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Topological Characterization of Scott Domains

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Abstract First we introduce the notion of super-coherent topology which does not depend on any ordering. Then we show that a topology is super-coherent if and only if it is the Scott topology over a suitable algebraic dcpo.

The main ideas of the paper are a by-product of the constructive approach to domain theory through information bases which we have proposed in a previous work, but the presentation here does not rely on that foundational framework.

Key words Scott Domain – Algebraic dcpo – Formal Topology

1 Introduction

The notion of Scott topology is a well established tool in theoretical computer science. The definitions or characterizations given in the literature (see for instance [1] and [2]) always assume the universe set to be equipped with a partial ordering. The aim of this short note is to point out a purely topological characterization of Scott topologies over algebraic dcpo's, independently of any ordering.

We briefly recall in this section the well known basic facts about domain theory which we are going to use.

Let $\mathcal{D} = \langle D, \leq \rangle$ be a partially ordered set. A subset U of D is called *directed* if U is inhabited and $(\forall u, v \in U)(\exists w \in U)\ (u \leq w \,\&\, v \leq w)$. A partially ordered set $\mathcal{D}$ is called *directed-complete* (briefly *dcpo*) if every directed subset has supremum. An element a of a dcpo $\mathcal{D}$ is called *compact* if, for any directed subset U of D, $a \leq \bigvee U$ implies that $(\exists u \in U)\ a \leq u$. Note that, whenever it exists, the supremum of any finite family of compact elements is compact. We will write K_D for the set of compact elements of D and we will reserve a, b, c, … to denote its elements, while we keep x, y, z, … for generic elements of D. A dcpo $\mathcal{D}$ is called *algebraic* if, for every $x \in D$, the set $\downarrow_K x \equiv \{a \in K_D |\ a \leq x\}$ is directed and $x = \bigvee \downarrow_K x$. In any algebraic dcpo $\mathcal{D}$ not only the elements but also their ordering relation can be recovered from the structure of K_D. In fact, it is routine to prove that, for any $x, y \in D$, $x \leq y$ if and only if $\downarrow_K x \subseteq \downarrow_K y$, that is $(\forall a \in K_D)\ (a \leq x\ \rightarrow\ a \leq y)$.

Let us now recall the definition of Scott topology on a dcpo. In any dcpo $\mathcal{D}$, $O \subseteq D$ is called *Scott open* if it is hereditary, that is if $x \in O$ and $x \leq y$ then $y \in O$, and it splits directed suprema, that is, for each directed subset U, if $\bigvee U \in O$ then $(\exists u \in U)\ u \in O$. It is easy to check that Scott opens of any dcpo $\langle D, \leq \rangle$ form a topology $\tau_\leq$, which is called the *Scott topology* on $\mathcal{D}$.

Any algebraic dcpo $\mathcal{D}$ is completely determined by its Scott topology since the order relation can be completely recovered because $x \leq y$ if and only if $(\forall O \in \tau_\leq)\ (x \in O \rightarrow y \in O)$. In fact, from left to right the result is an obvious consequence of hereditarity. To prove the other implication, one should note that, for any $a \in K_D$, the subset $\uparrow a \equiv \{x \in D |\ a \leq x\}$ is a Scott open and hence the assumption

$$(\forall O \in \tau_\leq)\ (x \in O \rightarrow y \in O)$$

yields

$$(\forall a \in K_D)\ (x \in \uparrow a \rightarrow y \in \uparrow a),$$

that is,

$$(\forall a \in K_D)\ (a \leq x \rightarrow a \leq y),$$

and we already observed that for any algebraic dcpo this is equivalent to $x \leq y$.

An immediate consequence of this observation is that, for any algebraic dcpo $\mathcal{D}$, the topology $\tau_\leq$ is T_0, that is, if $(\forall O \in \tau_\leq)\ (x \in O\ \leftrightarrow\ y \in O)$ then $x = y$.

The family $\uparrow K_D \equiv \{\uparrow a |\ a \in K_D\}$ is a base for the topology $\tau_\leq$ because, for any $x \in D$ there exists a compact element a such that $a \leq x$, that is $x \in \uparrow a$, and, supposing $x \in \uparrow a$ and $x \in \uparrow b$, that is $a \in \downarrow_K x$ and $b \in \downarrow_K x$, there exist an element $c \in \downarrow_K x$ such that $a \leq c$ and $b \leq c$, that is $\uparrow c \subseteq \uparrow a \cap \uparrow b$, since $\downarrow_K x$ is directed. Finally, supposing $O \subseteq D$ is any Scott open, $O = \bigcup_{a \in O} \uparrow a$. Moreover such a base has the interesting property that for any subset U of K_D and for any $a \in K_D$, $\uparrow a \subseteq \bigcup_{b \in U} \uparrow b$ if and only if $(\exists b \in U)\ \uparrow a \subseteq \uparrow b$. This is a very strong compactness property: a basic element is covered by a family of basic elements if and only if it is covered by *exactly* one of them. The proof is almost immediate: if $\uparrow a \subseteq \bigcup_{b\in U} \uparrow b$ then $a \in \bigcup_{b\in U} \uparrow b$ and hence $(\exists b \in U)\ a \in \uparrow b$ which gives $(\exists b \in U) \uparrow a \subseteq \uparrow b$; the other implication is straightforward.

Given a topological space (X, τ) and any base $\mathcal{B}_\tau$ for the topology τ, we can define the set $\mathsf{Pt}(\mathcal{B}_\tau)$ of *formal points* of the topology τ [3,4]. Its elements are the non-empty subsets α of $\mathcal{B}_\tau$ such that:

$$\emptyset \notin \alpha \qquad \frac{U \in \alpha \qquad V \in \alpha}{(\exists W \in \mathcal{B}_\tau)\ W \in \alpha \ \&\ W \subseteq U \cap V} \qquad \frac{U \in \alpha \quad U \subseteq \bigcup_{i\in I} V_i}{(\exists i \in I)\ V_i \in \alpha}$$

The *canonical* map $\phi : X \to \mathsf{Pt}(\mathcal{B}_\tau)$ is defined by putting

$$\phi(x) = \{U \in \mathcal{B}_\tau \mid x \in U\}$$

It is straightforward to show that, for any point $x \in X$, the set

$$\{U \in \mathcal{B}_\tau \mid x \in U\}$$

is indeed a formal point.

Moreover, if the topology τ is T_0 then the map ϕ is injective, since $\phi(x) = \phi(y)$, i.e. $(\forall U \in \mathcal{B}_\tau)\ (x \in U \ \leftrightarrow\ y \in U)$, yields $x = y$.

If the map ϕ is also surjective the topology τ is said *sober* since in this case no new formal point is added in $\mathsf{Pt}(\mathcal{B}_\tau)$ which "is" not already a point in X.

It is possible to show that, for any algebraic dcpo $\langle D, \leq\rangle$, the Scott topology $\tau_\leq$ with base $\uparrow K_D$ is sober. In this case formal points are the non-empty subsets α of $\uparrow K_D$ such that

$$\frac{\uparrow a \in \alpha \qquad \uparrow b \in \alpha}{(\exists c \in K_D) \uparrow c \in \alpha \ \&\ a \leq c \ \&\ b \leq c} \qquad \frac{\uparrow a \in \alpha \quad b \leq a}{\uparrow b \in \alpha}$$

In fact, the first condition on formal points is not necessary here since all the elements of $\uparrow K_D$ are not empty. Moreover, for any $a, b \in K_D$, $\uparrow a \subseteq \uparrow b$ if and only if $b \leq a$ and hence the first condition here is just a re-writing of the second condition on the formal points of a generic topological space while the third condition is here substituted by a simpler one because of the strong compactness property of the base $\uparrow K_D$. Observe that, for any formal point α, the subset $U_\alpha \equiv \{a \in K_D \mid \uparrow a \in \alpha\}$ is directed, because if $a, b \in U_\alpha$, that is $\uparrow a, \uparrow b \in \alpha$, then there exists $c \in K_D$ such that $\uparrow c \in \alpha$, that is $c \in U_\alpha$, and $a \leq c$ and $b \leq c$. Hence $\bigvee_{\uparrow a \in \alpha} a$ exists. Now we can prove that the map $\phi : D \to \mathsf{Pt}(\uparrow K_D)$ is surjective by showing that, for any formal point α, $\alpha = \phi(\bigvee U_\alpha)$. In fact, supposing $\uparrow a \in \phi(\bigvee U_\alpha)$, that is $a \leq \bigvee_{\uparrow b \in \alpha} b$, we obtain that $(\exists b \in K_D) \uparrow b \in \alpha \ \&\ a \leq b$, since a is compact and U_α is directed, and hence $(\exists b \in K_D) \uparrow b \in \alpha \ \&\ \uparrow b \subseteq \uparrow a$, which shows that $\uparrow a \in \alpha$, since α is a formal point; the other inclusion is immediate.

2 Super-coherent topologies

The properties of the Scott topology over an algebraic dcpo suggest the following definition (cf. the definition of coherent topology in [3]):

Definition 1 (Super-compact open set and super-coherent topology) *Let (X, τ) be a topological space over the set X. Then an open set U is called* super-compact *if, for any family of open subsets $(V_i)_{i \in I}$, if $U \subseteq \bigcup_{i \in I} V_i$ then $(\exists i \in I)\ U \subseteq V_i$ and τ is called* super-coherent *if it is sober and has a base of super-compact opens.*

We have shown that, for any algebraic dcpo $\mathcal{D}$, the corresponding Scott topology is super-coherent. The main result of this paper is that the converse holds:

Theorem 1 *Any super-coherent topological space (X, τ) coincides with the Scott topology of a suitable algebraic dcpo over X.*

The idea of the proof is that τ coincides with the Scott topology $\tau_{\sqsubseteq_\tau}$ induced by the well known *specialization ordering* over X, defined, for any $x, y \in X$, by

$$x \sqsubseteq_\tau y \equiv (\forall O \in \tau)\ (x \in O \ \rightarrow \ y \in O)$$

It is convenient to split the proof into some steps.

It is straightforward to see that $\sqsubseteq_\tau$ is a pre-order relation, i.e. it is reflexive and transitive. To obtain an order relation, i.e. anti-symmetry, it is then sufficient that τ is T_0, which holds for any super-coherent topological space.

Lemma 1 *Let (X, τ) be a super-coherent topological space with base of super-compact opens $\mathcal{B}_\tau$. Then $\langle \mathsf{Pt}(\mathcal{B}_\tau), \subseteq \rangle$ is an algebraic dcpo.*

Proof We first show that $\langle \mathsf{Pt}(\mathcal{B}_\tau), \subseteq \rangle$ is a dcpo, then we will define its compact elements and finally we will prove that it is algebraic. Suppose $\{\alpha_i \mid i \in I\}$ is a directed family of formal points, then $\bigcup_{i \in I} \alpha_i$ is a formal point and it is the supremum of the family.

Now, let us write, for any $U \in \mathcal{B}_\tau$, $\Uparrow U$ for the set $\{V \in \mathcal{B}_\tau \mid U \subseteq V\}$. Then, the compact elements of $\langle \mathsf{Pt}(\mathcal{B}_\tau), \subseteq \rangle$ are the subsets $\Uparrow U$ for any not-empty $U \in \mathcal{B}_\tau$. Indeed, it is immediate to verify that $\Uparrow U$ is a formal point, i.e. an element of $\mathsf{Pt}(\mathcal{B}_\tau)$, and that, for any directed family $\{\alpha_i \mid i \in I\}$ of formal points, if $\Uparrow U \subseteq \bigcup_{i \in I} \alpha_i$ then $(\exists i \in I)\ \Uparrow U \subseteq \alpha_i$. Finally, it is easy to show that, for any formal point α, $\alpha = \bigcup_{U \in \alpha} \Uparrow U$ and that, for any compact element γ of $\mathsf{Pt}(\mathcal{B}_\tau)$, there exists $U \in \mathcal{B}_\tau$ such that $\gamma = \Uparrow U$.

Now we can prove that $\langle X, \sqsubseteq_\tau \rangle$ is an algebraic dcpo by showing that it is isomorphic to $\langle \mathsf{Pt}(\mathcal{B}_\tau), \subseteq \rangle$.

Lemma 2 *Let* (X, τ) *be a super-coherent topological space with base of super-compact opens* $\mathcal{B}_\tau$*. Then* $\langle X, \sqsubseteq_\tau\rangle$ *and* $\langle \mathsf{Pt}(\mathcal{B}_\tau), \subseteq\rangle$ *are isomorphic algebraic dcpo's.*

Proof By assumption τ is sober, and hence the map $\phi : X \to \mathsf{Pt}(\mathcal{B}_\tau)$ is a bijection between X and $\mathsf{Pt}(\mathcal{B}_\tau)$. Thus we only have to show that, for any $x, y \in X$, $x \sqsubseteq_\tau y$ if and only if $\phi(x) \subseteq \phi(y)$. But this is immediate since $x \sqsubseteq_\tau y$ if and only if $(\forall U \in \mathcal{B}_\tau)\ (x \in U \ \rightarrow\ y \in U)$ if and only if $\phi(x) \subseteq \phi(y)$.

The next step is to obtain a purely topological characterization of the compact elements of the algebraic dcpo $\langle X, \sqsubseteq_\tau\rangle$.

Lemma 3 *Let* (X, τ) *be a super-coherent topological space with base of super-compact opens* $\mathcal{B}_\tau$*. Then* $x \in X$ *is a compact element of the algebraic dcpo* $\langle X, \sqsubseteq_\tau\rangle$ *if and only if* $\bigcap\{V \in \mathcal{B}_\tau |\ x \in V\}$ *is an element of the base* $\mathcal{B}_\tau$*.*

Proof Lemma 2 shows that the two dcpo's $\langle X, \sqsubseteq_\tau\rangle$ and $\langle \mathsf{Pt}(\mathcal{B}_\tau), \subseteq\rangle$ are isomorphic because of the map $\phi : X \to \mathsf{Pt}(\mathcal{B}_\tau)$; hence an element $x \in X$ is compact in $\langle X, \sqsubseteq_\tau\rangle$ if and only if $\phi(x)$ is compact in $\langle \mathsf{Pt}(\mathcal{B}_\tau), \subseteq\rangle$, i.e. there exists $U_x \in \mathcal{B}_\tau$ such that $\phi(x) = \Uparrow U_x$, that is

$$\{V \in \mathcal{B}_\tau |\ x \in V\} = \{V \in \mathcal{B}_\tau |\ U_x \subseteq V\},$$

that is $(\forall V \in \mathcal{B}_\tau)\ (x \in V \ \leftrightarrow\ U_x \subseteq V)$. Now we will show that the last proposition is equivalent to assert that $\bigcap_{x\in V} V$ is an element of the base $\mathcal{B}_\tau$. In fact, suppose that there exists $U_x \in \mathcal{B}_\tau$ such that

$$(\forall V \in \mathcal{B}_\tau)\ (x \in V \ \leftrightarrow\ U_x \subseteq V)$$

Then

$$(\forall V \in \mathcal{B}_\tau)\ (x \in V \ \leftarrow\ U_x \subseteq V)$$

yields $x \in U_x$, and hence

$$\bigcap\{V \in \mathcal{B}_\tau |\ x \in V\} \subseteq U_x,$$

while

$$(\forall V \in \mathcal{B}_\tau)\ (x \in V \ \rightarrow\ U_x \subseteq V)$$

yields

$$U_x \subseteq \bigcap\{V \in \mathcal{B}_\tau |\ x \in V\}$$

Thus $\bigcap\{V \in \mathcal{B}_\tau |\ x \in V\}$ is equal to U_x and hence it is an element in the base $\mathcal{B}_\tau$. On the other hand if $\bigcap\{V \in \mathcal{B}_\tau |\ x \in V\}$ is a basic open set in $\mathcal{B}_\tau$ then it is the set U_x, corresponding to x, we are looking for since $(\forall V \in \mathcal{B}_\tau)\ (x \in V \ \leftrightarrow\ \bigcap\{V \in \mathcal{B}_\tau |\ x \in V\} \subseteq V)$ holds.

We can now conclude the proof that any super-coherent topological space is the Scott topology of a suitable algebraic dcpo.

Theorem 2 *Let (X, τ) be a super-coherent topological space. Then the topologies τ and $\tau_{\sqsubseteq_\tau}$ coincide.*

Proof Assume $\mathcal{B}_\tau$ is a base of super-compact opens for the topology τ. Then $\uparrow_{\sqsubseteq_\tau} K_X \equiv \{\uparrow_{\sqsubseteq_\tau} x |\ x$ compact in $\langle X, \sqsubseteq_\tau \rangle\}$ is a base for the Scott topology $\tau_{\sqsubseteq_\tau}$, where $\uparrow_{\sqsubseteq_\tau} x \equiv \{y \in X |\ x \sqsubseteq_\tau y\}$. We will now show that the bases $\mathcal{B}_\tau$ and $\uparrow_{\sqsubseteq_\tau} K_X$ coincide.

First observe that, if $x \in X$, then, for any $y \in X$, $y \in \uparrow_{\sqsubseteq_\tau} x$ if and only if $x \sqsubseteq_\tau y$ if and only if $(\forall V \in \mathcal{B}_\tau)\ (x \in V \ \rightarrow\ y \in V)$ if and only if $y \in \bigcap \{V \in \mathcal{B}_\tau |\ x \in V\}$. That is, we proved that

$$\uparrow_{\sqsubseteq_\tau} x = \bigcap \{V \in \mathcal{B}_\tau |\ x \in V\}$$

Now, let $U \in \mathcal{B}_\tau$. To prove that $U = \uparrow_{\sqsubseteq_\tau} u$ for some compact element u of the algebraic dcpo $\langle X, \sqsubseteq_\tau \rangle$ it is convenient to use again the dcpo $\langle \mathsf{Pt}(\mathcal{B}_\tau), \subseteq \rangle$ which is isomorphic to $\langle X, \sqsubseteq_\tau \rangle$. In fact, we have already seen that if $U \in \mathcal{B}_\tau$ then $\Uparrow U$ is compact in $\langle \mathsf{Pt}(\mathcal{B}_\tau), \subseteq \rangle$ and hence, by the isomorphism in lemma 2, there exists a compact element u of $\langle X, \sqsubseteq_\tau \rangle$ such that $\Uparrow U = \phi(u)$. Then

$$\{V \in \mathcal{B}_\tau |\ U \subseteq V\} \equiv \Uparrow U = \phi(u) \equiv \{V \in \mathcal{B}_\tau |\ u \in V\}$$

yields

$$\bigcap \{V \in \mathcal{B}_\tau |\ U \subseteq V\} = \bigcap \{V \in \mathcal{B}_\tau |\ u \in V\}$$

that is

$$U = \uparrow_{\sqsubseteq_\tau} u,$$

i.e. $U \in \uparrow_{\sqsubseteq_\tau} K_X$.

Conversely, let x be any compact element of the dcpo $\langle X, \sqsubseteq_\tau \rangle$. Then, by lemma 3, $\bigcap \{V \in \mathcal{B}_\tau |\ x \in V\} \in \mathcal{B}_\tau$. But we have proved above that $\bigcap \{V \in \mathcal{B}_\tau |\ x \in V\} = \uparrow_{\sqsubseteq_\tau} x$ and hence $\uparrow_{\sqsubseteq_\tau} x \in \mathcal{B}_\tau$.

3 Towards Scott Domains

Let $\mathcal{D} = \langle D, \leq \rangle$ be a partially ordered set. Then a subset U of D is called *(upper) bounded* if there exists an element x such that $(\forall u \in U)(u \leq x)$, and $\mathcal{D}$ is called *coherent* if every bounded subset of D has a supremum. A Scott domain is a coherent algebraic dcpo which has a minimum element $\perp$. It is worth noting that $\perp$ is a compact element and also that, for an algebraic dcpo the condition of being coherent can be weakened to that of binary coherence. Indeed, an algebraic dcpo is *binary coherent* if any bounded pair of compact elements has a supremum. Then, any coherent algebraic dcpo is trivially binary coherent. To prove the other implication suppose that U is a subset of D bounded by z. Consider now the subset $W = \bigcup_{u \in U} \downarrow_K u$. We can obtain a directed subset W^d out of W by adding to W the suprema of all its finite subsets, which exist and are compact since we are supposing that $\mathcal{D}$ is binary coherent and all the elements in W are smaller then z. Then the supremum of W^d exists and it is obviously equal to the supremum of U.

Now, we want to prove that not only the condition of being an algebraic dcpo can be characterized in a purely topological way but also the conditions of being binary coherent and having a least element.

First observe that, for any binary coherent algebraic dcpo $\mathcal{D}$ and for any $a, b \in K_D$, if a and b are bounded then $\uparrow a \cap \uparrow b = \uparrow(a \vee b)$. In fact, in this case $a \vee b$ exists; moreover $x \in \uparrow a \cap \uparrow b$ if and only if $a \leq x$ and $b \leq x$ if and only if $a \vee b \leq x$ if and only if $x \in \uparrow(a \vee b)$. On the other hand, if a and b are not bounded then $\uparrow a \cap \uparrow b$ is empty. Thus, if $\mathcal{D}$ is a binary coherent algebraic dcpo, then we can obtain a base closed under intersection of super-compact basic opens for its Scott topology just by adding the empty set to $\uparrow K_D$.

Also the other way around holds, that is, for any topological space (X, τ) such that the topology τ is sober and has a base closed under intersection of super-compact opens, the algebraic dcpo $\langle X, \sqsubseteq_\tau \rangle$ is binary coherent. In fact, let $\mathcal{B}_\tau$ be the considered base for the topology τ; then we can prove the result for the dcpo $\langle \mathsf{Pt}(\mathcal{B}_\tau), \subseteq \rangle$ since, after lemma 2, we know that it is isomorphic to $\langle X, \sqsubseteq_\tau \rangle$. So, suppose that $\Uparrow U$ and $\Uparrow V$ are two compact elements of $\mathsf{Pt}(\mathcal{B}_\tau)$ bounded by some point α. Then $U \in \alpha$ and $V \in \alpha$ and hence $U \cap V \in \alpha$ since $\mathcal{B}_\tau$ is supposed to be closed under intersection. Thus $U \cap V \neq \emptyset$ and hence $\Uparrow(U \cap V)$ is a compact point.

We will show now that $\Uparrow(U \cap V)$ is the suprema of the compact points $\Uparrow U$ and $\Uparrow V$, that is, for every point β, $\Uparrow(U \cap V) \subseteq \beta$ if and only if $\Uparrow U \subseteq \beta$ and $\Uparrow V \subseteq \beta$. One direction is immediate since $U \cap V \subseteq U$ and $U \cap V \subseteq V$ yield $\Uparrow U \subseteq \Uparrow(U \cap V)$ and $\Uparrow V \subseteq \Uparrow(U \cap V)$ and hence $\Uparrow(U \cap V) \subseteq \beta$ yields $\Uparrow U \subseteq \beta$ and $\Uparrow V \subseteq \beta$. To prove the other implication, note that $\Uparrow U \subseteq \beta$ and $\Uparrow V \subseteq \beta$ yield $U \in \beta$ and $V \in \beta$; hence $U \cap V \in \beta$ and thus $\Uparrow(U \cap V) \subseteq \beta$.

Finally, we want to find a topological characterization of the presence of the bottom element in an binary coherent algebraic dcpo. Here the solution is immediate: we have just to require that the base closed under intersection of super-compact elements contains also the whole set X. In fact, if $\mathcal{D}$ has a bottom element $\perp$ then it is a compact element and hence $\uparrow\perp(= D)$ is an element of the base $\uparrow K_D$. On the other hand, supposing (X, τ) is a topological space such that the topology τ is sober and has a base $\mathcal{B}_\tau$ closed under intersection of super-compact opens which also contains the whole set X, then it is immediate to verify that X is a formal point in $\mathsf{Pt}(\mathcal{B}_\tau)$ and it is clearly the bottom element in the dcpo $\langle \mathsf{Pt}(\uparrow K_D), \subseteq \rangle$ since X belongs to any other formal point.

We are thus arrived to the main theorem.

Theorem 3 *Let (X, τ) be a topological space such that the topology τ is sober and has a base closed under intersection of super-compact opens which contains the whole set X. Then $\langle X, \sqsubseteq_\tau \rangle$ is a Scott domain and any Scott domain is obtained in this way.*

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