frac{a \lhd U \qquad a \lhd V}{a \lhd U \downarrow V}$$

and Pos *is a predicate over S satisfying the following conditions:*

$$\text{(monotonicity)} \quad \frac{\mathsf{Pos}(a) \qquad a \lhd U}{(\exists u \varepsilon U)\ \mathsf{Pos}(u)}$$

$$\text{(positivity)} \quad \frac{\mathsf{Pos}(a) \to a \lhd U}{a \lhd U}$$

1.3 Formal points

When working in formal topology one is in general interested to those properties of a topological space $(X, \Omega(X))$ which make no reference to the points, that is the elements of X. Thus, he is dispensed of the collection X and it is possible to work by using only the set of the names for the basic opens. But this does not mean that points are out of reach. Indeed, a point can be identified with the filter of all the basic opens which, in the concrete case, contain it. In fact, supposing $(X, S, \Vdash)$ is a concrete topological space, we can associate to any point $x \in X$, the following subset of S

$$\alpha_x \equiv \{a \in S \mid x \Vdash a\}$$

which contains all the basic opens which *contain* x. Then, since from a topological point of view we can *see* only those points which can be distinguished by using the open sets, we are led to identify the concrete points with the subsets α_x, for some $x \in X$.

If we want to move to the formal side, we have to find those properties which characterize such subsets and are expressible in our language. Here, supposing x is any element in X, we propose the following

$$\text{(point not-emptiness)} \quad (\exists a \in S)\ a \varepsilon \alpha_x$$

$$\text{(directness)} \quad \frac{a \varepsilon \alpha_x \qquad b \varepsilon \alpha_x}{(\exists c \varepsilon a \downarrow b)\ c \varepsilon \alpha_x}$$

$$\text{(completeness)} \quad \frac{a \varepsilon \alpha_x \qquad \mathsf{ext}(a) \subseteq \mathsf{Ext}(U)}{(\exists u \varepsilon U)\ u \varepsilon \alpha_x}$$

$$\text{(point positivity)} \quad \frac{a \varepsilon \alpha_x}{(\exists x \in X)\ x \Vdash a}$$

In fact, *point not-emptiness* is just the condition B_1, *directness* is an immediate consequence of the condition B_2, and *completeness* and *point positivity* can be proved by using a bit of intuitionistic logic.

Thus, we are led to the following definition.

Definition 1.7 *Let* $(S, \lhd, \mathsf{Pos})$ *be a formal topology. Then a* formal point *is any non-empty subset* α *of* S *which, for any* $a, b \in S$ *and any* $U \subseteq S$*, satisfies the following conditions:*

$$\text{(directness)} \quad \frac{a \varepsilon \alpha \qquad b \varepsilon \alpha}{(\exists c \varepsilon a \downarrow b)\ c \varepsilon \alpha}$$

$$\text{(completeness)} \quad \frac{a \varepsilon \alpha \qquad a \lhd U}{(\exists u \varepsilon U)\ u \varepsilon \alpha}$$

$$\text{(point positivity)} \quad \frac{a \varepsilon \alpha}{\mathsf{Pos}(a)}$$

In the following we will call $\mathsf{Pt}(S)$ the collection of formal points of the formal topology $\mathcal{S}$. We can give $\mathsf{Pt}(S)$ the structure of a topological space if we mimic the situation of a concrete topological even if $\mathsf{Pt}(S)$ is a collection and not a set. Thus, we put, for any $a \in S$,

$$\mathsf{ext}(a) \equiv \{\alpha \in \mathsf{Pt}(S) |\ a \varepsilon \alpha\}$$

and consider the set-indexed family $(\mathsf{ext}(a))_{a \in S}$. Then the fact that formal points are not-empty subsets shows that the condition (B_1) holds for this set-indexed family since for any formal point α there is an element $a \in S$ such that $\alpha \in \mathsf{ext}(a)$. In a similar way *directness*, together with *completeness*, shows that the condition (B_2) holds; in fact, if $\alpha \in \mathsf{ext}(a)$ and $\alpha \in \mathsf{ext}(b)$ then $a \varepsilon \alpha$ and $b \varepsilon \alpha$ and hence there exists $c \in S$ such that $c \varepsilon a \downarrow b$ and $c \varepsilon \alpha$, that is $c \lhd a$ and $c \lhd b$, i.e. $\mathsf{ext}(c) \subseteq \mathsf{ext}(a)$ and $\mathsf{ext}(c) \subseteq \mathsf{ext}(b)$, and $\alpha \in \mathsf{ext}(c)$. Thus $(\mathsf{ext}(a))_{a \in S}$ is a base for a topology on $\mathsf{Pt}(S)$. Moreover *completeness* shows that this topology *conforms* with the formal topology we started with since $a \lhd U$ yields $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$. To prove that the other implication holds, that is if $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$ then $a \lhd U$, is one of the most important problem in the constructive approach to topology since it requires the existence of "enough" formal points (cf. [Joh82]).

1.4 Continuous functions

A morphism between the topological space $\mathcal{X}$ and the topological space $\mathcal{Y}$ is a functions $\phi : X \to Y$ such that, for any basic open subset $\mathcal{B}$ in $\mathcal{Y}$, the subset $\phi^{-1}(\mathcal{B}) \equiv \{x \in X |\ \phi(x) \in \mathcal{B}\}$ is an open set of $\mathcal{X}$. If we write this condition for the concrete topological spaces $(X, S, \Vdash_1)$ and $(Y, T, \Vdash_2)$ we obtain that the condition for a function $\phi : X \to Y$ to be continuous becomes

$$(\forall b \in T)(\exists U \subseteq S)\ \phi^{-1}(\mathsf{ext}(b)) = \mathsf{Ext}(U)$$

There is only one possible constructive meaning for this sentence, that is, there exists a *function* $F : (b : T)(a : S)$ prop such that, for any $b \in T$, $F(b)$ is a subset of S which satisfies the required condition, that is, it contains the basic opens $a \in S$ such that the image of all the points in $\mathsf{ext}(a)$ is in $\mathsf{ext}(b)$.

Thus, we can simple state that the formal counterpart of a continuous function ϕ between X and Y is a relation F between elements of S and elements of T such that

$$aFb \text{ if and only if } (\forall x \in X)\ x \Vdash a \rightarrow \phi(x) \Vdash b$$

or, equivalently,

$$aFb \text{ if and only if } \mathsf{ext}(a) \subseteq \phi^{-1}(\mathsf{ext}(b))$$

We will use in the following the fact that a function ϕ is continuous if and only if for any $b \in T$ if, for some $x \in X$, $\phi(x) \Vdash b$ then there exists $a \in S$ such that $x \Vdash a$ and the image of any point in $\mathsf{ext}(a)$ is a point in $\mathsf{ext}(b)$, that is aFb. We can state this fact by using a formula in the language of concrete topological spaces if we say that ϕ is continuous if and only if

$$(\forall b \in T)(\forall x \in X)\ \phi(x) \Vdash b \rightarrow (\exists a \in S)\ x \Vdash a \ \&\ aFb$$

The problem with this formulation of the continuity of the function ϕ is that this condition is not expressed completely within the formal side. Hence, in order to develop our constructive approach to topology, we need to find suitable *formal* conditions, which do not rely on the presence of the set of concrete points in order to be formulated, that a relation F has to satisfy to be the formal counterpart of a continuous function. To achieve this result we will proceed as follows. First we will define a function ϕ_F between $\mathsf{Pt}(S)$ and $\mathsf{Pt}(T)$, associated with the relation F. Then, we will look for the conditions on F which are both expressible in the language of formal topologies and allow to prove that ϕ_F is a continuous function between $\mathsf{Pt}(S)$ and $\mathsf{Pt}(T)$. And finally, we will check the validity of such conditions in every concrete topological space.

So, let us suppose that F is a relation between two formal topologies. Then we want to define a map ϕ_F between $\mathsf{Pt}(S)$ and $\mathsf{Pt}(T)$ whose intended meaning is that aFb if and only if, for any formal point $\alpha \in \mathsf{Pt}(S)$, $\alpha \in \mathsf{ext}(a)$ yields $\phi_F(\alpha) \in \mathsf{ext}(b)$. But $\phi_F(\alpha) \in \mathsf{ext}(b)$ if and only if $b\varepsilon\phi_F(\alpha)$ and hence we are forced to the following definition

$$\phi_F(\alpha) \equiv \bigcup_{a\varepsilon\alpha} Fa$$

where $Fa \equiv \{b \in T |\ aFb\}$.

Now we will look for the conditions which are necessary to be able to state that ϕ_F is a continuous function from $\mathsf{Pt}(S)$ into $\mathsf{Pt}(T)$. So, the first step is to prove that the image of a formal point of S is a formal point of T. Then, supposing α is a formal point, we have to prove that $\phi_F(\alpha)$ is inhabited, that is $(\exists b \in T)\ b\varepsilon\phi_F(\alpha)$. Now, we know that there is some element a of S which is in α and hence in order to obtain the result it is sufficient to require that $S \lhd F^-(T)$, where, for any subset V of T, $F^-(V) \equiv \{c \in S |\ (\exists v\varepsilon V)\ cFv\}$. In fact, $a\varepsilon\alpha$ and $S \lhd F^-(T)$ yield $(\exists c\varepsilon F^-(T))\ c\varepsilon\alpha$, i.e. $(\exists c \in S)(\exists b \in T)\ cFb \ \&\ c\varepsilon\alpha$, that is $(\exists b \in T)\ b\varepsilon\phi_F(\alpha)$. Finally, we have to check that $S \lhd F^-(T)$ is valid for any concrete topological space. So, let us assume that $(X, S, \Vdash)$ and $(Y, T, \Vdash)$ are two concrete topological spaces, ϕ is a continuous map from X to Y and F is a relation between S and T such that aFb if and only in $(\forall x \in X)\ x \Vdash a \rightarrow \phi(x) \Vdash b$. Moreover let us suppose that $a \in S$, $x \in X$ and $x \Vdash a$. Then $\phi(x) \in Y$ and hence, by the condition (B_1) there exists $b \in T$ such that $\phi(x) \Vdash b$. Then the continuity of ϕ yields that there exists $c \in S$ such that $x \Vdash c$ and cFb, i.e. $c \varepsilon F^-(T)$.

The second condition is that, supposing $b \varepsilon \phi_F(\alpha)$ and $d \varepsilon \phi_F(\alpha)$, there exists $f \varepsilon b \downarrow d$ such that $f \varepsilon \phi_F(\alpha)$. To obtain this result it is sufficient to require the following two conditions:

$$\text{(saturation)} \qquad \frac{a \lhd W \qquad (\forall w \varepsilon W)\ wFb}{aFb}$$

$$\text{(down-closure)} \qquad \frac{aFb \qquad aFd}{a \lhd F^-(b \downarrow d)}$$

In fact $b \varepsilon \phi_F(\alpha)$ and $d \varepsilon \phi_F(\alpha)$ yield that there are $a \varepsilon \alpha$ and $c \varepsilon \alpha$ such that aFb and cFd, and hence there is also $e \varepsilon a \downarrow c$ such that $e \varepsilon \alpha$. So, by using *saturation*, we obtain that eFb and eFd, which, by *down-closure*, yield $e \lhd F^-(b \downarrow d)$. Then $(\exists h \varepsilon F^-(b \downarrow d))\ h \varepsilon \alpha$, that is $(\exists f \varepsilon b \downarrow d)\ f \varepsilon \phi_F(\alpha)$. Also in this case it is necessary to check that the two required conditions are valid. *Saturation* is just a consequence of intuitionistic logic and the condition which links F and ϕ. Thus, let us prove the validity of *down-closure*. Suppose $x \in X$ and $x \Vdash a$, then aFb yields $\phi(x) \Vdash b$ and aFd yields $\phi(x) \Vdash d$; then by (B_2), there exists $k \varepsilon b \downarrow d$ such that $\phi(x) \Vdash k$. Then the continuity of ϕ yields that there exists $h \in S$ such that $x \Vdash h$ and hFk, that is $h \varepsilon F^-(b \downarrow d)$.

The third condition for ϕ_F be a formal point is that, supposing $b \varepsilon \phi_F(\alpha)$ and $b \lhd V$, there exists $v \varepsilon V$ such that $v \varepsilon \phi_F(\alpha)$. The necessary condition is

$$\text{(continuity)} \qquad \frac{aFb \qquad b \lhd V}{a \lhd F^-(V)}$$

In fact, $b \varepsilon \phi_F(\alpha)$ yields that there is $a \in \alpha$ such that aFb and hence *continuity* shows that there exists $c \varepsilon F^-(V)$ which is an element of α, that is, there is $v \varepsilon V$ such that cFv and $c \varepsilon \alpha$. The proof of validity of this condition is immediate. If aFb, that is $x \Vdash a \rightarrow \phi(x) \Vdash b$, and $b \lhd V$, that is $y \Vdash b \rightarrow (\exists v \varepsilon V)\ y \Vdash v$, then there is $v \varepsilon V$ such that $\phi(x) \Vdash v$ and hence, by continuity of ϕ, there is $c \in S$ such that $x \Vdash c$ and cFv.

Finally, we have to prove that if $b \varepsilon \phi_F(\alpha)$ then $\mathsf{Pos}(b)$. The condition here is

$$\text{(function monotonicity)} \qquad \frac{\mathsf{Pos}(a) \qquad aFb}{\mathsf{Pos}(b)}$$

because $b \varepsilon \phi_F(\alpha)$ means that there is $a \varepsilon \alpha$, and hence $\mathsf{Pos}(a)$ holds, such that aFb. Its validity is immediate; it can be proved by using intuitionistic logic and the intended meaning of the positivity predicate.

Now we have to look for the conditions that are needed to prove continuity of the map ϕ_F between the topological space $\mathsf{Pt}(S)$ and $\mathsf{Pt}(T)$, that is, the conditions that are necessary in order to be able to state that, for any $b \in T$, $\phi_F^-(\mathsf{ext}(b))$ is an open set. First note that

$$\begin{array}{rcl} \phi_F^-(\mathsf{ext}(b)) & = & \{\alpha \in \mathsf{Pt}(S) |\ \phi_F(\alpha) \in \mathsf{ext}(b)\} \\ & = & \{\alpha \in \mathsf{Pt}(S) |\ b\varepsilon\phi_F(\alpha)\} \\ & = & \{\alpha \in \mathsf{Pt}(S) |\ (\exists a\varepsilon\alpha)\ aFb\} \end{array}$$

Then, to prove that $\phi_F^-(\mathsf{ext}(b))$ is an open set, we have to check that it coincides with its interior, that is, for any point α of $\phi_F^-(\mathsf{ext}(b))$ there exists a basic open which contains α and is included in $\phi_F^-(\mathsf{ext}(b))$. So, let us assume that $\alpha \in \phi_F^-(\mathsf{ext}(b))$, that is, there exists $a\varepsilon\alpha$ such that aFb. Then $\alpha \in \mathsf{ext}(a)$. Now, $\mathsf{ext}(a)$ is the basic open that we are looking for. In fact, if γ is any other point in $\mathsf{ext}(a)$, that is $a\varepsilon\gamma$, then aFb yields that $\gamma \in \phi_F^-(\mathsf{ext}(b))$, that is we have shown that $\mathsf{ext}(a)$ is a basic open which contains α and is a sub-collection of $\phi_F^-(\mathsf{ext}(b))$.

So we need no new condition to obtain continuity of the map ϕ_F, but still there are valid condition on a continuous map between two concrete topological spaces that we should require. A remarkable one is the following

$$\text{(function positivity)}\quad \frac{\mathsf{Pos}(a) \to aFb}{aFb}$$

which allows to see the strong connection between the cover relation in a formal topology and a continuous relation. Indeed, the cover relation is the simplest among the continuous relation, that is, it is the continuous relation associated with the identity map. In fact, supposing $\mathcal{S}$ is any formal topology, if we put $aF_{\lhd}b \equiv a \lhd \{b\}$ we obtain a continuous relation such that, for any $\alpha \in \mathsf{Pt}(S)$, $\phi_{F_{\lhd}}(\alpha) = \alpha$. The validity of *function positivity* can be proved as follows: suppose $\mathsf{Pos}(a) \to aFb$, that is $((\exists x \in X)\ x \Vdash a) \to (\forall x \in X)\ x \Vdash a \to \phi(x) \Vdash b$. Then, assuming x is an element of X such that $x \Vdash a$, we obtain that $(\exists x \in X)\ x \Vdash a$ and hence $(\forall x \in X)\ x \Vdash a \to \phi(x) \Vdash b$ and so, by using again the assumption $x \Vdash a$, $\phi(x) \Vdash b$. Thus $(\forall x \in X)\ x \Vdash a \to \phi(x) \Vdash b$, that is aFb, holds.

Thus, we arrived to the following definition.

Definition 1.8 (Continuous relation) *Suppose* $\mathcal{S} = (S, \lhd_S, \mathsf{Pos}_S)$ *and* $\mathcal{T} = (T, \lhd_T, \mathsf{Pos}_T)$ *are two formal topologies. Then a* continuous relation *between* $\mathcal{S}$ *and* $\mathcal{T}$ *is any binary proposition* aFb prop $[a : S, b : T]$ *such that the following* *conditions are satisfied:*

(function non-emptiness)	$S \lhd_S F^-(T)$
(down-closure)	$\dfrac{aFb \qquad aFd}{a \lhd_S F^-(b \downarrow d)}$
(saturation)	$\dfrac{a \lhd_S W \qquad (\forall w \varepsilon W)\ wFb}{aFb}$
(continuity)	$\dfrac{aFb \qquad b \lhd_T V}{a \lhd_S F^-(V)}$
(function monotonicity)	$\dfrac{\mathsf{Pos}_S(a) \qquad aFb}{\mathsf{Pos}_T(b)}$
(function positivity)	$\dfrac{\mathsf{Pos}(a) \rightarrow aFb}{aFb}$

2 Fixed-points of continuous relations

Let us suppose that $\mathcal{S}$ is a formal topology and F is a continuous relation of $\mathcal{S}$ into itself. In this section we want to find sufficient conditions to be able to say that the function ϕ_F has a fixed point.

Supposing that n is a natural number greater then 0 (we will indicate the set of the natural number greater then 0 by ω), we put, for any $a, b \in S$,

$$aF^n b \;\equiv\; (\exists c_1, \ldots, c_n \in S)\ a = c_1 F c_2 F \ldots F c_{n-1} F c_n = b$$

that is

$$\begin{cases} aF^1 b & = \quad aFb \\ aF^{n+1} b & = \quad (\exists c \in S)\ aF^n c \;\&\; cFb \end{cases}$$

Hence

$$F^n a \equiv \{b \in S |\ (\exists c_1, \ldots, c_n \in S)\ (a = c_1 F c_2 F \ldots c_{n-1} F c_n = b)\}$$

Finally we put

$$F^\infty a \equiv \bigcup_{n \in \omega} F^n a$$

We can generalize this notation to subset by putting, for any $U \subseteq S$,

$$FU \equiv \bigcup_{u \varepsilon U} Fu$$

and hence

$$F^n U \equiv \bigcup_{u \varepsilon U} F^n u$$

Let us now suppose that $a \in S$; then we have

$$\begin{array}{rcl} F(F^\infty a) & = & F(\bigcup_{n\in\omega} F^n a) \\ & = & F(\{c \in S|\ (\exists n \in \omega)\ aF^n c\}) \\ & = & \bigcup_{(\exists n\in\omega)\ aF^n c} Fc \\ & \subseteq & \bigcup_{n\in\omega} F^n a \\ & = & F^\infty a \end{array}$$

where the inclusion is proved as follows. Suppose $b\varepsilon \bigcup_{(\exists n\in\omega)\ aF^n c} Fc$, then there exists $c \in S$ such that, for some $n \in \omega$, $aF^n c$ and cFb; hence $aF^{n+1}b$, that is $b\varepsilon \bigcup_{n\in\omega} F^n a$. In order to obtain the other inclusion, and hence to transform the inclusion into an equivalence, it is sufficient to require that there is $k \in \omega$ such that $aF^k a$; in fact, in this case $b\varepsilon \bigcup_{n\in\omega} F^n a$, that is $aF^n b$ for some $n \in \omega$, yields $aF^k aF^n b$ and hence there exists $c \in S$ such that $aF^{k+n-1}c$, and cFb, that is $b\varepsilon \bigcup_{(\exists n\in\omega)\ aF^n c} Fc$.

Now, supposing that, for some $a \in S$, $F^\infty a$ is a formal point and that $F(F^\infty a) = F^\infty a$, it is immediate to verify that $F^\infty a$ is a fixed point of ϕ_F. In fact

$$\begin{array}{rcl} \phi_F(F^\infty a) & = & \bigcup_{b\varepsilon F^\infty a} Fb \\ & = & \bigcup_{(\exists n\in\omega)\ aF^n b} Fb \\ & = & F^\infty a \end{array}$$

Then a sufficient condition for finding a fixed-point for ϕ_F is that there exists $a \in S$ such that $F^\infty a$ is a formal point and there exists $k \in \omega$ such that $aF^k a$. We will prove that to satisfy these two conditions it is necessary and sufficient to require that for any $V \subseteq S$ the following condition, which is expressed completely within the language of formal topologies, holds for a:

$$\text{(shrinking)} \qquad \frac{a \lhd V}{(\exists n \in \omega)(\exists v\varepsilon V)\ aF^n v}$$

besides the conditions that aFa and $\mathsf{Pos}(a)$. The meaning of these three conditions can be understood by thinking of a as of a basic open which *contains* the fixed-point. Then the condition $\mathsf{Pos}(a)$ is obvious since it just states that a is inhabited, and it is indeed inhabited by the fixed-point. Note that this condition is also necessary in order $F^\infty(a)$ be a formal point; in fact, we will show that $a\varepsilon F^\infty(a)$ and hence *point positivity* requires that $\mathsf{Pos}(a)$ holds. The condition that aFa will be necessary just in one point of the next proof and it can be substituted almost everywhere by a weaker condition which is a consequence of *shrinking*. The intended meaning of aFa is that all the point in a are mapped by the function ϕ_F inside a, and hence not only the fixed-point is going to stay within the basic open a but also all of the points which are near to the fixed-point. Note that this condition is clearly not necessary in order to have a fixed-point. Consider for instance the map f from the set of rational numbers into itself defined by putting $f(x) = \alpha x$ for $\alpha > 1$. Then f has clearly a fixed point, that is 0, but no basic open interval is mapped by f into itself. Finally the intended meaning of the *shrinking* condition is that the map ϕ_F is a shrinking map since all of the points in the basic open a are sooner or later mapped into an arbitrary chosen basic open v, provided that the fixed-point is contained in v. It is worth noting that *shrinking* is a necessary condition for $F^\infty a$ being a formal point. In fact, let us suppose that $a \lhd V$; then we will show that $a \varepsilon F^\infty a$ and hence there must exist $v \varepsilon V$ such that $v \varepsilon F^\infty a$, that is $(\exists v \in V)(\exists n \in \omega)\ v \in F^n a$.

It is interesting to note that an immediate consequence of *shrinking* is the fact that there exists $k \in \omega$ such that $aF^k a$ holds since we know that $a \lhd \{a\}$ holds. Thus, even if the condition aFa is sufficient to state that there exists $k \in \omega$ such that $aF^k a$, and hence to prove that $F(F^\infty a) = F^\infty a$, it is not necessary if also *shrinking* is assumed.

Let us check now that $F^\infty a$ is a formal point. We have first to prove that $F^\infty a$ is not empty. But, we know that, for some $k \in \omega$, $aF^k a$ holds and hence $a \varepsilon F^k a \subseteq F^\infty a$. The second condition is that, supposing $b \varepsilon F^\infty a$ and $d \varepsilon F^\infty a$, there exists $c \varepsilon b \downarrow d$ such that $c \varepsilon F^\infty a$. But $b \varepsilon F^\infty a$ means that there exists $n \in \omega$ such that $aF^n b$ and $d \varepsilon F^\infty a$ means that there exists $m \in \omega$ such that $aF^m d \equiv aFk_1 \ldots k_m Fb$. We can suppose without lack of generality that $n \leq m$ and hence, by using the condition that aFa, $aF^{m-n} aF^n b$, that is $aF^m b \equiv aFh_1 \ldots h_m Fb$ where $h_i \equiv a$ for $1 \leq i \leq m-n$. Now, consider $h_i F h_{i+1}$ and $k_i F k_{i+1}$; then, if $x \varepsilon h_i \downarrow k_i$, that is $x \lhd h_i$ and $x \lhd k_i$, then xFh_{i+1} and xFk_{i+1} by *saturation*; hence $(\forall x \varepsilon h_i \downarrow k_i)\ x \lhd F^-(h_{i+1} \downarrow k_{i+1})$, by *down-closure*. So the *shrinking* condition yields

$$(\forall x \varepsilon h_i \downarrow k_i)(\exists n_i \in \omega)(\exists y \varepsilon F^-(h_{i+1} \downarrow k_{i+1}))\ xF^{n_i} y$$

that is

$$(\forall x \varepsilon h_i \downarrow k_i)(\exists n'_i (= n_i + 1) \in \omega)(\exists z \varepsilon h_{i+1} \downarrow k_{i+1})\ xF^{n'_i} z$$

On the other hand, from aFh_1 and aFk_1, we get $a \lhd F^-(h_1 \downarrow k_1)$ by *down-closure* and hence the *shrinking* condition yields

$$(\exists n_0 \in \omega)(\exists y \varepsilon F^-(h_1 \downarrow k_1))\ aF^{n_0} y$$

that is

$$(\exists n'_0 (= n_0 + 1) \in \omega)(\exists z \varepsilon h_1 \downarrow k_1))\ aF^{n'_0} z$$

Then, by using intuitionistic logic, we obtain that

$$(\exists n''_2 (= n'_0 + n'_1) \in \omega)(\exists z \varepsilon h_2 \downarrow k_2)\ aF^{n''_2} z$$

If we proceed in this way we obtain

$$(\exists n''_{i+1} (= n''_i + n'_i) \in \omega)(\exists z \varepsilon h_{i+1} \downarrow k_{i+1})\ aF^{n''_{i+1}} z$$

and finally

$$(\exists l (= n''_m + n'_m) \in \omega)(\exists z \varepsilon b \downarrow d)\ aF^l z$$

that is, by exchange the existential quantifiers,

$$(\exists z \varepsilon b \downarrow d)\ z \varepsilon F^{\infty} a$$

The third condition is that, supposing $b \varepsilon F^{\infty} a$ and $b \lhd V$, there exists $v \varepsilon V$ such that $v \varepsilon F^{\infty} a$. Now, $b \varepsilon F^{\infty} a$ means that there is $n \in \omega$ such that $aF^n b$ and hence, $b \lhd V$ yields, by using *continuity*, that, for some element $h \in S$, $aF^{n-1}h$ and $h \lhd F^{-}(V)$. Then the *shrinking* condition shows that there exists $m \in \omega$ such that $(\exists x \varepsilon F^{-}(V))\ hF^m x$; hence $(\exists m'(= m+1) \in \omega)(\exists v \varepsilon V)\ hF^{m'} v$ and thus $(\exists k(= n + m' - 1) \in \omega)(\exists v \varepsilon V)\ aF^k v$, that is, by exchanging the existential quantifiers, $(\exists v \varepsilon V)\ v \varepsilon F^{\infty} a$.

Finally, supposing $b \varepsilon F^{\infty} a$, we have to prove that $\mathsf{Pos}(b)$. But $b \varepsilon F^{\infty} a$ means that there exists $n \in \omega$ such that $aFh_1F \ldots Fh_nFb$ and hence the condition $\mathsf{Pos}(a)$ yields, one after the other, $\mathsf{Pos}(h_1)$, …, $\mathsf{Pos}(h_n)$ and finally $\mathsf{Pos}(b)$.

3 Applications

We want to show an example of application of the general technique for finding fixed-points of continuous function that we introduced in the previous section. First of all let us note that, supposing $\mathcal{S}$ is a formal topology, F is a continuous relation of S into itself and $a \in S$ is an element which satisfies the conditions that we suggested for the existence of a fixed-point, the fixed-point $F^{\infty} a$ is the minimal fixed-point of F with respect to the inclusion between points which contain a. In fact, let us suppose that α is a fixed-point for F, that is $\alpha = F(\alpha) = \bigcup_{c \varepsilon \alpha} Fc$, and that $a \varepsilon \alpha$; then $Fa \subseteq F(\alpha) \equiv \alpha$ because F is a monotone operator over subsets of S. In fact, if $U \subseteq V$ and $c \varepsilon F(U)$, then there exists $u \varepsilon U$ such that uFc, but then $u \varepsilon V$ and hence $c \varepsilon F(V)$. But, for the same reason, we also obtain that $F^2 a \subseteq F(\alpha) = \alpha$ and in general, for any $n \in \omega$, $F^n a \subseteq \alpha$; hence $F^{\infty} a = \bigcup_{n \in \omega} F^n a \subseteq \alpha$. Then our approach can be used when we are looking for a constructive proof of existence of a minimal fixed-point.

Of course, the problem is to be able to present the fixed-point problem that we want to solve in the framework of formal topologies and their morphisms. In the following we will illustrate a typical example.

3.1 Inductive generation of formal topologies

One of the main tools in formal topology is inductive generation of the cover relation since this allows to develop proofs by induction. The problem of inductively generate formal topologies has been completely solved and the reader can look in [CSSV] and [Val99] for a detailed discussion of the problems that an inductive generation of formal topologies requires to solve and for their solutions. We will recall here, without any proofs, only the results that we will use in the next sections.

The conditions appearing in the definition of formal topology, though written in the shape of rules, must be understood as requirements of validity: if the premises hold, also the conclusion must hold. As they stand, they are by no means acceptable rules to generate inductively a cover relation. For instance, the operation $\downarrow$ among subsets, which occurs in the conclusion of $\downarrow$*-right*, is not even well defined unless we already have a complete knowledge of the cover. Another problem is that admitting *transitivity* as acceptable rule for an inductive definition is equivalent to a well-known fix-point principle, which does not have a predicative justification (see [CSSV] and [Val99]). So, to transform the axiomatic definition into good inductive rules we need to face with these problems.

An inductive definition of a cover will start from some axioms, which at the moment we assume to be given by means of any relation $R(a, U)$ for $a \in S$ and $U \subseteq S$. We thus want to generate the least cover $\lhd_R$ which satisfies the following condition:

$$\text{(axioms)} \qquad \frac{R(a, U)}{a \lhd_R U}$$

From an impredicative point of view, $\lhd_R$ is easily obtained "from above" simply as the intersection of the collection $\mathcal{C}_R$ of all the reflexive, transitive infinitary relations containing R. In fact, it is clear that the total relation is in $\mathcal{C}_R$ and that the intersection preserves all such conditions.

Predicatively the method of defining $\lhd_R$ as the intersection of $\mathcal{C}_R$ is not acceptable, since there is no way of producing $\mathcal{C}_R$ above as a set-indexed family and hence to define its intersection.

Therefore, we must obtain $\lhd_R$ "from below" by means of some introductory rules. The first naive idea is that of using axioms, *reflexivity* and *transitivity* for this purpose. But then a serious problem emerges: in the premises of *transitivity*, that is

$$\frac{a \lhd_R V \qquad V \lhd_R U}{a \lhd_R U}$$

there is a subset V which does not appear in the conclusion. This means that the tree of possible premises to conclude that $a \lhd_R U$ has an unbounded branching: each subset V satisfying $a \lhd_R V$ and $V \lhd_R U$ would be enough to obtain $a \lhd_R U$, and there is no way to survey them all. Also, a dangerous vicious circle seems to be present: the subset V, whose existence would be enough to obtain $a \lhd_R U$, could be defined by means of the relation $\lhd_R$ itself which we are trying to construct. In this way, the instructions to try to build up $\lhd_R$ would not be fixed in advance, but change along their application.

This is the reason why we have to put some constrains on the infinitary relation $R(a, U)$. Thus, we are going to generate a cover relation only when we have a *stock of axioms*, that is a set-indexed family $I(a)$ set $[a : S]$ and an indexed family $C(a, i) \subseteq S$ $[a : S, i : I(a)]$ of subsets of S, whose intended meaning is to state that, for all $i \in I(a)$, $a \lhd C(a, i)$. Then, provided we have a stock of axioms I and C, a safe infinitary relation is

$$R_{\lhd}(a, U) \equiv (\exists i \in I(a))\ C(a, i) \subseteq U$$

In fact, in this case we can generate the cover relation which satisfies *reflexivity* and *transitivity* by using the following rules

$$\text{(reflexivity)}\ \frac{a \varepsilon U}{a \lhd U} \quad \text{and} \quad (\lhd\text{-infinity})\ \frac{i \in I(a) \qquad C(a,i) \lhd U}{a \lhd U}$$

In this way any reference to the subset V disappeared and the implicit use of an existential quantification on the collection $\mathcal{P}(S)$ is transformed into an existential quantification on the elements of the set $I(a)$.

We want now to extend the previous rules into new ones which allow to generate a cover relation which satisfies also $\downarrow$*-right*. As we observed, the definition of the operation $\downarrow$ among subsets depends on the covers and it requires the cover to be known. However, a crucial observation is that only the trace of the cover on elements is sufficient. The idea is then to separate covers between elements, that is $a \lhd \{b\}$, from those $a \lhd U$ with an arbitrary subset U on the right, so that we can block the former, require $\downarrow$ on it and then generate the latter. So, we must add, to those of a formal topology, an extra primitive expressing what in the concrete case is $\mathsf{ext}(a) \subseteq \mathsf{ext}(b)$. We can obtain this by adding directly a pre-order relation $a \leq b$ among names. Thus we obtain the following definition.

Definition 3.1 *A* formal topology with pre-order*, shortly a* $\leq$-formal topology*, is a quadruple* $(S, \leq, \lhd, \mathsf{Pos})$ *where* S *is a set,* $\leq$ *is a pre-order relation over* S*, that is* $\leq$ *is reflexive and transitive, and* $\lhd$ *is a relation between elements and subsets of* S *which satisfies* reflexivity*,* transitivity *and the two following conditions*

$$(\leq\text{-left})\ \frac{a \leq b \qquad b \lhd U}{a \lhd U} \qquad (\leq\text{-right})\ \frac{a \lhd U \qquad a \lhd V}{a \lhd \downarrow U \cap \downarrow V}$$

where $\downarrow U \equiv \{c : S |\ (\exists u \varepsilon U)\ c \leq u\}$*. Finally,* Pos *is a positivity predicate with respect to* $\lhd$*, that is it satisfies* monotonicity *and* positivity*.*

It is straightforward to verify that the new conditions are valid in any concrete topological space under the intended interpretation.

The condition $\leq$*-left* is clearly equivalent to the fact that $\leq$ respects $\lhd$, that is

$$\frac{a \leq b}{a \lhd \{b\}}$$

Since $\leq$ respects $\lhd$, for any subset U we have that $\downarrow U \subseteq \downarrow^{\lhd} U$, where $\downarrow^{\lhd} U \equiv \{c : S |\ (\exists u \varepsilon U)\ c \lhd \{u\}\}$. Thus $\downarrow U \cap \downarrow V \subseteq \downarrow^{\lhd} U \cap \downarrow^{\lhd} V \equiv U \downarrow V$, so that $\leq$*-right* implies $\downarrow$*-right*. Thus any $\leq$-formal topology is a formal topology. The converse is trivial: given any formal topology $(S, \lhd)$, all we need to do is to define

$$a \leq b \equiv a \lhd \{b\}$$

and we obtain a $\leq$-formal topology with a cover relation coinciding with the original ones. Now, let us suppose that $I(a)$ set $[a : S]$ together with $C(a,i) \subseteq S$ $[a : S, i : I(a)]$ is a stock of axioms for a cover relation. Moreover, let us suppose that the following *axiom condition* is satisfied.

Definition 3.2 *If $c \leq a$ and $a \lhd C(a,i)$ is an axiom for some $i \in I(a)$, then there exists $j \in I(c)$ such that $(\forall x \varepsilon C(c,j))(\exists y \varepsilon C(a,i))\ x \leq c \ \&\ x \leq y$.*

Then, we can generate a cover relation $\lhd$ by using the following introduction rules:

$$\text{(reflexivity)} \quad \frac{a \varepsilon U}{a \lhd U}$$

$$(\leq\text{-left}) \quad \frac{a \leq b \qquad b \lhd U}{a \lhd U}$$

$$(\lhd\text{-infinity}) \quad \frac{i \in I(a) \qquad C(a,i) \lhd U}{a \lhd U}$$

In fact closure under *transitivity* and *$\leq$-right* can be proved by induction and the proof of the axioms, that is, for any $i \in I(a)$, $a \lhd C(a,i)$, is immediate (see [CSSV]).

3.2 The formal topology of recursive and recursive enumerable subsets of the natural numbers

We can now introduce the formal topology of the recursively enumerable subsets of the natural numbers and the formal topology of the recursive subsets of the natural numbers. First note that we can extend the set N of the natural numbers to a new set N_+ by adding it the new element $*$. To obtain such an extension we can use the set constructor Suc whose introduction rules are the following, for any set A (see [NPS90]):

$$0_{\mathsf{Suc}(A)} \in \mathsf{Suc}(A) \qquad\qquad \frac{a \in A}{\mathsf{suc}_{\mathsf{Suc}(A)}(a) \in \mathsf{Suc}(A)}$$

It is now possible to put $\mathsf{N}_+ \equiv \mathsf{Suc}(\mathsf{N})$, $* \equiv 0_{\mathsf{Suc}(\mathsf{N})}$ and forget of the distinction between the natural number n and the element $\mathsf{suc}_{\mathsf{Suc}(\mathsf{N})}(n)$ of $\mathsf{Suc}(\mathsf{N})$ since, given any element $c \in \mathsf{N}_+$, if $c \neq *$, the elimination rule for the set N_+ allows to obtain the number n such that $\mathsf{suc}_{\mathsf{Suc}(\mathsf{N})}(n) = c$.

Now, we will say that a subset U of N, that is, a propositional function over N, is *recursively enumerable* if there is a function $f : \mathsf{N} \to \mathsf{N}_+$ such that $U = \mathsf{Im}[f] \setminus \{*\}$, where $\mathsf{Im}[f] \equiv \{x \in \mathsf{N}_+ | \ (\exists y \in \mathsf{N})\ x =_{\mathsf{N}_+} f(y)\}$ is the image of the function f. It should be clear the need for the extension from N to N_+: it allows to obtain that the empty subset is recursively enumerable. In a similar way, we will say that a subset U of N is *recursive* if it is the image of an increasing function $f : \mathsf{N} \to \mathsf{N}_+$, that is a function such that, for any $x, y \in \mathsf{N}$, if $x < y$ then $f(x) <^+ f(y)$ where $<^+$ is the extension of the usual order relation between natural numbers to the element of N_+ such that, for any $x \in \mathsf{N}_+$, $x <^+ *$.

Note that we can identify the finite subsets of N_+ with the elements of the set N_+^* of the lists of elements in N_+. Hence, also the finite subsets of N can be identified with such lists since the occurrences of $*$ within a list are effectively recognizable. Thus we can define a membership relation by putting, for any $n \in \mathsf{N}$ and $\sigma \in \mathsf{N}_+^*$,

$$n \varepsilon \sigma \equiv (\exists x \in \mathsf{N})\ x < \mathsf{len}(\sigma)\ \&\ \mathsf{suc}_{\mathsf{N}_+}(n) =_{\mathsf{N}_+} \sigma[x]$$

Now, let us introduce the family of axioms that we will use to generate a formal topology $\mathsf{RecEnum}$ over N_+^* whose points correspond to the recursively enumerable subsets of N. We put, for any $\sigma \in \mathsf{N}_+^*$,

$$I(\sigma) \equiv \{*\}$$
$$C(\sigma, *) \equiv \{\tau \in \mathsf{N}_+^* |\ (\exists k \in \mathsf{N}_+)\ \tau =_{\mathsf{N}_+^*} \sigma k\}$$

that is, for any $\sigma \in \mathsf{N}_+^*$ there is only one axiom, namely

$$\sigma \lhd \{\tau \in \mathsf{N}_+^* |\ (\exists k \in \mathsf{N}_+)\ \tau =_{\mathsf{N}_+^*} \sigma k\}$$

It is immediate to verify that this family of axioms satisfy the axiom conditions in the previous section and hence $\mathsf{RecEnum}$ can be inductively generated by using *reflexivity*, $\leq$*-left*, where the order relation $\sigma_1 \leq \sigma_2$ states that the list σ_2 is an initial segment of the list σ_1, and $\lhd$*-infinity*. We can associate with any formal point α of this formal topology a function $f_\alpha : \mathsf{N} \to \mathsf{N}_+$ such that, if $\sigma\varepsilon\alpha$, then $\sigma \cdot f_\alpha(\mathsf{len}(\sigma))$ is the only list of length $\mathsf{len}(\sigma) + 1$ which is contained in α. Moreover, given any function $f : \mathsf{N} \to \mathsf{N}_+$, we obtain a formal point by putting

$$\alpha_f \equiv \{\sigma \in \mathsf{N}_+^* |\ (\forall k \in \mathsf{N})\ (k < \mathsf{len}(\sigma)) \to \sigma[k] =_{\mathsf{N}_+} f(k)\}$$

Indeed, the proof that *point not-emptiness*, *up-closure*, *consistency* and *completeness* hold is almost immediate. Finally, it is obvious that the correspondence between functions and formal points is biunivocal, that is, for any $g \in \mathsf{N} \to \mathsf{N}_+$, $f_{\alpha_g} = g$ and, for any formal point β, $\alpha_{f_\beta} = \beta$.

Now, given a formal point α of this formal topology we can obtain a subset of N by putting:

$$U_\alpha \equiv \{n \in \mathsf{N} |\ (\exists \sigma\varepsilon\alpha)\ n\varepsilon\sigma\}$$

and U_α is recursively enumerable since $U_\alpha = \mathsf{Im}[f_\alpha] \setminus \{*\}$.

On the other hand, for any recursively enumerable subset U of S, we can define a formal point by putting

$$\alpha_U \equiv \alpha_{f_U}$$

where f_U is the function from N into N_+ which shows that U is a recursively enumerable subset of N.

In a similar way one can obtain a formal topology Rec whose formal points corresponds to the recursive subsets of the natural numbers if he uses the set $\mathsf{N}_+^{*<}$, that is the set whose introduction rules are:

$$\mathsf{nil} \in \mathsf{N}_+^{*<} \qquad \frac{a \in \mathsf{N}_+}{\mathsf{nil}\ a \in \mathsf{N}_+^{*<}}$$

$$\frac{a \in \mathsf{N}_+ \quad a_1 \ldots a_n \in \mathsf{N}_+^{*<} \quad a_n <^+ a\ \mathsf{true}}{a_1 \ldots a_n a \in \mathsf{N}_+^{*<}}$$

instead that the set N_+^*.

The collection of the formal points of Rec corresponds to the set of increasing functions from N into N_+ whose introduction rule is

$$\frac{\begin{array}{cc} [x:\mathsf{N}]_1 & [x:\mathsf{N}]_1 \\ \vdots & \vdots \\ b(x) \in \mathsf{N}_+ & b(x+1) <^* b(x)\ \mathsf{true} \end{array}}{\lambda x.b(x) \in \mathsf{N} \to^< \mathsf{N}_+}\ 1$$

3.3 Fixed-point of monotone operator which maps recursive subsets into recursive subsets

Let us now go back to our original problem, that is, to find the least fixed-point of an operator $\tau : \mathcal{P}(\mathsf{N}) \to \mathcal{P}(\mathsf{N})$. Of course, we are not going to present a general solution since there is no predicative justification for the existence of least fixed-point also supposing τ monotone, but we will state suitable conditions for some interesting case. In particular, let us suppose that the operator τ is mapping recursively enumerable subsets of N into recursively enumerable subsets of N (respectively, recursive subsets of N into recursive subsets of N). Then, we can introduce a new operator τ_α from Pt(RecEnum) into Pt(RecEnum) (from Pt(Rec) into Pt(Rec)) defined by putting, for any formal point β, $\tau_\alpha(\beta) \equiv \alpha_{\tau(U_\beta)}$. For semplicity sake, in the following we will forget of the index and we will confuse the operator on subsets and that one on formal points. Hence we can look for the least fixed-point of τ by studing with the tecnique of the previous section the continuous relation

$$\sigma_1 F_\tau \sigma_2 \equiv \text{for all } \alpha \in \mathsf{Pt(RecEnum)},\ \sigma_1 \varepsilon \alpha \to \sigma_2 \varepsilon \tau(\alpha)$$

The problem with this definition is that it is not espressable within type theory since a quantification over the *collection* of formal points is used. But we already noticed that the collection Pt(RecEnum) (respectively Pt(Rec)) can be indexed by using the *set* of the funtions from N into N_+ (the *set* of the incresing function $\mathsf{N} \to^< \mathsf{N}_+$); hence the previous definition can be transformed into an equivalent one which is completely within type theory. Thus, for any $\sigma_1, \sigma_2 \in \mathsf{N}_+^*$ we put

$$\sigma_1 F_\tau \sigma_2 \equiv (\forall f \in \mathsf{N} \to \mathsf{N}_+)\ \sigma_1 \sqsubseteq_\omega f \to \sigma_2 \sqsubseteq_\omega \tau(f)$$

where $\sigma \sqsubseteq_\omega f \equiv (\forall x \in \mathsf{N})\ x < \mathsf{len}(\sigma) \to \sigma[x] = f(x)$ means that the list σ is an initial segment of the function f and where we continue to use the symbol τ for a function from $\mathsf{N} \to \mathsf{N}_+$ into $\mathsf{N} \to \mathsf{N}_+$ instead that for the corresponding map from formal points into formal points, that is, given any function $g \in \mathsf{N} \to \mathsf{N}_+$ we will write $\tau(g)$ instead that $f_{\tau(\alpha_g)}$.

Now, the three required conditions for the existence of a fixed-point are that there exists an element σ in N_+^* such that $\mathsf{Pos}(\sigma)$, $\sigma F_\tau \sigma$ and, assuming $\sigma \lhd U$, there exists $n \in \mathsf{N}$ and $u \varepsilon U$ such that $\sigma F_\tau^n u$. A forced choice for σ is the empty list nil which guarantees the validity of the first two conditions. Let us then analize what are suitable requirements for the validity of the third one; it states that, for any subset U of lists, there must exist a natural number n such that there exists $u \varepsilon U$ such that $\mathsf{nil} F_\tau^n u$, that is,

$$(\forall f \in \mathsf{N} \to \mathsf{N}_+)\ \mathsf{nil} \sqsubseteq_\omega f \to u \sqsubseteq_\omega \tau^n(f)$$

which can be semplified into

$$(\forall f \in \mathsf{N} \to \mathsf{N}_+)\ u \sqsubseteq_\omega \tau^n(f)$$

since the antecedent in the implication is always true. In the general case this last condition is not very clear, but let us observe that in the case the function τ is mapping recursive subsets into recursive subsets we can obtain a result analogous to the usual Tarsky theorem for monotone maps. To this aim let us say that for any $f, g \in \mathsf{N} \to^< \mathsf{N}_+$

$$f \subset g \equiv \mathsf{Im}[f] \setminus \{*\} \subset \mathsf{Im}[g] \setminus \{*\}$$

It is then clear that $\subset$ is a pre-order relation among functions and that the function $\mathsf{empty} \equiv \lambda x.*$ is the minimum element in such an order relation since $\mathsf{empty} \subseteq f$ for every $f \in \mathsf{N} \to^< \mathsf{N}_+$.

Now, the function $\tau \in (\mathsf{N} \to^< \mathsf{N}_+) \to (\mathsf{N} \to^< \mathsf{N}_+)$ is monotone if $f \subset g$ yields $\tau(f) \subset \tau(g)$. It is immediate to see that if τ is monotone and $\mathsf{empty} \subset \tau(\mathsf{empty})$ then, for any natural number n, $\tau^n(\mathsf{empty}) \subset \tau^{n+1}(\mathsf{empty})$. Moreover, for any list σ, $f \subseteq g$ yields $\sigma \sqsubseteq_\omega f \to \sigma \sqsubseteq_\omega g$. Thus, in the case of a monotone operator which maps recursive subsets into recursive subsets, the condition $(\forall f \in \mathsf{N} \to^< \mathsf{N}_+)\ u \sqsubseteq_\omega \tau^n(f)$ can be substituted by the simpler and equivalent condition $u \sqsubseteq_\omega \tau^n(\mathsf{empty})$. Hence the third condition for the existence of a fixed-point becomes

$$\frac{\mathsf{nil} \lhd U}{(\exists n \in \mathsf{N})(\exists u \varepsilon U)\ u \sqsubseteq_\omega \tau^n(\mathsf{empty})}$$

We can show its validity by proving a stronger condition, i.e.

$$\frac{\tau^k(\mathsf{empty}) \restriction_k \sqsubseteq_\omega \sigma \quad \sigma \lhd U}{(\exists n \in \mathsf{N})(\exists u \varepsilon U)\ u \sqsubseteq_\omega \tau^{k+n}(\mathsf{empty})}$$

where $f \restriction_n$ is the list obtained by considering the first n values of the function f. In fact the former condition is an immediate consequence of the latter when we instantiate k to 0.

The proof is by induction on the length of the derivation of $\sigma \lhd U$. If $\sigma \lhd U$ has been obtained by *reflexivity* from $\sigma \varepsilon U$ then the result is obvious by taking $n = 0$ and $u = \sigma$. If $\sigma \lhd U$ has been obtained by $\leq$-*left* from $\sigma \sqsubseteq \sigma'$ and $\sigma' \lhd U$ then the result is immediate by inductive hypothesis since $\tau^k(\mathsf{empty}) \restriction_k \sqsubseteq \sigma$ and $\sigma \sqsubseteq \sigma'$ yield $\tau^k(\mathsf{empty}) \restriction_k \sqsubseteq \sigma'$. Finally, if $\sigma \lhd U$ has been obtained by *infinity* from $\sigma \cdot x \lhd U\ [x \in \mathsf{N}_+]$, then from $\tau^k(\mathsf{empty}) \restriction_k \sqsubseteq \sigma$ we obtain that there exists $x \in \mathsf{N}_+$ such that $\tau^{k+1}(\mathsf{empty}) \restriction_{k+1} \sqsubseteq \sigma \cdot x$ and hence, by inductive hypothesis, from $\sigma \cdot x \lhd U\ [x \in \mathsf{N}_+]$ we obtain that $(\exists n \in \mathsf{N})(\exists u \varepsilon U)\ u \sqsubseteq \tau^{k+1+n}(\mathsf{empty})$ and hence the thesis by $\exists$-*elimination*.

4 Conclusion

We showed some sufficient conditions to prove the existence of fixed-points for the function ϕ_F associated to the continuous relation F. But we also proved that such conditions are not necessary. Thus the main problem now is to find necessary and sufficient conditions for the existence of fixed-points.

Another open problem concerns the uniqueness of the fixed-point. In fact, it is clear that the condition that we stated just allow to prove the existence of a fixed-point, but give no information about the uniqueness. Consider for instance the identity function over the set of rational numbers: the relation associated to it clearly satisfies our conditions, but the fixed point is surely not unique. We have only a small suggestion: if both a and b satisfy our conditions for the existence of a fixed-point and there exists $n \in \omega$ such that $b \varepsilon F^n a$ then $F^\infty b \subseteq F^\infty a$; in fact, if $c \varepsilon F^\infty b$ then $(\exists k \in \omega)\ c \varepsilon F^k b$ and hence $(\exists k \in \omega)\ c \varepsilon F^{n+k} a$ which yields $c \in F^\infty a$. Then two fixed-points $F^\infty a$ and $F^\infty b$ coincide if and only if $a \varepsilon F^\infty b$ and $b \varepsilon F^\infty a$.

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