Archive for Mathematical Logic manuscript No.
(will be inserted by the editor)

The problem of completeness of formal topologies with a binary positivity predicate and their inductive generation

Silvio Valentini

Dipartimento di Matematica Pura ed Applicata, Università di Padova, via G. Belzoni n.7, I–35131 Padova, Italy, e-mail: silvio@math.unipd.it

The date of receipt and acceptance will be inserted by the editor

Abstract Formal topologies are today an established topic in the development of constructive mathematics and many classical results of general topology have been already brought into the realm of constructive mathematics by using this approach. One of the main tools in formal topology is inductive generation and it has been completely solved the problem of inductively generating formal topologies with a cover relation and a unary positivity predicate, namely, the mathematical structures where the properties of the open subsets of a topological space can be expressed. Anyhow, in order to deal both with open and closed subsets, a binary positivity predicate has to be considered and a suitable axiomatization must be provided. In this paper we will show how to adapt to this framework the method used for the inductive generation of formal topologies with a unary positivity predicate since we have now to take into account both the cover relation and the positivity predicate in a general setting where both of them have proper axioms. Indeed, we will show that it is necessary to work in such a generality because the lack of a complete axiomatization of the topological spaces in the present definition of formal topology does not allow to determine the open and the closed subsets by using only the basic opens as it can be done in the classical case.

1 Introduction

The aim of formal topology is to develop topology in a constructive framework where "constructive" is meant to include both intuitionistic and predicative.

One of the main tools in formal topology is inductive generation since many formal topologies can be presented in a predicative way by an inductive generation and thus their properties can be proved inductively. The problem of inductively generating formal topologies with a cover relation and a unary positivity predicate, that is, the mathematical structures where the properties of the open subsets of a topological space can be expressed, has been dealt with and completely solved in [CSSV].

Anyhow, in order to deal both with open and closed subsets, a binary positivity predicate has to be considered and a suitable axiomatization must be provided [GS99]. In order to make the paper self contained, we will give in the next sections an introduction to formal topologies with a binary positivity predicate; these sections contain some observations which cannot be found in the papers on formal topology which have been published till now.

The main problem with the present definition of formal topology is that only some of the valid conditions on the cover relation and the positivity predicate are considered but still it is far from a complete formalization of the topological spaces. This is confirmed by the fact that, while the basic opens are sufficient to determine in the case of a topological space both the open and the closed subsets this does not happen for a formal topology. We will make this problem explicit by showing how to generate by induction both a cover relation which satisfies *reflexivity*, $\lhd$-*transitivity* and $\downarrow$-*right* and how to generate by co-induction a positivity predicate which satisfies *anti-reflexivity*, Pos-*transitivity* and *compatibility* with respect to such a cover relation in such a way that both of them have proper and independent axioms. [1] This will show in particular that many different positivity predicates are compatible with the same cover relation. So we will give a complete solution to the problem of an inductive generation of a formal topology with a binary positivity predicate, but such a solution will show that the cover relation and the positivity predicate are not linked strongly enough and that some more connections should be looked for. To avoid to repeat what can be found in [CSSV], we will skip the proofs of the results which are already presented there when they are not directly relevant to the development of this paper.

The reader interested to have other details on formal topology and a deeper analysis of the foundational motivations for the formal de-

[1] Our starting point for the inductive generation have been some notes of Per Martin-Löf and Giovanni Sambin on the inductive generation of a reflexive and transitive cover relation and a compatible binary positivity predicate with no axioms.

velopment of topology within Martin-Löf's constructive type theory [ML84] is invited to look at the first sections of [CSSV].

1.1 Concrete topological spaces

The classical definition of topological space reads: $(X, \Omega(X))$ is a topological space if X is a set and $\Omega(X)$ is a subset of $\mathcal{P}(X)$ which satisfies:

- (Ω_1) $\emptyset, X \in \Omega(X)$;
- (Ω_2) $\Omega(X)$ is closed under finite intersection;
- (Ω_3) $\Omega(X)$ is closed under arbitrary union.

Usually, elements of X are called points and elements of $\Omega(X)$ are called opens.

The quantification implicitly used in (Ω_3) is of the third order, since it says $(\forall F \subseteq \Omega(X)) \bigcup F \in \Omega(X)$, that is,

$$(\forall F \in \mathcal{P}(\mathcal{P}(X)))\ (F \subseteq \Omega(X) \to \bigcup F \in \Omega(X))$$

We can "go down" one step by thinking of $\Omega(X)$ as a family of subsets indexed by a set S through a map $\mathsf{n} : S \to \mathcal{P}(X)$, since we can now quantify on S rather than on $\Omega(X)$. But we still have to say $(\forall U \in \mathcal{P}(S))(\exists c \in S)\ (\cup_{a \varepsilon U} \mathsf{n}(a) = \mathsf{n}(c))$, which is still impredicative[2].

We can "go down" another step by defining opens to be of the form $\mathsf{N}(U) \equiv \cup_{a \varepsilon U} \mathsf{n}(a)$ for an arbitrary subset U of S. In this way $\emptyset$ is open, because $\mathsf{N}(\emptyset) = \emptyset$, and closure under union is automatic, because obviously $\cup_{i \in I} \mathsf{N}(U_i) = \mathsf{N}(\cup_{i \in I} U_i)$. So, all we have to do is to require $\mathsf{N}(S)$ to be the whole X and closure under finite intersections. It is not difficult to realize that this amounts to the standard definition saying that $\{\mathsf{n}(a) \subseteq X |\ a \in S\}$ is a base (see for instance [Eng77]). So, we reach the following definition:

Definition 1 *A* concrete topological space *is a triple $\mathcal{X} \equiv (X, S, \mathsf{n})$ where X is a set of* concrete points*, S is a set of* names for basic open subsets*, n is a map from S into subsets of X, called the* neighborhood map*, which associates the names with the basic open subsets and*

[2] All the set-theoretical notions that we use conform to the subset theory for Martin-Löf's type theory as presented in [SV98]. In particular, we will use the symbol $\in$ for the membership relation between an element and a set or a collection and ε for the membership relation between an element and a subset, which is never a set but a propositional function.

satisfies

$$(B_1)\ X = \cup_{a \in S} \mathsf{n}(a)$$

$$(B_2)\ (\forall a, b \in S)(\forall x \in X)\ (x \varepsilon \mathsf{n}(a) \cap \mathsf{n}(b) \rightarrow (\exists c \in S)\ (x \varepsilon \mathsf{n}(c)\ \&\ \mathsf{n}(c) \subseteq \mathsf{n}(a) \cap \mathsf{n}(b)))$$

Now, a map $\mathsf{n} : S \rightarrow \mathcal{P}(X)$ is a propositional function with two arguments, i.e. $\mathsf{n}(a)(x)\ \mathsf{prop}\ [a : S, x : X]$, that is a binary relation. Then we can write it as

$$x \Vdash a\ \mathsf{prop}\ [x : X, a : S]$$

and read it "x lies in a".

It is convenient to use the following two abbreviations:

$$\mathsf{ext}(a) \equiv \{x : X |\ x \Vdash a\}$$
$$\mathsf{Ext}(U) \equiv \cup_{a \varepsilon U} \mathsf{ext}(a)$$

Hence $x \Vdash a$ is the same as $x \varepsilon \mathsf{ext}(a)$ and thus the map n coincides with ext.

Then (B_1) and (B_2) can be rewritten as

$$(B_1)\ X = \mathsf{Ext}(S)$$

$$(B_2)\ (\forall a, b \in S)(\forall x \in X)\ ((x \varepsilon \mathsf{ext}(a) \cap \mathsf{ext}(b)) \rightarrow (\exists c \in S)\ (x \varepsilon \mathsf{ext}(c)\ \&\ \mathsf{ext}(c) \subseteq \mathsf{ext}(a)\ \&\ \mathsf{ext}(c) \subseteq \mathsf{ext}(b)))$$

We can make (B_2) a bit shorter by introducing another abbreviation, that is

$$a \downarrow b \equiv \{c : S |\ \mathsf{ext}(c) \subseteq \mathsf{ext}(a)\ \&\ \mathsf{ext}(c) \subseteq \mathsf{ext}(b)\}$$

so that it becomes

$$(B_2)\ (\forall a, b \in S)\ \mathsf{ext}(a) \cap \mathsf{ext}(b) \subseteq \mathsf{Ext}(a \downarrow b)$$

which looks much better.

Note that $c \varepsilon a \downarrow b$ implies that $\mathsf{ext}(c) \subseteq \mathsf{ext}(a) \cap \mathsf{ext}(b)$, so that $\mathsf{Ext}(a \downarrow b) \equiv \cup_{c \varepsilon a \downarrow b} \mathsf{ext}(c) \subseteq \mathsf{ext}(a) \cap \mathsf{ext}(b)$. Then the definition of concrete topological space can be rewritten as follows:

Definition 2 *A* concrete topological space *is a triple* $\mathcal{X} \equiv (X, S, \Vdash)$ *where* X *and* S *are sets and* $\Vdash$ *is a binary relation from* X *to* S *satisfying:*

$$(B_1)\ X = \mathsf{Ext}(S)$$

$$(B_2)\ (\forall a, b \in S)\ \mathsf{ext}(a) \cap \mathsf{ext}(b) = \mathsf{Ext}(a \downarrow b)$$

1.2 Formal topologies with a binary positivity predicate

The notion of formal topology arises by describing as well as possible the structure induced by a concrete topological space on the *formal side*, that is the side of the set S of the names, and then by taking the result as an axiomatic definition. The reason for such a move is that the definition of concrete topological space is too restrictive, given that in the most interesting cases of topological space we do not have, from a constructive point of view, a *set* of points to start with and in the definition of concrete topological space we have to require that X and S are sets; in fact, we need to quantify over their elements in order to state the conditions (B_1) and (B_2) and this has a constructive meaning only if they are sets.

Then we introduce the two main operators which link the *concrete side*, that is the side of the set X of the concrete points, with the formal side. There intended meaning is to allow to speak of the concrete open and closed subsets of the topological space X by means of the names of the basic opens, which live in the set S.

The problem to identify the open sets is easily solved. Since the elements in S are names for basic opens of the topology on X, then we can obtain their *extension*, that is the concrete basic open, by using the operator ext. Now, by definition, any open set is the union of basic opens and hence it can be specified in the formal side by using the subset of all the (names of the) basic opens which are used to form it.

It is possible to check that, provided the conditions (B_1) and (B_2) are satisfied, in this way we really obtain a topology on the set X. Indeed, the whole space X is an open set, since $X = \mathsf{Ext}(S)$ because of (B_1), and $\emptyset = \mathsf{Ext}(\emptyset)$ is an open set as well; moreover, an arbitrary union of open sets is an open set since $\bigcup_{i\in I} \mathsf{Ext}(U_i) = \mathsf{Ext}(\bigcup_{i\in I} U_i)$ can be proved by using a bit of intuitionistic logic; finally, also finite intersection of open sets is an open set since, as a consequence of (B_2), we can prove that

$$\mathsf{Ext}(U) \cap \mathsf{Ext}(V) = \mathsf{Ext}(U \downarrow V)$$

where

$$U \downarrow V \equiv \{c \in S|\ (\exists u \varepsilon U)\ \mathsf{ext}(c) \subseteq \mathsf{ext}(u)\ \&\ (\exists v \varepsilon V)\ \mathsf{ext}(c) \subseteq \mathsf{ext}(v)\}$$

In fact

$$\mathsf{Ext}(U) \cap \mathsf{Ext}(V) \equiv \cup_{a\varepsilon U} \mathsf{ext}(a) \cap \cup_{b\varepsilon V} \mathsf{ext}(b) = \cup_{a\varepsilon U} \cup_{b\varepsilon V} \mathsf{ext}(a) \cap \mathsf{ext}(b)$$

So, by (B_2), $\mathsf{Ext}(U) \cap \mathsf{Ext}(V) \subseteq \cup_{a\varepsilon U} \cup_{b\varepsilon V} \mathsf{Ext}(a \downarrow b)$ and hence $\mathsf{Ext}(U) \cap \mathsf{Ext}(V) \subseteq \mathsf{Ext}(U \downarrow V)$ follows since Ext distributes unions. The other inclusion is trivial.

We can now show that not only we obtain a topology on X but also that we obtain the *correct* topology, namely, that any open subset of the topology whose base is the family $(\mathsf{ext}(a))_{a\in S}$ is extensionally equal to the extension of some subset of S. In fact, from a topological point of view, an open subset A of X is characterized by the property of being the union of all the basic opens that it contains or, equivalently, to coincide with its interior $\mathsf{Int}(A)$, where

$$\mathsf{Int}(A) \equiv \{x \in X | \ (\exists a \in S) \ x \Vdash a \ \& \ \mathsf{ext}(a) \subseteq A\}$$

Of course, for any $A \subseteq X$, $\mathsf{Int}(A) \subseteq A$ and thus a subset A is open if and only if $A \subseteq \mathsf{Int}(A)$.

Theorem 1 *Let $A \subset X$. Then A is an open subset if and only if there exists a subset U of S such that $A = \mathsf{Ext}(U)$.*

Proof. First, let us observe that for any subset U of S, $\mathsf{Ext}(U)$ is an open subset. In fact, suppose that $x\varepsilon\mathsf{Ext}(U)$ then there exists $a \in S$ such that $x \Vdash a$ and $a\varepsilon U$; but the latter yields $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$ and hence $x\varepsilon\mathsf{Int}(\mathsf{Ext}(U))$, that is, $\mathsf{Ext}(U)$ is open. Now, let A be an open subset of X and consider the subset $U_A \equiv \{a \in S | \ \mathsf{ext}(a) \subseteq A\}$. Then $A = \mathsf{Ext}(U_A)$. In fact $\bigcup_{a\varepsilon U_A} \mathsf{ext}(a) \subseteq A$ is obvious and if $x\varepsilon A$ then there exists $a \in S$ such that $x \Vdash a$ and $\mathsf{ext}(a) \subseteq A$, since A is open; hence $a\varepsilon U_A$ and so $x\varepsilon\mathsf{Ext}(U_A)$.

The proof of the previous theorem shows how to find, for any given open subset A of X, a suitable subset U_A of S such that A and $\mathsf{Ext}(U_A)$ are extensionally equal; we chose the *biggest* among the possible subsets, that is the one which contains *all* of the suitable basic opens. It is clear that in general this is not the only choice and that it is well possible that two different subsets of S have the same extension. Thus we don't have a bijective correspondence between concrete opens and subsets of S and we need to introduce an equivalence relation on the formal side if we want to obtain such a correspondence. What we need is a relation which identifies the subsets U and V when $\mathsf{Ext}(U) = \mathsf{Ext}(V)$. Of course, within a constructive set theory, we cannot introduce such a relation among subsets since the collection of the subsets of a set is not a set, but we can simplify a bit the problem if we realize that $\mathsf{Ext}(U) = \mathsf{Ext}(V)$ holds if and only if, for any $a \in S$, $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$ if and only if $\mathsf{ext}(a) \subseteq \mathsf{Ext}(V)$.

Theorem 2 *Let U and V be subsets of S. Then* $\mathsf{Ext}(U) = \mathsf{Ext}(V)$ *if and only if* $(\forall a \in S) \ (\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)) \leftrightarrow (\mathsf{ext}(a) \subseteq \mathsf{Ext}(V))$.

Proof. From left to right the statement is obvious. On the other hand, let us suppose that $x\varepsilon\mathsf{Ext}(U)$, then $x\varepsilon\mathsf{ext}(u)$ for some $u\varepsilon U$; but $u\varepsilon U$ yields $\mathsf{ext}(u) \subseteq \mathsf{Ext}(U)$ and hence the assumption yields $\mathsf{ext}(u) \subseteq \mathsf{Ext}(V)$ and thus $x\varepsilon\mathsf{Ext}(V)$ follows from $x\varepsilon\mathsf{ext}(u)$. The proof of the other inclusion is completely similar.

Thus we need just to introduce, in the formal side, a new relation

$$a \lhd U \ \mathsf{prop}\ [a : S, U \subseteq S]$$

whose intended meaning is that

$$a \lhd U \text{ if and only if } \mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$$

In fact, after the previous theorem 2, we can define an equivalence relation over the subsets of S by putting

$$U =_{\lhd} V \equiv (\forall a \in S)\ a \lhd U \leftrightarrow a \lhd V$$

and this is the equivalence relation that we were looking for since it is immediate to prove the following theorem.

Theorem 3 *Let U and V be two subsets of S. Then $U =_{\lhd} V$ if and only if* $\mathsf{Ext}(U) = \mathsf{Ext}(V)$.

Now, in order to obtain a bijective correspondence between formal and concrete open subsets, we could simply state that a formal open is an equivalence class of the relation $=_{\lhd}$. But, since we prefer to avoid to deal with collections of collections of subsets, we will follow a slightly different approach and we simply choose the “fullest” among the representative of an equivalence class. So, let us put

$$\lhd(U) \equiv \{a \in S |\ a \lhd U\}$$

and let us say that a *formal open* is any subset $\lhd(U)$ for some subset U.

Then we can prove that, given any concrete open subset A, the subset $U_A \equiv \{a \in S |\ \mathsf{ext}(a) \subseteq A\}$ is the *only* formal open which corresponds to A, that is, such that $\mathsf{Ext}(U_A) = A$. Indeed, we can easily prove that U_A is a formal open. In fact, after theorem 1, we know that $\mathsf{Ext}(U_A) = A$ and hence, for any $a \in S$, $a\varepsilon \lhd (U_A)$ iff $a \lhd U_A$ iff $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U_A)$ iff $\mathsf{ext}(a) \subseteq A$ iff $a\varepsilon U_A$, that is, we proved that $\lhd(U_A) = U_A$ which means that U_A is a formal open.

By using again theorem 1, we can also prove that the correspondence is injective because if $U_A = U_B$ then $\mathsf{Ext}(U_A) = \mathsf{Ext}(U_B)$ that yields $A = B$.

Finally, we can prove that the correspondence is surjective as a consequence of the following two conditions:

$$\text{(Reflexivity)}\quad \frac{a\varepsilon U}{a \lhd U} \qquad (\lhd\text{-transitivity})\quad \frac{a \lhd U \qquad U \lhd V}{a \lhd V}$$

where $U \lhd V$ is a short-hand for a derivation of $u \lhd V$ under the assumption that $u\varepsilon U$. These conditions are valid in any concrete topological space. In fact, *reflexivity* holds since if $a\varepsilon U$ then $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$ and $\lhd$-*transitivity* holds because the first assumption means that $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$ and the second one yields that $\mathsf{Ext}(U) \subseteq \mathsf{Ext}(V)$. It is convenient to re-write such conditions in an equivalent set-theoretical notation, namely

$$\text{(Reflexivity)}\quad U \subseteq \lhd(U) \qquad (\lhd\text{-transitivity})\quad \frac{U \subseteq \lhd(V)}{\lhd(U) \subseteq \lhd(V)}$$

Now, let us suppose that $\lhd(U)$ is any formal open, then we can prove that $\lhd(U)$ is the image of the open subset $\mathsf{Ext}(\lhd(U))$. In fact, $\lhd$-*transitivity* and *reflexivity* show that $\lhd(\lhd(U)) = \lhd(U)$; hence, for any $a \in S$, $a\varepsilon U_{\mathsf{Ext}(\lhd(U))}$ iff $\mathsf{ext}(a) \subseteq \mathsf{Ext}(\lhd(U))$ iff $a \lhd \lhd(U)$ iff $a\varepsilon \lhd(U)$.

So, we solved completely the problem of dealing with concrete open subsets on the formal side. Let us now turn our attention to closed subsets.

As we will see, we have here to face the problem that from an intuitionistic point of view we cannot simply identify the closed subsets with the complements of the open subsets. Thus our plan is to follow an approach similar to the one that we used for the open subsets and hence we need a primitive definition for them too. Of course the problem is that we want to identify a closed subset by using only the basic opens, which are the only subsets that have a name in the formal side. But note that a subset $A \subseteq X$ is closed if and only if any point which cannot be separated from A by mean of a basic open is inside A, or equivalently if A is equal to its closure $\mathsf{Cl}(A)$ defined by putting

$$\mathsf{Cl}(A) \equiv \{x \in X|\ (\forall a \in S)\ x \Vdash a \rightarrow \mathsf{ext}(a) \between A\}$$

where $\mathsf{ext}(a) \between A$ is a shorthand for $(\exists y \in X)\ y\varepsilon\mathsf{ext}(a)\ \&\ y\varepsilon A$ that we will read $\mathsf{ext}(a)$ *meets* A. It is straightforward to verify that, for any subset A of X, $A \subseteq \mathsf{Cl}(A)$ and hence a subset is closed if and only if it contains its closure.

We can now prove that from an intuitionistic point of view there is no direct correspondence between concrete open subsets and concrete closed subsets. Indeed, for any given open subset A of X it is possible to prove that the subset $\mathsf{C}(A) \equiv \{x \in X|\ \neg(x \varepsilon A)\}$ is closed. In fact, if $x \varepsilon \mathsf{Cl}(\mathsf{C}(A))$, then, for any $a \in S$ such that $x \Vdash a$, it holds that $\mathsf{ext}(a) \between \mathsf{C}(A)$. Let us now suppose that $x \varepsilon A$, then there is $a \in S$ such that $x \Vdash a$ and $\mathsf{ext}(a) \subseteq A$, because A is an open subset, and hence we obtain that $\mathsf{ext}(a) \between \mathsf{C}(A)$, but of course it is not possible that both $\mathsf{ext}(a) \subseteq A$ and $\mathsf{ext}(a) \between \mathsf{C}(A)$ hold and hence we arrived to a contradiction; thus $\neg(x \varepsilon A)$ holds, that is, $x \varepsilon \mathsf{C}(A)$ and this means that the subset $\mathsf{C}(A)$ is closed. On the other hand a similar proof cannot be developed to prove that a subset $\mathsf{C}(A)$ is open when the subset A is closed, since it would be necessary to infer that a certain basic open exists which satisfies the required condition out of the fact that it is not possible that such a condition fails for all the basic open subsets. Thus, there are *more* concrete closed subsets then concrete open subsets.

The key observation to find a formal characterization of the concrete closed subsets is that a closed subset is completely determined by the collection of the basic opens which meet it, that is, the following theorem holds.

Theorem 4 *Let A and B be two closed subsets of the concrete topological space on the set X whose base is the family $(\mathsf{ext}(a))_{a \in S}$. Then A and B are equal if and only if, for any $a \in S$,*

$$(\mathsf{ext}(a) \between A) \leftrightarrow (\mathsf{ext}(a) \between B)$$

Proof. If $A = B$ then obviously $(\mathsf{ext}(a) \between A) \leftrightarrow (\mathsf{ext}(a) \between B)$ holds for any $a \in S$. On the other hand, if we assume that, for any $a \in S$, $(\mathsf{ext}(a) \between A) \leftrightarrow (\mathsf{ext}(a) \between B)$, then, by using the fact that A and B are closed subsets, for any $x \in X$, we obtain that $x \varepsilon A$ iff $x \varepsilon \mathsf{Cl}(A)$ iff $(\forall a \in S)\ x \Vdash a \rightarrow \mathsf{ext}(a) \between A$ iff $(\forall a \in S)\ x \Vdash a \rightarrow \mathsf{ext}(a) \between B$ iff $x \varepsilon \mathsf{Cl}(B)$ iff $x \varepsilon B$, that is, A and B are equal.

So, in order to have a complete information on a concrete closed subset we can simply collect, in the formal side, all the basic opens which meet it. It is then necessary to introduce a new operator, besides Ext, which links the formal side with the concrete one and which allows to obtain back the closed subset when we are given with the collection of the basic opens which meet it. Thus, for any $F \subseteq S$, we put

$$\mathsf{Rest}(F) \equiv \{x \in X|\ (\forall a \in S)\ x \Vdash a \ \rightarrow\ a \varepsilon F\}$$

and we can finally prove the following theorem.

Theorem 5 *Let A be a subset of X. Then A is a closed subset if and only if there exists a subset F of S such that $A = \mathsf{Rest}(F)$.*

Proof. Let us first prove that, if F is any subset of S, then $\mathsf{Rest}(F)$ is a closed subset of X. In fact, suppose $x\varepsilon\mathsf{Cl}(\mathsf{Rest}(F))$, then for any $a \in S$, if $x \Vdash a$ then $\mathsf{ext}(a) \between \mathsf{Rest}(F)$; now the latter yields that $a\varepsilon F$, since $\mathsf{ext}(a) \between \mathsf{Rest}(F)$ means that there exists a point $y \in X$ such that $y\varepsilon\mathsf{ext}(a)$ and $y\varepsilon\mathsf{Rest}(F)$, i.e. $(\forall b \in S)\ y \Vdash b \rightarrow b\varepsilon F$; hence for all $a \in S$, if $x \Vdash a$ then $a\varepsilon F$, i.e. $x\varepsilon\mathsf{Rest}(F)$ which means that $\mathsf{Rest}(F)$ is a closed subset. Now, let A be a closed subset of X and consider the subset $F_A \equiv \{a \in S |\ \mathsf{ext}(a) \between A\}$. Then $A = \mathsf{Rest}(F_A)$. In fact, supposing $x\varepsilon A$, for any $a \in S$, if $x \Vdash a$ then $\mathsf{ext}(a) \between A$ and hence $a\varepsilon F_A$ and thus $x\varepsilon\mathsf{Rest}(F_A)$; let us now suppose that $x\varepsilon\mathsf{Rest}(F_A)$, thus, for all $a \in S$, if $x \Vdash a$ then $a\varepsilon F_A$, that is $\mathsf{ext}(a) \between A$, thus $x\varepsilon\mathsf{Cl}(A)$ which yields $x\varepsilon A$ since A is closed.

Thus, we have solved the problem of dealing with concrete closed subsets by using only subsets of names of basic opens, but, as in the previous case with open subsets, in the proof of the above theorem we chose a suitable subset which corresponds to a given concrete closed subset and there may well be other subsets which correspond to the same closed subset. So, also in this case we need to define an equivalence relation between subsets of S such that two subsets F and G are equal if and only if $\mathsf{Rest}(F) = \mathsf{Rest}(G)$. We can simplify the problem because of the following theorem.

Theorem 6 *Let F and G be two subsets of S. Then* $\mathsf{Rest}(F) = \mathsf{Rest}(G)$ *if and only if* $(\forall a \in S)(\mathsf{ext}(a) \between \mathsf{Rest}(F)) \leftrightarrow (\mathsf{ext}(a) \between \mathsf{Rest}(G))$.

Proof. We have already proved that, for any $F, G \subseteq S$, $\mathsf{Rest}(F)$ and $\mathsf{Rest}(G)$ are closed subsets. Then the result is immediate by theorem 4.

Then it is clear that in order to have a completely formal counterpart of a closed subset we need to formalize the condition which states that the concrete subset $\mathsf{ext}(a) \cap \mathsf{Rest}(F)$ is inhabited. To this aim we introduce a new proposition $\mathsf{Pos}(a, F)\ \mathsf{prop}\ [a : S, F \subseteq S]$ whose intended meaning is that

$$\mathsf{Pos}(a, F) \text{ if and only if } \mathsf{ext}(a) \between \mathsf{Rest}(F)$$

The name Pos is used to recall that this relation is a *positive* way to state that the intersection between $\mathsf{ext}(a)$ and $\mathsf{Rest}(F)$ is inhabited instead that simply saying that $\mathsf{ext}(a) \cap \mathsf{Rest}(F) \neq \emptyset$ which would amount to state that $\neg(\forall x \in X)\ \neg(x\varepsilon\mathsf{ext}(a) \cap \mathsf{Rest}(F))$.

After the previous theorem 6, and in analogy to the previous case with the open subsets, we put

$$F =_{\mathsf{Pos}} G \equiv (\forall a \in S)\ \mathsf{Pos}(a, F)\ \leftrightarrow\ \mathsf{Pos}(a, G)$$

and it is clear that we obtain an equivalence relation. Moreover, this is the relation we were looking for since the following theorem is straightforward.

Theorem 7 *Let F and G be two subsets of S. Then $F =_{\mathsf{Pos}} G$ if and only if* $\mathsf{Rest}(F) = \mathsf{Rest}(G)$.

We want now to introduce the notion of formal closed subset in such a way that there is a bijective correspondence between concrete closed subsets and formal closed subsets. In analogy with what we did in the case of the formal open subsets, we put, for any subset F of S,

$$\mathsf{Pos}(F) \equiv \{c \in S \mid \mathsf{Pos}(c, F)\}$$

and we will say that a *formal closed* is any subset $\mathsf{Pos}(F)$ for some subset F.

We will show now that, for any concrete closed subset A of X, the subset $F_A \equiv \{c \in S \mid \mathsf{ext}(c) \between A\}$ is the *only* formal closed such that $\mathsf{Rest}(F_A) = A$.

Indeed, it is easy to see that $F_A = \mathsf{Pos}(F_A)$ and hence that F_A is a formal closed. In fact, theorem 5 shows that $A = \mathsf{Rest}(F_A)$ and hence, for any $c \in S$, $c \varepsilon \mathsf{Pos}(F_A)$ iff $\mathsf{Pos}(c, F_A)$ iff $\mathsf{ext}(c) \between \mathsf{Rest}(F_A)$ iff $\mathsf{ext}(c) \between A$ iff $c \varepsilon F_A$.

Moreover, the correspondence is injective because $F_A = F_B$ yields that $\mathsf{Rest}(F_A) = \mathsf{Rest}(F_B)$ and hence $A = B$ by theorem 5.

Finally the correspondence is surjective. To prove this fact, let us note first that the following two conditions hold in any concrete topological space:

$$\text{(Anti-reflexivity)} \qquad \frac{\mathsf{Pos}(a, F)}{a \varepsilon F}$$

$$\text{(}\mathsf{Pos}\text{-transitivity)} \qquad \frac{\mathsf{Pos}(a, F) \qquad b \varepsilon G \ [b : S, \mathsf{Pos}(b, F)]}{\mathsf{Pos}(a, G)}$$

In fact, *anti-reflexivity* is valid since we already showed that $a \varepsilon F$ holds if $\mathsf{ext}(a) \between \mathsf{Rest}(F)$. And Pos-*transitivity* is valid since $\mathsf{Pos}(a, F)$ means that there exists a point $y \in X$ such that $y \Vdash a$ and $y \varepsilon \mathsf{Rest}(F)$; now, by assuming $b \in S$ and $y \Vdash b$ we obtain that $\mathsf{Pos}(b, F)$ and thus the second premise yields $b \varepsilon G$; so, by discharging the assumptions $b \in S$ and $y \Vdash b$, we get $y \varepsilon \mathsf{Rest}(G)$ which, together with $y \Vdash a$, yields $\mathsf{Pos}(a, G)$.

Repeating what we did with the cover relation, it is convenient also in this case, to re-write *anti-reflexivity* and Pos-*transitivity* in a set-theoretical notation.

$$\text{(Anti-reflexivity) } \mathsf{Pos}(F) \subseteq F \quad (\mathsf{Pos}\text{-transitivity}) \ \frac{\mathsf{Pos}(F) \subseteq G}{\mathsf{Pos}(F) \subseteq \mathsf{Pos}(G)}$$

In fact, given any formal closed $\mathsf{Pos}(F)$, by using *anti-reflexivity* and Pos-*transitivity* we now obtain that $\mathsf{Pos}(\mathsf{Pos}(F)) = \mathsf{Pos}(F)$ and hence we can prove that $\mathsf{Pos}(F)$ is the image of the concrete closed subset $\mathsf{Rest}(\mathsf{Pos}(F))$ because, for any $c \in S$, $c \varepsilon F_{\mathsf{Rest}(\mathsf{Pos}(F))}$ if and only if $\mathsf{ext}(c) \between \mathsf{Rest}(\mathsf{Pos}(F))$ if and only if $c \varepsilon \mathsf{Pos}(\mathsf{Pos}(F))$ if and only if $c \varepsilon \mathsf{Pos}(F)$.

Thus, we found two relations, that is $\lhd$ and Pos, and some conditions on them, that is *reflexivity*, $\lhd$-*transitivity*, *anti-reflexivity* and Pos-*transitivity*, which allow to deal on the formal sides with concrete open and closed subsets. But these conditions are not sufficient to describe completely the concrete situation; for instance there is no condition which describe formally the conditions (B_1) and (B_2) and no condition which connects $\lhd$ and Pos, that is, which states that $\lhd(U)$ and $\mathsf{Pos}(F)$ are respectively a formal open and a formal closed subset of the *same* topology.

Anyhow, let us observe that *anti-reflexivity* is strictly connected with the condition (B_1). In fact, the condition (B_1) is (classically) equivalent to the fact that the empty set is a closed set, namely, $\mathsf{Cl}(\emptyset) = \emptyset$ or, equivalently, $\mathsf{Rest}(\emptyset) = \emptyset$. Now, *anti-reflexivity* yields that $\neg\mathsf{Pos}(a, \emptyset)$ holds, that is, for any $a \in S$, $\neg(\mathsf{ext}(a) \between \mathsf{Rest}(\emptyset))$, and this suggests that $\mathsf{Rest}(\emptyset)$ should be empty.

To formulate (B_2) completely within the formal side, we can use the fact that $\mathsf{Ext}(U) \cap \mathsf{Ext}(V) \subseteq \mathsf{Ext}(U \downarrow V)$, that we already proved. In fact, supposing $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$ and $\mathsf{ext}(a) \subseteq \mathsf{Ext}(V)$, we immediately obtain that $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U) \cap \mathsf{Ext}(V)$ and hence $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U \downarrow V)$. Its formal counterpart is

$$(\downarrow\text{-right}) \qquad \frac{a \lhd U \qquad a \lhd V}{a \lhd U \downarrow V}$$

The link between $\lhd$ and Pos is expressed by the following condition

$$\text{(Compatibility)} \qquad \frac{\mathsf{Pos}(a, F) \quad a \lhd U}{(\exists b \varepsilon U)\ \mathsf{Pos}(b, F)}$$

whose validity is straightforward.

If we collect all the valid conditions that we found till now, we arrive at the complete definition of formal topology; it was first proposed in [GS99], even if there a slightly different path was followed to arrive to the same outcome.

Definition 3 *A* formal topology *with a binary positivity predicate is a triple* $\mathcal{A} \equiv (S, \lhd, \mathsf{Pos})$ *where* S *is a set,* $\lhd$ *is an infinitary relation between elements and subsets of* S*, that is* $a \lhd U$ prop $[a : S, U \subseteq S]$*, satisfying the following conditions:*

$$\text{(reflexivity)} \quad \frac{a \varepsilon U}{a \lhd U}$$

$$(\lhd\text{-transitivity}) \quad \frac{a \lhd U \qquad U \lhd V}{a \lhd V}$$

$$(\downarrow\text{-right}) \quad \frac{a \lhd U \qquad a \lhd V}{a \lhd U \downarrow V}$$

and Pos *is an infinitary relation between elements and subsets of* S*, which satisfies the following conditions*

$$\text{(Anti-reflexivity)} \quad \frac{\mathsf{Pos}(a, F)}{a \varepsilon F}$$

$$(\mathsf{Pos}\text{-transitivity}) \quad \frac{\mathsf{Pos}(a, F) \qquad b \varepsilon G \ [b : S, \mathsf{Pos}(b, F)]}{\mathsf{Pos}(a, G)}$$

$\lhd$ *and* Pos *are called respectively* cover relation *and* positivity predicate *and they are linked by the following condition*

$$\text{(Compatibility)} \quad \frac{\mathsf{Pos}(a, F) \quad a \lhd U}{(\exists b \varepsilon U)\ \mathsf{Pos}(b, F)}$$

The main problem with this definition is that we just collected *some* of the valid conditions on the cover relation and the positivity predicate, but still we are far from a complete formalization of concrete topological spaces. This fact is confirmed by the fact that, while the basic opens are sufficient to determine in the case of a concrete topological space both the open and the closed subsets (as we proved in theorems 1, 4 and 5), if we consider a formal topology there are many possible positivity predicates for the *same* cover relation. For instance, given any positivity predicate Pos for a given cover relation it is not difficult to verify that also

$$\mathsf{Pos}_H(a, F) \equiv \mathsf{Pos}(a, F \cap H)$$

is a positivity predicate for the same cover relation. In fact, *anti-reflexivity* is immediate because $\mathsf{Pos}_H(F) = \mathsf{Pos}(F \cap H) \subseteq F \cap H \subseteq F$. As regard to the validity of Pos_H*-transitivity*, let us first observe that $\mathsf{Pos}(F \cap H) \subseteq F \cap H \subseteq H$; then, if we assume that $\mathsf{Pos}_H(F) \subseteq G$ holds, that is $\mathsf{Pos}(F \cap H) \subseteq G$, then we obtain that $\mathsf{Pos}(F \cap H) \subseteq G \cap H$ which yields $\mathsf{Pos}(F \cap H) \subseteq \mathsf{Pos}(G \cap H)$ by Pos-*transitivity*, that is, $\mathsf{Pos}_H(F) \subseteq \mathsf{Pos}_H(G)$. Finally, the proof that *compatibility* holds for Pos_H is straightforward. Thus the correspondence between concrete and formal closed subsets cannot be completely precise and this means that we were not able to completely formalize by using the positivity predicate $\mathsf{Pos}(a, F)$ the condition that $\mathsf{ext}(a) \between \mathsf{Rest}(F)$.

On the other hand, the definition of formal topology here proposed has the advantage to be applicable in many more cases than just the concrete topological spaces and this fact has already proved to be useful.

Anyhow, a consequence of the present definition is that in order to specify a cover relation and a binary positivity predicate it is not sufficient to give only axioms for the cover relation but also specific axioms are necessary for the positivity predicate in order to determine one of them among the many possible choices. We will see in the next sections that this problem is affecting also the method to be used for an inductive generation of a formal topology.

2 Inductive generation of a cover relation and a positivity predicate

The conditions appearing in definition 3 of formal topology, though written in the shape of rules, must be understood as requirements of validity: if the premises hold, also the conclusion must hold. As they stand, they are by no means acceptable rules to generate inductively a cover and a positivity predicate. This is obvious if one notes that the operation $\downarrow$ among subsets, which occurs in the conclusion of $\downarrow$-*right*, is not even well defined unless we already have a complete knowledge of the cover.

As we will see, another problem is that admitting $\lhd$-*transitivity* and Pos-*transitivity* as acceptable rules for an inductive definition is equivalent to well-known fix-point principles, which do not have a predicative justification.

So, to transform the axiomatic definition into good inductive rules we need to face with these problems.

2.1 The predicativity problem

An inductive definition of a cover will start from some axioms, which we can assume to be given by means of a relation

$$R(a, U) \ \mathsf{prop} \ [a : S, U \subseteq S]$$

We thus want to generate the least cover $\lhd_R$ which satisfies the following condition:

$$\text{(axioms)} \qquad \frac{R(a,U)}{a \lhd_R U}$$

As we will see in section 2.3, the task of forcing $\lhd_R$ to satisfy $\downarrow$-*right* is essentially only technical once it is clear that $\lhd_R$ satisfies *reflexivity* and $\lhd$-*transitivity*. So we concentrate in this section on the conceptual problem of constructing the minimal infinitary relation $\lhd_R$ which satisfies *reflexivity*, $\lhd$-*transitivity* and the axioms given by R.

From an impredicative point of view, $\lhd_R$ is easily obtained "from above" simply as the intersection of the collection $\mathcal{C}_R$ of all the reflexive and transitive infinitary relations containing R. In fact, it is clear that the total relation is in $\mathcal{C}_R$ and that the intersection preserves all such conditions (see [CSSV] for details).

Predicatively the method of defining $\lhd_R$ as the intersection of $\mathcal{C}_R$ is not acceptable, since there is no way of producing $\mathcal{C}_R$ above as a set-indexed family and hence to define its intersection.

Therefore, we must obtain $\lhd_R$ "from below" by means of some introductory rules. The first naive idea is that of using axioms, *reflexivity* and $\lhd$-*transitivity* for this purpose. But then a serious problem emerges: in the premises of $\lhd$-*transitivity*, that is

$$\frac{a \lhd_R V \qquad V \lhd_R U}{a \lhd_R U}$$

there is a subset V which does not appear in the conclusion. This means that the tree of possible premises to conclude that $a \lhd_R U$ has an unbounded branching: each subset V satisfying $a \lhd_R V$ and $V \lhd_R U$ would be enough to obtain $a \lhd_R U$, and there is no way to survey them all. Also, a dangerous vicious circle seems to be present: the subset V, whose existence would be enough to obtain $a \lhd_R U$, could be defined by means of the relation $\lhd_R$ itself which we are trying to construct. In this way, the instructions to try to build up $\lhd_R$ would not be fixed in advance, but change along their application.

A similar problem arises when one tries to generate a binary positivity predicate. In fact, the first idea one can think of in order to add axioms on the positivity predicate is to require that $R(a,F) \to \mathsf{Pos}(a,F)$ holds for some suitable relations $R(a,F)$ prop $[a : S, F \subseteq S]$. Anyhow this approach does not fit well with the idea that Pos is going to be generated by co-induction and hence it is not convenient to add an introduction rule for it. In this case it is much more natural to add axioms in the shape of an elimination rule, that is, $\mathsf{Pos}(a,F) \to R(a,F)$, which can be used to state when $\mathsf{Pos}(a,F)$ does not holds.

Thus, in order to concentrate on the problems due to $\lhd$-*transitivity* and Pos-*transitivity*, let us suppose to work in the case of a cover relation which has to satisfy only *reflexivity* and $\lhd$-*transitivity* and a positivity predicate which has to satisfy only *anti-reflexivity* and Pos-*transitivity*, besides some axioms.

Then, supposing $R_\lhd(a, U)$ prop $[a : S, U \subseteq S]$ is the infinitary relation which we want to use to state the axioms on $\lhd$, the only condition on the cover relation, besides *reflexivity* and $\lhd$-*transitivity*, is the following

$$(\lhd\text{-general axioms})\ \frac{R_\lhd(a, U)}{a \lhd U}$$

and, supposing $R_{\mathsf{Pos}}(a, F)$ prop $[a : S, F \subseteq S]$ is the infinitary relation which we want to use to state the axioms on Pos then, besides *anti-reflexivity* and Pos-*transitivity*, we have to require only the following condition:

$$(\mathsf{Pos}\text{-general axioms})\ \frac{\mathsf{Pos}(a, F)}{R_{\mathsf{Pos}}(a, F)}$$

Let us re-write the previous conditions by using the set-theoretical notation that we already used for $\lhd$ and Pos in the previous section, that is, let us introduce the following definitions

$$R_\lhd(U) \equiv \{a \in S |\ R_\lhd(a, U)\} \qquad R_{\mathsf{Pos}}(F) \equiv \{a \in S |\ R_{\mathsf{Pos}}(a, F)\}$$

Then, we obtain that the previous conditions on the cover relation can be re-written according to the following table $(1_\lhd)$:

$$\begin{array}{ll} (\lhd\text{-general axiom}) & R_\lhd(U) \subseteq \lhd(U) \\ (\text{reflexivity}) & U \subseteq \lhd(U) \\ (\lhd\text{-transitivity}) & \dfrac{V \subseteq \lhd(U)}{\lhd(V) \subseteq \lhd(U)} \end{array}$$

and those on the positivity predicate according to the following table (1_{Pos})

$$\begin{array}{ll} (\mathsf{Pos}\text{-general axiom}) & \mathsf{Pos}(F) \subseteq R_{\mathsf{Pos}}(F) \\ (\text{anti-reflexivity}) & \mathsf{Pos}(F) \subseteq F \\ (\mathsf{Pos}\text{-trans}) & \dfrac{\mathsf{Pos}(G) \subseteq F}{\mathsf{Pos}(G) \subseteq \mathsf{Pos}(F)} \end{array}$$

Now the duality between the rules for $\lhd$ and Pos is evident.

After [CSSV], we know that the rules $(1_\lhd)$ cannot be used to predicatively generate a cover relation but let us analyze again the proof since this will be useful to discover that also the rules (1_{Pos}) should not be used to generate predicatively a positivity predicate.

In fact, the following rules $(2_{\lhd})$ and (2_{Pos}), provided we could use them in a generation process, are equivalent to the previous $(1_{\lhd})$ and (1_{Pos}).

$$\text{(reflexivity)} \quad U \subseteq \lhd(U) \quad (\lhd\text{-infinity}) \quad \frac{V \subseteq \lhd(U)}{R_{\lhd}(V) \subseteq \lhd(U)}$$

$$\text{(anti-reflexivity)} \; \mathsf{Pos}(F) \subseteq F \; (\mathsf{Pos}\text{-infinity}) \; \frac{\mathsf{Pos}(G) \subseteq F}{\mathsf{Pos}(G) \subseteq R_{\mathsf{Pos}}(F)}$$

The meaning of the previous sentence "if these rules could be used in a generation process" is that the rules in $(2_{\lhd})$ should be considered as introduction rules for the *minimal* proposition which satisfies them, that is, the following rule should be valid

$$\text{(minimality)} \qquad \frac{U \subseteq P \qquad R_{\lhd}(V) \subseteq P \; [V \subseteq P]}{\lhd(U) \subseteq P}$$

In a similar way, the rules concerning the positivity predicate Pos have to be considered like elimination rules for the *maximal* proposition which satisfies them, that is, the following rule should be valid

$$\text{(maximality)} \qquad \frac{Q \subseteq F \qquad Q \subseteq R_{\mathsf{Pos}}(H) \; [Q \subseteq H]}{Q \subseteq \mathsf{Pos}(F)}$$

Now, we can prove that $(1_{\lhd})$ implies $(2_{\lhd})$, that is, the rules $(2_{\lhd})$ are valid. *Reflexivity* is the same in both cases, thus let us prove $\lhd$*-infinity*.

$$\frac{\dfrac{\lhd\text{-gen-ax}}{R_{\lhd}(V) \subseteq \lhd(V)} \qquad \dfrac{V \subseteq \lhd(U)}{\lhd(V) \subseteq \lhd(U)}}{R_{\lhd}(V) \subseteq \lhd(U)} \; \lhd\text{-tran}$$

By using *minimality*, we can prove also the other implication, namely that $(2_{\lhd})$ implies $(1_{\lhd})$. The result is obvious for *reflexivity* and immediate for $\lhd$*-general axioms*

$$\frac{\dfrac{\text{refl}}{U \subseteq \lhd(U)}}{R_{\lhd}(U) \subseteq \lhd(U)} \; \lhd\text{-inf}$$

while $\lhd$*-transitivity* is the point where *minimality* is required. In fact let us put $P \equiv \lhd(U)$ in the *minimality* rule, then we obtain

$$\frac{V \subseteq \lhd(U) \qquad \dfrac{[T \subseteq \lhd(U)]_1}{R_{\lhd}(T) \subseteq \lhd(U)} \; \lhd\text{-inf}}{\lhd(V) \subseteq \lhd(U)} \; 1 \quad \text{minimality}$$

A similar proof can be carried on to show the equivalence of the rules (1_{Pos}) and (2_{Pos}). Let us show first that (1_{Pos}) implies (2_{Pos}). *Anti-reflexivity* is present in both the cases, thus let us consider Pos-*infinity*:

$$\dfrac{\dfrac{\mathsf{Pos}(G) \subseteq F}{\mathsf{Pos}(G) \subseteq \mathsf{Pos}(F)}\ \mathsf{Pos}\text{-tran} \qquad \dfrac{}{\mathsf{Pos}(F) \subseteq R_{\mathsf{Pos}}(F)}\ \mathsf{Pos}\text{-gen-ax}}{\mathsf{Pos}(G) \subseteq R_{\mathsf{Pos}}(F)}$$

It is clear that this proof is just the analogous of the one we used in the case of $\lhd$ but we preferred to write it explicitly to show how the duality works for us. The same is going to happen in the next proof that shows that (2_{Pos}) and *maximality* imply (1_{Pos}). In fact *anti-reflexivity* is assumed and Pos-*general axioms* is immediate. Finally, Pos-*transitivity* follows by putting $Q \equiv \mathsf{Pos}(G)$ in the *maximality* rule.

$$\dfrac{\mathsf{Pos}(G) \subseteq F \qquad \dfrac{[\mathsf{Pos}(G) \subseteq H]_1}{\mathsf{Pos}(G) \subseteq R_{\mathsf{Pos}}(H)}\ \mathsf{Pos}\text{-inf}}{\mathsf{Pos}(G) \subseteq \mathsf{Pos}(F)}\ 1 \quad \text{maximality}$$

In [CSSV] it is shown that it is not a good idea to use the rules ($2_\lhd$) and *minimality* for every operator $R_\lhd$ because it is then possible to show that every monotone operator $R_\lhd$ over $\mathcal{P}(S)$ has a least fix-point, a principle which does not have a predicative justification. Here is a quick proof. The least fix-point is $\lhd(\emptyset)$. In fact

$$\dfrac{\lhd(\emptyset) \subseteq \lhd(\emptyset)}{R_\lhd(\lhd(\emptyset)) \subseteq \lhd(\emptyset)}\ \lhd\text{-inf}$$

and

$$\dfrac{\emptyset \subseteq R_\lhd(\lhd(\emptyset)) \qquad \dfrac{\dfrac{[V \subseteq R_\lhd(\lhd(\emptyset))]_1 \qquad \dfrac{\lhd(\emptyset) \subseteq \lhd(\emptyset)}{R_\lhd(\lhd(\emptyset)) \subseteq \lhd(\emptyset)}\ \lhd\text{-inf}}{V \subseteq \lhd(\emptyset)}}{R_\lhd(V) \subseteq R_\lhd(\lhd(\emptyset))}\ \text{monotonicity of } R_\lhd}{\lhd(\emptyset) \subseteq R_\lhd(\lhd(\emptyset))}\ 1 \quad \text{minimality}$$

Thus we showed that $R_\lhd(\lhd(\emptyset)) = \lhd(\emptyset)$, that is, $\lhd(\emptyset)$ is a fix-point for $R_\lhd$. We can use again *minimality* to show that it is the least fix-point. In fact, let us suppose that Z is a fix-point for $R_\lhd$, that is $R_\lhd(Z) = Z$, and consider the following derivation

$$\dfrac{\emptyset \subseteq Z \qquad \dfrac{\dfrac{[V \subseteq Z]_1}{R_\lhd(V) \subseteq R_\lhd(Z)}\ \text{monotonicity of } R_\lhd \qquad R_\lhd(Z) = Z}{R_\lhd(V) \subseteq Z}}{\lhd(\emptyset) \subseteq Z}\ 1 \quad \text{minimality}$$

It is possible to develop an analogous proof by using the rules (2_{Pos}) and *maximality* in order to show that for every monotone operator R_{Pos} over $\mathcal{P}(S)$ the greatest fix-point is $\mathsf{Pos}(S)$.

It is worth noting that there is a deeper connection between the existence of least and greatest fix-points for monotone operators and the existence of the propositions $a \lhd U$ and $\mathsf{Pos}(a, F)$.

Let us first consider any operator $R_\lhd$ which is both *monotone*, that is, such that $(U \subseteq V) \to (R_\lhd(U) \subseteq R_\lhd(V))$ holds, and *reflexive*, that is, such that $U \subseteq R_\lhd(U)$ holds. Then, for any subset U we can define a new operator R_U by putting

$$R_U(V) \equiv U \cup R_\lhd(V)$$

and it is immediate to prove that also R_U is monotone and reflexive.

Now, let us suppose that the principle of existence of least fix-point holds for monotone operators, that is, let us suppose that for any monotone operator R there exists a subset P^R such that $R(P^R) = P^R$ and $R(Z) = Z \to P^R \subseteq Z$. Then we can define a proposition $a \lhd U$ prop $[a : S, U \subseteq S]$, that is, an operator $\lhd: \mathcal{P}(S) \to \mathcal{P}(S)$, by putting

$$\lhd(U) \equiv P^{R_U}$$

and prove that it respects *reflexivity*, $\lhd$*-infinity* and *minimality*.

In fact, *reflexivity* is immediate because

$$U \subseteq U \cup R_\lhd(P^{R_U}) \equiv R_U(P^{R_U}) = P^{R_U} \equiv \lhd(U)$$

and $\lhd$*-infinity* can be proved as follows: let us suppose that $V \subseteq \lhd(U)$, then $R_\lhd(V) \subseteq R_\lhd(\lhd(U))$ since $R_\lhd$ is monotone, but

$$R_\lhd(\lhd(U)) \subseteq U \cup R_\lhd(\lhd(U)) \equiv R_U(\lhd(U)) \equiv R_U(P^{R_U}) = P^{R_U} \equiv \lhd(U)$$

and hence $R_\lhd(V) \subseteq \lhd(U)$. Finally, to prove *minimality*, that is

$$\frac{U \subseteq P \qquad R_\lhd(V) \subseteq P\ [V \subseteq P]}{\lhd(U) \subseteq P}$$

first observe that, provided $R_\lhd$ is monotone, *minimality* is equivalent to

$$\frac{U \subseteq P \qquad R_\lhd(P) \subseteq P}{\lhd(U) \subseteq P}$$

since obviously if, for any V, $R_\lhd(V) \subseteq P\ [V \subseteq P]$ then $R_\lhd(P) \subseteq P$, and if $V \subseteq P$ and $R_\lhd$ is monotone then $R_\lhd(V) \subseteq R_\lhd(P)$ and then $R_\lhd(V) \subseteq P$ is a consequence of $R_\lhd(P) \subseteq P$. It is interesting to observe that if $R_\lhd$ is monotone then the impredicative assumption $R_\lhd(V) \subseteq P\ [V \subseteq P]$, which requires a universal quantification over all the subsets V, is not necessary and all our proofs are fully predicative; and indeed we will see that the only impredicative step is to obtain a monotone operator when we are given with a generic one. Now, if $R_\lhd$ is also reflexive, *minimality* is also equivalent to the following rule

$$\frac{U \subseteq P \qquad R_\lhd(P) = P}{\lhd(U) \subseteq P}$$

since in this case $R_\lhd(P) \subseteq P$ if and only if $R_\lhd(P) = P$. Thus, we have just to prove the validity of the former rule, but this is immediate since, supposing $U \subseteq P$ and $R_\lhd(P) = P$ we can prove that P is a fix-point for R_U, because

$$\begin{aligned} R_U(P) &\equiv U \cup R_\lhd(P) && \text{(by definition)} \\ &= U \cup P && \text{(because } R_\lhd(P) = P\text{)} \\ &= P && \text{(because } U \subseteq P\text{)} \end{aligned}$$

and hence $\lhd(U) \equiv P^{R_U} \subseteq P$, because P^{R_U} is the least fix-point of R_U.

Thus, we proved that, provided $R_\lhd$ is a monotone and reflexive operator, the principle of existence of the least fix-point is sufficient to define a cover relation. But, at least from an impredicative point of view, this is all what we need since given any operator $R_\lhd$ we can define a new operator $R^*_\lhd$ which is monotone and reflexive and generates the same cover. In fact, let us define first the "reflexivization" $R^r_\lhd$ of $R_\lhd$ by putting

$$R^r_\lhd(U) \equiv U \cup R_\lhd(U)$$

It is obvious that $R^r_\lhd$ is a reflexive operator which contains $R_\lhd$, where we say that the operator R_1 *contains* the operator R_2 when, for any subset U, $R_1(U) \subseteq R_2(U)$. The second step, that is, the "monotonization" of $R^r_\lhd$, is impredicative. In fact, by using an impredicative existential quantification on the collection of all the subsets of S, we can put

$$\begin{aligned} R^*_\lhd(U) &\equiv \bigcup_{V \subseteq U} R^r_\lhd(V) \\ &\equiv \{a \in S \mid (\exists V \subseteq S)\ a \varepsilon R^r_\lhd(V)\ \&\ V \subseteq U\} \end{aligned}$$

Then $R^*_\lhd$ is a monotone and reflexive operator which contains $R_\lhd$ and which is *minimal*, that is, it is contained in any other monotone and reflexive operator which contains $R_\lhd$. In fact, let us suppose $W_1 \subseteq W_2$; then any subset V which is contained in W_1 is also contained in W_2 and hence $R^*_\lhd(W_1) \equiv \bigcup_{V \subseteq W_1} R^r_\lhd(V) \subseteq \bigcup_{V \subseteq W_2} R^r_\lhd(V) \equiv R^*_\lhd(W_2)$, that is $R^*_\lhd$ is monotone. Moreover, $R^r_\lhd$ is obviously contained in $R^*_\lhd$ and hence $R^*_\lhd$ is reflexive and contains $R_\lhd$. Finally, it is the smallest monotone and reflexive relation containing $R_\lhd$. In fact, let us suppose that $a \varepsilon R^*_\lhd(U)$ and that P is any monotone and reflexive operator containing $R_\lhd$. Then there exists a subset V such that $a \varepsilon R^r_\lhd(V)$ and $V \subseteq U$; hence $a \varepsilon P(V)$, since P contains $R^r_\lhd$ because it is reflexive and contains $R_\lhd$, and thus $a \varepsilon P(U)$, since P is monotone.

To conclude we have just to show that $R_\lhd$ and $R^*_\lhd$ generates the same cover relation; to this aim, observe that $\lhd$ is the minimal infinitary relation obtained by closing $R_\lhd$ under *reflexivity* and $\lhd$-*transitivity* and hence that $\lhd$ is a reflexive and monotone relation; thus if $\lhd$ contains $R_\lhd$ then it also contains $R^*_\lhd$ and the result follows by minimality.

We can deal with in a similar way with the positivity predicate. In fact, let us consider any operator R_{Pos} which is *monotone*, that is, such that $(F \subseteq G) \to (R_{\mathsf{Pos}}(F) \subseteq R_{\mathsf{Pos}}(G))$ holds, and *anti-reflexive*, that is, such that $R_{\mathsf{Pos}}(F) \subseteq F$ holds. Then, for any subset F we can define a new operator R_F by putting

$$R_F(G) \equiv F \cap R_{\mathsf{Pos}}(G)$$

and it is immediate to prove that also R_F is monotone and anti-reflexive.

Now, let us suppose that the principle of existence of greatest fix-point holds for monotone operators, that is, let us suppose that for any monotone operator R there exists a subset M^R such that $R(M^R) = M^R$ and $R(Z) = Z \to Z \subseteq M^R$. Then, we can define a proposition $\mathsf{Pos}(a, F)$ prop $[a : S, F \subseteq S]$, that is an operator $\mathsf{Pos} : \mathcal{P}(S) \to \mathcal{P}(S)$, by putting

$$\mathsf{Pos}(F) \equiv M^{R_F}$$

and prove as above that it respects *anti-reflexivity*, Pos-*infinity* and *maximality*.

Thus, provided R_{Pos} is a monotone and anti-reflexive operator, the principle of existence of the greatest fix-point is sufficient to define a positivity predicate. But, at least from an impredicative point of view, this is all what we need since given any operator R_{Pos} we can define a new operator R^*_{Pos} which is monotone and anti-reflexive and which generates the same positivity predicate. In fact, let us define first the "anti-reflexivization" R^a_{Pos} of R_{Pos} by putting

$$R^a_{\mathsf{Pos}}(F) \equiv F \cap R_{\mathsf{Pos}}(F)$$

It is obvious that R^a_{Pos} is an anti-reflexive operator which is contained in R_{Pos}. The second step, that is, the "monotonization" of

R^a_{Pos} is impredicative. In fact, let us use an impredicative universal quantification on the collection of the subsets of S and put

$$\begin{aligned} R^*_{\mathsf{Pos}}(F) &\equiv \bigcap_{F\subseteq H} R^a_{\mathsf{Pos}}(H) \\ &\equiv \{a \in S |\ (\forall H)\ F \subseteq H \rightarrow a\varepsilon R^a_{\mathsf{Pos}}(H)\} \end{aligned}$$

Then R^*_{Pos} is a monotone and anti-reflexive operator which is contained in R_{Pos} and which is *maximal*, that is it contains any other monotone and anti-reflexive operator which is contained in R_{Pos}.

To conclude we have just to show that R_{Pos} and R^*_{Pos} generate the same positivity predicate. Pos is the maximal infinitary relation obtained by closing R_{Pos} under *anti-reflexivity* and Pos-*transitivity* and hence it is an anti-reflexive and monotone relation; thus if Pos is contained in R_{Pos} then it is also contained in R^*_{Pos} and the result follows by maximality.

So we proved that, from a predicative point of view, it is not safe to generate a cover relation or a positivity predicate by using the rules (2) in their general form, that is, for any infinitary relation R. Thus we have to choose the suitable relations with more care. To find the correct constrains on R we are going to use, like a suggestion, what was already discovered in [CSSV].

2.2 Safe axioms for $\lhd$ *and* Pos

In the previous section we showed that if we want to generate a cover relation and a positivity predicate in a predicative way, we have to put some constrains on the infinitary relation to be used in $\lhd$-*infinity* and Pos-*infinity*, otherwise *minimality* and *maximality* cannot be constructively justified. This is the reason why we are going to generate a cover relation only when we have an *axiom-set*, that is, a set-indexed family $I(a)$ set $[a : S]$ and an indexed family $C(a,i) \subseteq S\ [a : S, i : I(a)]$ of subsets of S, whose intended meaning is to state that, for all $i \in I(a)$, $a \lhd C(a,i)$. Thus, provided we have an axiom-set, a safe infinitary relation is

$$R_\lhd(a,U) \equiv (\exists i \in I(a))\ C(a,i) \subseteq U$$

and in general are safe all the relations $R_\lhd$ such that there exist I and C such that

$$R_\lhd(a,U) \text{ if and only if } (\exists i \in I(a))\ C(a,i) \subseteq U$$

In fact, in this case we can generate a cover relation by using the rules $(2_\lhd)$ of the previous section, that we can write as follows

$$\text{(reflexivity)}\quad \frac{a\varepsilon U}{a \lhd U} \qquad (\lhd\text{-infinity})\quad \frac{R_\lhd(a,V) \qquad V \lhd U}{a \lhd U}$$

In order to convince ourselves that the predicativity problem is solved, let us give a simpler form to $\triangleleft$-*infinity*. We know that $R_{\triangleleft}(a, V)$ holds only if there exists $i \in I(a)$ such that $C(a, i) \subseteq V$. Then we can transform $\triangleleft$-*infinity* into

$$(\triangleleft\text{-infinity}) \quad \frac{i \in I(a) \qquad C(a,i) \triangleleft U}{a \triangleleft U}$$

and in this way any reference to the subset V is disappeared and the implicit use of an existential quantification on the collection $\mathcal{P}(S)$ is transformed into an existential quantification on the elements of the set $I(a)$.

What was described above is essentially the approach to a predicative inductive generation of a cover relation that was suggested in [CSSV]. Now we want to use a similar approach to generate a positivity predicate. Thus, let us say that an *axiom-set* for the positivity predicate is a set-indexed family $J(a)$ set $[a : S]$ and an indexed family $D(a, j) \subseteq S$ $[a : S, j : J(a)]$ of subsets of S. Since Pos is going to be generated by co-induction it is not immediate to state the intended meaning of these axioms; in fact we can just know that they are going to be used in some consequence of the fact that the positivity predicate holds for some element and some subset of S. We will deal again with this topic later, after having shown suitable rules of elimination and co-induction for Pos.

Thus, let us continue to use the duality between the cover relation and the positivity predicate like a guide to state which are safe infinitary relations for the positivity predicate. Provided that we have an axiom-set J and D to generate the positivity predicate, a safe infinitary relation is

$$R_{\mathsf{Pos}}(a, F) \equiv (\forall j \in J(a))\ F \between D(a, j)$$

where $F \between G$ is a shorthand for $(\exists a \in S)\ a\varepsilon F\ \&\ a\varepsilon G$. In general are safe all the infinitary relations R_{Pos} such that there exist J and D such that

$$R_{\mathsf{Pos}}(a, F) \text{ if and only if } (\forall j \in J(a))\ F \between D(a, j)$$

In this case, we can generate the positivity predicate by using the rules (2_{Pos}) of the previous section that we can write as follows

$$(\text{anti-reflexivity}) \quad \frac{\mathsf{Pos}(a, F)}{a\varepsilon F}$$

$$(\mathsf{Pos}\text{-infinity}) \quad \frac{\mathsf{Pos}(a, G) \qquad x\varepsilon F\ [\mathsf{Pos}(x, G)]}{R_{\mathsf{Pos}}(a, F)}$$

Also in this case we can simplify a bit Pos-*infinity*. In fact $R_{\mathsf{Pos}}(a, F)$ means $(\forall j \in J(a))\ F \between D(a, j)$ and thus we can re-write Pos-*infinity* as follows

$$(\mathsf{Pos}\text{-infinity})\ \frac{\mathsf{Pos}(a, G) \qquad x \varepsilon F\ [\mathsf{Pos}(x, G)]}{(\forall j \in J(a))\ F \between D(a, j)}$$

and then it is obvious that we can avoid the universal quantification in the conclusion if we write

$$(\mathsf{Pos}\text{-infinity})\ \frac{\mathsf{Pos}(a, G) \qquad j \in J(a) \qquad x \varepsilon F\ [\mathsf{Pos}(x, G)]}{F \between D(a, j)}$$

Now, we can simplify it a bit more by realizing that a single instance of this rule is sufficient to obtain its full strength. In fact let us consider the case $F \equiv \mathsf{Pos}(G)$. Then the assumption $x \varepsilon F\ [\mathsf{Pos}(x, G)]$ is trivially satisfied and we obtain the simpler

$$(\mathsf{Pos}\text{-infinity})\ \frac{\mathsf{Pos}(a, G) \qquad j \in J(a)}{\mathsf{Pos}(G) \between D(a, j)}$$

which is sufficient to obtain the previous version of the rule because, assuming $\mathsf{Pos}(G) \subseteq F$, if $\mathsf{Pos}(G) \between D(a, j)$ then $F \between D(a, j)$.

This new form of Pos-*infinity* gives some suggestions on the intended meaning of the axioms we introduced. Since the positivity predicate is going to be generated by co-induction, that is, we have elimination rules for it and only out of them we can obtain an introduction rule, at any step of the generation process what we can know is just whether some element a of S is *not* positive with some subset F of S, that is, one of the elimination rule for Pos yields to a contradiction when used on $\mathsf{Pos}(a, F)$. Thus, we have to expect that also the axioms are going to give us this kind of information. And indeed this is the information that we reach by using Pos-*infinity*: let a be any element of S and let us suppose that there is some $j \in J(a)$ such that $\neg(F \between D(a, j))$, then $\neg\mathsf{Pos}(a, F)$. Thus, we can use the axioms in order to exclude that the positivity predicate holds for some element and some subset. For instance, whatever are the chosen axioms, $\mathsf{Pos}(a, \emptyset)$ will never hold, but we can also state that an element a is not positive, that is, $\neg\mathsf{Pos}(a, S)$ holds, by stating for instance that, for some $j \in J(a)$, the subset $D(a, j)$ is empty.

Let us analyze the effects of the modifications that we made on Pos-*infinity* on the *maximality* rule, that is, on co-induction on Pos. The rule of co-induction is

$$\frac{Q \subseteq F \qquad Q \subseteq R_{\mathsf{Pos}}(H)\ [Q \subseteq H]}{Q \subseteq \mathsf{Pos}(F)}$$

It is clear that it uses an implicit universal quantification on all subsets, that is, the subset H appears as a free parameter in the minor premise. Due to the new form of Pos-*infinity*, we can modify *maximality* into

$$\frac{Q \subseteq F \qquad Q \between D(x,j)\ [x \varepsilon Q, j : J(x)]}{Q \subseteq \mathsf{Pos}(F)}$$

or, equivalently,

$$\text{(maximality)} \qquad \frac{a \varepsilon Q \qquad Q \subseteq F \qquad Q \between D(x,j)\ [x \varepsilon Q, j : J(x)]}{\mathsf{Pos}(a,F)}$$

and in this way the universal quantification on the collection of all the subsets of S has been transformed into a universal quantification on the elements of the set $J(x)$ for $x \varepsilon Q$.

Now, we will prove that this rule is sufficient, together with the elimination rules for Pos, to show that Pos is a positivity predicate which satisfies the axioms.

First we give the proof of validity of the new elimination rules that we are going to use. As usual there is nothing to prove as regard to *anti-reflexivity* while the following is the proof of the validity of Pos-*infinity*.

$$\frac{\dfrac{\dfrac{\dfrac{\mathsf{Pos}(a,G) \quad \mathsf{Pos}(G) \subseteq \mathsf{Pos}(G)}{\mathsf{Pos}(a,\mathsf{Pos}(G))}\ \mathsf{Pos}\text{-transitivity}}{R_{\mathsf{Pos}}(a,\mathsf{Pos}(G))}\ \text{axiom}}{(\forall j \in J(a))\ \mathsf{Pos}(G) \between D(a,j)} \qquad j \in J(a)}{\mathsf{Pos}(G) \between D(a,j)}$$

Now let us prove that the previous rules are sufficient. Nothing has to be proved for *anti-reflexivity*. Pos-*transitivity* is proved by co-induction, by putting $Q \equiv \mathsf{Pos}(G)$ in the *maximality* rule.

$$\frac{\dfrac{\mathsf{Pos}(a,G)}{a \varepsilon \mathsf{Pos}(G)} \quad \mathsf{Pos}(G) \subseteq F \quad \dfrac{\dfrac{[x \varepsilon \mathsf{Pos}(G)]_1}{\mathsf{Pos}(x,G)} \quad [j \in J(x)]_1}{\mathsf{Pos}(G) \between D(x,j)}\ \mathsf{Pos}\text{-inf}}{\mathsf{Pos}(a,F)}\ 1 \quad \text{maximality}$$

Finally, closure under axioms, that is $\mathsf{Pos}(a, G) \to R_{\mathsf{Pos}}(a, G)$, is proved as follows by using Pos-*infinity*

$$\dfrac{\dfrac{\dfrac{\mathsf{Pos}(a,G) \quad [j \in J(a)]_1}{\mathsf{Pos}(G) \between D(a,j)}\,\mathsf{Pos}\text{-inf} \qquad \dfrac{\text{anti-ref}}{\mathsf{Pos}(G) \subseteq G}}{\dfrac{G \between D(a,j)}{(\forall j \in J(a))\; G \between D(a,j)}\,1}}{R_{\mathsf{Pos}}(a,G)}$$

2.3 Inductive generation of formal topologies with a binary positivity predicate

In this section we will show how to generate inductively a cover relation which satisfies some given axioms, *reflexivity*, $\lhd$-*transitivity* and $\downarrow$-*right* and a positivity predicate which is compatible with this cover relation and which satisfies some given axioms. Thus, we are going to give a complete solution to the problem of a predicative inductive generation of formal topologies with a binary positivity predicate.

As we observed in the beginning of section 2, the definition of the operation $\downarrow$ among subsets depends on the covers and it requires the cover to be known. However, a crucial observation is that only the trace of the cover on elements is sufficient. The idea is then to separate covers between elements, that is $a \lhd \{b\}$, from those $a \lhd U$ with an arbitrary subset U on the right, so that we can block the former, require $\downarrow$-*right* on it and then generate the latter. So, we must add, to those of a formal topology, an extra primitive expressing what in the concrete case is $\mathsf{ext}(a) \subseteq \mathsf{ext}(b)$. We can obtain this by adding a pre-order relation $a \leq b$ among names. Thus we obtain the following definition.

Definition 4 *A* formal topology with pre-order*, shortly a* $\leq$-formal topology*, is a quadruple* $(S, \leq, \lhd, \mathsf{Pos})$ *where* S *is a set,* $\leq$ *is a pre-order relation over* S*, that is,* $\leq$ *is reflexive and transitive,* $\lhd$ *is a relation between elements and subsets of* S *which satisfies* reflexivity*,* $\lhd$-transitivity *and the two following conditions*

$$(\leq\text{-left})\quad \frac{a \leq b \qquad b \lhd U}{a \lhd U} \qquad\qquad (\leq\text{-right})\quad \frac{a \lhd U \qquad a \lhd V}{a \lhd \downarrow U \cap \downarrow V}$$

where $\downarrow U \equiv \{c : S |\ (\exists u \varepsilon U)\ c \leq u\}$*, and* Pos *is a binary positivity predicate with respect to* $\lhd$*, that is, it satisfies* anti-reflexivity*,* Pos-transitivity*,* compatibility *and the following condition:*

$$(\leq\text{-monotonicity})\qquad \frac{a \leq b \qquad \mathsf{Pos}(a, F)}{\mathsf{Pos}(b, F)}$$

It is straightforward to verify that all the new conditions are valid in any concrete topological space under the intended interpretation.

The condition $\leq$-*left* is clearly equivalent to the fact that $\leq$ respects $\lhd$, that is

$$\frac{a \leq b}{a \lhd \{b\}}$$

Since $\leq$ respects $\lhd$, for any subset U we have $\downarrow U \subseteq \downarrow^{\lhd} U$, where $\downarrow^{\lhd} U$ is a shorthand for the subset $\{c : S|\ (\exists u \varepsilon U)\ c \lhd \{u\}\}$.

Thus $\downarrow U \cap \downarrow V \subseteq \downarrow^{\lhd} U \cap \downarrow^{\lhd} V \equiv U \downarrow V$, and hence $\leq$-*right* implies $\downarrow$-*right*. So any $\leq$-formal topology is a formal topology. The converse is trivial: given any formal topology $(S, \lhd, \mathsf{Pos})$, all we need to do is to define

$$a \leq b \equiv a \lhd \{b\}$$

and we obtain a $\leq$-formal topology with a cover and a binary positivity predicate coinciding with the original ones.

Now, let us suppose that $I(a)$ set $[a : S]$ and $C(a, i) \subseteq S$ $[a : S, i : I(a)]$ is an axiom-set for the cover relation. Then, we can generate a cover relation $\lhd$ by using the following introduction rules:

$$\text{(reflexivity)}\ \frac{a \varepsilon U}{a \lhd U}$$

$$(\leq\text{-left})\ \frac{a \leq b \qquad b \lhd U}{a \lhd U}$$

$$(\leq\text{-infinity})\ \frac{a \leq b \qquad i \in I(b) \qquad \downarrow C(b,i) \cap \downarrow\{a\} \lhd U}{a \lhd U}$$

The proof that $\leq$-*infinity* is valid in any $\leq$-formal topology has been already given in [CSSV] and thus we will not repeat it here. Since these rules are acceptable for an inductive generation of a proposition $\lhd$ we can assume also the following elimination rule, namely *minimality*, that is, *induction on* $\lhd$:

$$\frac{a \lhd U \quad U \subseteq P \quad \begin{array}{c}[x \leq y, y \varepsilon P]_1 \\ \vdots \\ x \varepsilon P\end{array} \quad \begin{array}{c}[x \leq y, i \in I(y), \downarrow C(y,i) \cap \downarrow\{x\} \subseteq P]_1 \\ \vdots \\ x \varepsilon P\end{array}}{a \varepsilon P}\ 1$$

Hence closure under $\lhd$-*transitivity* and $\leq$-*right* can be proved by induction and the proof of the axioms, that is, for any $i \in I(a)$, $a \lhd C(a, i)$, is immediate (see [CSSV]).

Now, supposing that $J(a)$ set $[a : S]$ and $D(a, j) \subseteq S$ $[a : S, j : J(a)]$ is an axiom-set for a positivity predicate, we can generate a positivity predicate Pos by using the following elimination rules

$$\text{(anti-reflexivity)} \quad \frac{\mathsf{Pos}(a,G)}{a\varepsilon G}$$

$$\text{($\leq$-monotonicity)} \quad \frac{a \leq b \qquad \mathsf{Pos}(a,F)}{\mathsf{Pos}(b,F)}$$

$$\text{(Pos-infinity)} \quad \frac{\mathsf{Pos}(a,G) \qquad j \in J(a)}{\mathsf{Pos}(G) \between D(a,j)}$$

$$\text{(compatibility on axioms)} \quad \frac{\mathsf{Pos}(a,G) \qquad a \leq b \qquad i \in I(b)}{\downarrow C(b,i) \cap \downarrow\{a\} \between \mathsf{Pos}(G)}$$

Anti-reflexivity and Pos-*infinity* are the same rules that we analyzed in the previous section while $\leq$-*monotonicity* and *compatibility on axioms* are the novelty of this section. It is immediate to prove that *compatibility on axioms* is valid in any $\leq$-formal topology by using $\leq$-*infinity* and *compatibility*. Since the previous elimination rules are acceptable for a predicative generation of an infinitary relation also the following co-induction *maximality* rule

$$\dfrac{a\varepsilon Q \quad Q \subseteq F \quad \begin{array}{c}[x \leq y, x\varepsilon Q]_1\\ \vdots \\ y\varepsilon Q\end{array} \quad \begin{array}{c}[x\varepsilon Q, j : J(x)]_1\\ \vdots \\ Q \between D(x,j)\end{array} \quad \begin{array}{c}[x\varepsilon Q, x \leq y, i : I(y)]_1\\ \vdots \\ \downarrow C(y,i) \cap \downarrow\{x\} \between Q\end{array}}{\mathsf{Pos}(a,F)}\ 1$$

can be used. The novelties with respect to the *maximality* condition that we used in the previous section are the two new minor premises which correspond to the rules of $\leq$-*monotonicity* and *compatibility on axioms*.

So, Pos is a binary infinitary relation which satisfies *anti-reflexivity* by definition, and Pos-*transitivity* because of the following proof by *maximality* obtained by adapting to this new setting the analogous proofs in the previous section.

$$\dfrac{\dfrac{\mathsf{Pos}(a,G)}{a\varepsilon\mathsf{Pos}(G)} \quad \mathsf{Pos}(G) \subseteq F \quad \dfrac{\dfrac{[x \leq y]_1 \quad \dfrac{[x\varepsilon\mathsf{Pos}(G)]_1}{\mathsf{Pos}(x,G)}}{\mathsf{Pos}(y,G)}}{y\varepsilon\mathsf{Pos}(G)} \quad \dfrac{\dfrac{[x\varepsilon\mathsf{Pos}(G)]_1}{\mathsf{Pos}(x,G)} \quad [j \in J(x)]_1}{\mathsf{Pos}(G) \between D(x,j)} \quad \dfrac{[x\varepsilon\mathsf{Pos}(G), x \leq y, i : I(y)]_1}{\downarrow C(y,i) \cap \downarrow\{x\} \between \mathsf{Pos}(G)}}{\mathsf{Pos}(a,F)}\ 1$$

Also closure under axioms, that is, $\mathsf{Pos}(a,G) \rightarrow R_{\mathsf{Pos}}(a,G)$, is proved as in the previous section by using Pos-*infinity*.

Finally, *compatibility* must be proved. Thus, let us suppose that $\mathsf{Pos}(a,F)$ and $a \lhd U$; then we must prove that $(\exists c\varepsilon U)\ \mathsf{Pos}(c,F)$

and this can be done by induction on the derivation of $a \lhd U$. In fact, if $a \lhd U$ was derived from $a \varepsilon U$ then obviously $(\exists c \varepsilon U)\ \mathsf{Pos}(c, F)$ holds, and if $a \lhd U$ was obtained from $a \leq b$ and $b \lhd U$ by using $\leq$-*left* then $\mathsf{Pos}(a, F)$ yields $\mathsf{Pos}(b, F)$ by $\leq$-*monotonicity* and hence $(\exists c \varepsilon U)\ \mathsf{Pos}(c, F)$ holds by inductive hypothesis. Finally if $a \lhd U$ was obtained from $a \leq b$, $i \in I(b)$ and $x \lhd U\ [x \varepsilon \downarrow C(b,i) \cap \downarrow \{a\}]$ by using $\leq$-*infinity*, then from $\mathsf{Pos}(a, F)$, $a \leq b$ and $i \in I(b)$ we get $(\exists x \varepsilon \downarrow C(b,i) \cap \downarrow \{a\})\ \mathsf{Pos}(x, F)$, by *compatibility on axioms*, and hence $(\exists c \varepsilon U)\ \mathsf{Pos}(c, F)$ by $\exists$-elimination and inductive hypothesis.

3 Conclusion

It is worth observing that inductive generation of a formal topology according to definition 3 makes even more clear that the lack of a complete formalization of the concrete topological spaces produces some unpleasent effects. In fact, the axioms for the positivity predicate are completely independent with respect to the axioms for the cover relation, and the only rule which links the positivity predicate and the cover relation, namely *compatibility on axioms*, depends *only* on the axioms on the cover relation. So, even if the axioms for the cover relation are fixed, one is still completely free in choosing the axioms for the positivity predicate and he is in any case going to obtain a formal topology with such a cover relation and a compatible positivity predicate.

One simple way out is of course to give up with the present definition of formal topology and say that a formal topology is any mathematical structure obtained by inductively generating a cover relation with a specific axiom-set and a positivity predicate with no axioms, as Martin-Löf seems to suggest, so that only one positivity predicate is associate with any cover relation. But in this way interesting mathematical structures can get lost.

A much more interesting approach is to think that a more deep connection between formal open and formal closed subsets is still waiting to be discovered and that the present definition of formal topology is still missing some important aspects of the concrete topological spaces that we should find and make explicit.

References

[CSSV] Coquand, T., Sambin, G., Smith, J. and Valentini, S., *Inductive generation of formal topologies*, to appear.

[Eng77] Engelking, R., *General Topology*, Polish Scientific Publisher, Warszawa, 1977.

[GS99] Gebellato, S., Sambin, G., *A preview of the basic picture: a new perspective on formal topology*, proceedings of the Annual Meeting of the Types Working Group "Types '98", T. Altenkirch and W. Naraschewski and B. Reus (eds.), Springer Lecture Notes in Computer Science, 1999, pp. 194-207.

[ML84] Martin-Löf, P., *Intuitionistic Type Theory, notes by G. Sambin of a series of lectures given in Padua*, Bibliopolis, Naples, 1984

[SV98] G. Sambin, S. Valentini, *Building up a tool-box for Martin-Löf intuitionistic type theory*, in "Twenty-five years of Constructive Type Theory", G. Sambin and J. Smith (eds.), Oxford logic guides 36, 1998, pp. 221-244.