A note on the Identity Criteria

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Abstract

In [G???] the notion of *identity criteria* is introduced in order to distinguish between predicates which carry an identity criteria and predicates which do not.

We will show here that, under some very usual assumptions, every predicate carries an identity criteria and we will characterize its general shape.

1 Every predicate carries an identity criteria

A predicate ψ, that is, a property over elements of some domain, carries an identity criteria ρ if and only if the following formula holds:

$$\forall x \forall y.\ \psi(x)\ \&\ \psi(y) \rightarrow (\rho(x,y) \leftrightarrow x = y)$$

The intended meaning is that ρ behaves like an equality relation when restricted to the elements which satisfy ψ.

Note, in particular, that if ρ is an identity criteria for ψ then ρ is *reflexive over* ψ, that is, $\forall x. \psi(x) \rightarrow \rho(x,x)$. In fact, if ρ is an identity criteria for ψ then

$$\forall x.\ \psi(x)\ \&\ \psi(x) \rightarrow (\rho(x,x) \leftrightarrow x = x)$$

holds and it yields immediately the result.

This observation is essential to see that a predicate ψ carries an identity criteria ρ if and only if it carries a *reflexive* identity criteria ρ^*, where we mean that a binary predicate η is reflexive if, for any element x in the domain, $\eta(x,x)$ holds. In fact, it is obvious that if ψ carries a reflexive identity criteria then it also carries an identity criteria; on the other hand, supposing ρ is any identity criteria for ψ we can define the binary predicate ρ^* by setting

$$\rho^*(x,y) \equiv \psi(x) \rightarrow \rho(x,y)$$

and it is immediate to prove that ρ^* is both a reflexive predicate and an identity criteria for ψ. In fact, $\rho^*(x,x)$ is $\psi(x) \rightarrow \rho(x,x)$ and hence it holds as a consequence of the above observation. Moreover, supposing $\psi(x)$ & $\psi(y)$ and $\rho^*(x,y)$, that is, $\psi(x) \rightarrow \rho(x,y)$, we immediately obtain $\rho(x,y)$ and hence $x = y$ from $\rho(x,y)$ and $\psi(x)$ & $\psi(y)$ since ρ is an identity criteria for ψ. On the other hand, supposing $\psi(x)$ & $\psi(y)$ and $x = y$, we obtain that $\rho(x,y)$ holds because ρ is an identity criteria for ψ and hence trivially $\psi(x) \rightarrow \rho(x,y)$. Thus, from now on we will consider only reflexive identity criteria.

The first consequence of this fact is the following simplification in the definition of identity criteria. Since the equality predicate satisfies the Leibniz rule, that is,

$$\frac{x = y \qquad \phi(x)}{\phi(y)}$$

the above characterization for an identity criteria is equivalent to the following:

$$\forall x \forall y.\ \psi(x)\ \&\ \psi(y) \rightarrow (\rho(x,y) \rightarrow x = y)$$

since the omitted implication is an immediate consequence of the reflexivity of ρ because $x = y$ and $\rho(x,x)$ yield $\rho(x,y)$ by the Leibniz rule.

We are now ready to show that if the underlying logical system includes first order classical logic, then all of the formulas definable in the logical language carry a reflexive identity criteria. Of course this result is obvious since clearly the equality predicate is an identity criteria for any predicate ψ. Our improvement here is that we will show the most general shape for an identity criteria for ψ.

Let ψ be an arbitrary formula in the logical language considered, $\phi(x,y)$ be any reflexive formula and consider the class of formulas $\rho_{\psi,\phi}$ defined by setting:

$$\rho_{\psi,\phi}(x,y) \equiv (\psi(x)\ \&\ \psi(y) \rightarrow x = y)\ \&\ (\neg(\psi(x)\ \&\ \psi(y)) \rightarrow \phi(x,y))$$

Then we can prove the following lemma

Lemma 1.1 *For any reflexive formula ϕ with two free variables, $\rho_{\psi,\phi}$ is a reflexive predicate which satisfies the condition for an identity criteria for ψ.*

Proof. Let ϕ be an arbitrary reflexive formula with two free variables. Then it is immediate to check that $\rho_{\psi,\phi}$ is reflexive. In fact $\rho_{\psi,\phi}(x,x)$ is $(\psi(x)\ \&\ \psi(x) \rightarrow x = x)\ \&\ (\neg(\psi(x)\ \&\ \psi(x)) \rightarrow \phi(x,x))$ and hence the first conjunct holds because the equality predicate is reflexive and the second one because ϕ is reflexive by hypothesis.

We will prove now that

$$\forall x \forall y.\ (\psi(x)\ \&\ \psi(y)) \rightarrow (\rho_{\psi,\phi}(x,y) \rightarrow x = y)$$

Assume $\psi(x)$ & $\psi(y)$ and $\rho_{\psi,\phi}(x,y)$; the latter means $(\psi(x)\ \&\ \psi(y) \rightarrow x = y)$ and $(\neg(\psi(x)\ \&\ \psi(y)) \rightarrow \phi(x,y))$ and hence the former yields $x = y$. ■

Lemma 1.2 *For any reflexive ρ which satisfies the condition for an identity criteria for ψ there exists a reflexive formula ϕ, such that ρ is logically equivalent to $\rho_{\psi,\phi}$, that is,*

$$\forall x \forall y.\ (\rho(x,y) \leftrightarrow \rho_{\psi,\phi}(x,y))$$

Proof. Let ρ satisfy the condition for an identity criteria for ψ. We will show that

$$\forall x \forall y. (\rho(x,y) \leftrightarrow \rho_{\psi,\rho}(x,y))$$

Let us assume that

$$\rho(x,y) \qquad (1)$$

holds. We have to prove both

$$\psi(x)\ \&\ \psi(y) \rightarrow x = y$$

and

$$\neg(\psi(x)\ \&\ \psi(y)) \rightarrow \rho(x,y)$$

Assume now $\psi(x)\ \&\ \psi(y)$. From (1) and the fact that ρ satisfies the condition for an identity criteria we immediately obtain $x = y$ and thus

$$\psi(x)\ \&\ \psi(y) \rightarrow x = y \qquad (2)$$

holds. Assume now $\neg(\psi(x)\ \&\ \psi(y))$. Since (1) is already assumed, we trivially obtain:

$$\neg(\psi(x)\ \&\ \psi(y)) \rightarrow \rho(x,y),$$

which together with (2) yields:

$$(\psi(x)\ \&\ \psi(y) \rightarrow x = y)\ \&\ (\neg(\psi(x)\ \&\ \psi(y)) \rightarrow \rho(x,y))$$

that is, $\rho_{\psi,\rho}(x,y)$.

To prove the other implication let us assume that

$$(\psi(x)\ \&\ \psi(y) \rightarrow x = y)\ \&\ (\neg(\psi(x)\ \&\ \psi(y)) \rightarrow \rho(x,y)) \qquad (3)$$

We will show that $\rho(x,y)$ holds. Assume $\psi(x)\ \&\ \psi(y)$. Then from (3) it follows that $x = y$ and thus $\rho(x,y)$ since the predicate ρ is reflexive. Consequently:

$$\psi(x)\ \&\ \psi(y) \rightarrow \rho(x,y) \qquad (4)$$

But, from (3) it follows immediately $\neg(\psi(x)\ \&\ \psi(y)) \rightarrow \rho(x,y))$. Thus

$$((\psi(x)\ \&\ \psi(y)) \vee \neg(\psi(x)\ \&\ \psi(y))) \rightarrow \rho(x,y)$$

holds and hence we obtain that $\rho(x,y)$ holds since we are supposing to work within classical logic. ■

Thus we proved that any predicate ψ carries an identity criteria and that any possible identity criteria is logically equivalent to a formula $\rho_{\psi,\phi}$ for some reflexive predicate ϕ.

2 Formalization problem

First order classical logic cannot distinguish between predicates which are logically equivalent since this logical system is sensible only to the truth value of a proposition.

On the other hand the formal proof above does not work if intuitionistic logic is used - but it is well possible that therein other tricks can be created.

We think that there is a need for a formal system which can define finer distinctions between predicates, than first order logic does.

References

[G???] Guarino, N. ?????