The binary modal logic of the intersection types

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Looking for a suitable logic for the subtype relation between the types of the intersection types lambda calculus we developed a modal logic with a two-places modality. We present here its main syntactical and semantical properties, that is, the completeness theorem, the finite model property, the cut-elimination theorem and a decision procedure for theoremhood.

1. INTERSECTION TYPES

Let us quickly recall the main ideas of intersection type lambda calculus (for a recent paper on this topic see [6]). It is well known that the pure lambda calculus Λ (see [2]) formalizes the notion of computable function without any reference to the concepts of domain and co-domain, contrary to what happens in the set theoretic or the categorical approach. The main advantage of this approach is the possibility of coding any recursive function within a very simple formalism. Indeed, a lambda term is built inductively, starting from variables, by means of lambda abstraction and a *free* form of application, that is, we have the following term formation rules:

$$\mathsf{Term} := \mathsf{Var} \mid (\lambda\, \mathsf{Var}.\mathsf{Term}) \mid \mathsf{Term}(\mathsf{Term})$$

where Var is a countable set whose elements are called variables.

Not only the syntax of the objects of Λ is simple, but also the notion of computation for this very abstract formalization of computable functions becomes the simple *β-reduction* (notation $\rightsquigarrow_\beta$). This is the relation between lambda terms obtained by closing under the term construction operations the relation of β-*contraction*, that is $(\lambda x.c)(a) \rightsquigarrow c[x := a]$.

The computation of the value of a lambda term is then defined as a *reduction process*, that is, successive steps of β-reduction, until a *normal form* of the term is possibly reached, that is, a form where no β-contraction can be applied. Given a lambda term c, there are in general many different reduction processes, according to the choice of the β-contraction to be expanded within c; hence, it is well possible that only some of the reduction processes eventually terminate into a normal form. Moreover, since it is possible to have a code within Λ for any recursive function, there is no possibility to know if a reduction process for c will eventually terminate, because of the halting problem.

On the other hand, in the usual mathematical practice - both in the set theoretic and in the categorical approach - and in many concrete algorithms, functions are intended to operate over objects of a certain type in order to produce objects of some other type. Following this idea, the rule of application should be no longer completely free; in fact a function should be applicable only to arguments of the correct type. Thus it will be no longer possible to build all the terms of Λ. However, a main advantage of this approach is the possibility to prove more properties on the terms which can be built because of the greater quantity of information. For instance, one of the main problems on the terms of Λ is to determine whether all the reduction processes for a certain term will eventually terminate, that is, the *strong normalization* problem. In the case of the lambda-calculi where functions and their arguments have a type there are suitable tools to deal with this problem.

In order to keep the good aspects of both the sides, a possible strategy is to find suitable typing systems for the terms of Λ. For instance, a possibility is to use *simply typed* lambda calculus $\Lambda_{\rightarrow}$; its rules of type formation are the following:

$$\mathsf{Type} := \mathsf{BasTypes} \mid \mathsf{Type} \rightarrow \mathsf{Type}$$

were $\mathsf{BasTypes}$ is a set whose elements are called *basic types*.

The intended meaning is that a type $\sigma \rightarrow \tau$ denotes a set of functions from elements of the set denoted by the type σ into elements of the set denoted by the type τ. Thus, in order to build the elements of these types, we use the following rules[1]:

(variable) $$\Gamma, x : \sigma \vdash x : \sigma$$

(lambda abstraction) $$\frac{\Gamma, x : \sigma \vdash c : \tau}{\Gamma \vdash \lambda x.c : \sigma \rightarrow \tau}$$

(application) $$\frac{\Gamma \vdash c : \tau \rightarrow \sigma \qquad \Gamma \vdash a : \tau}{\Gamma \vdash c(a) : \sigma}$$

[1] To be more precise we should speak here of typing system *a la* Curry versus a typing system *a la* Church where all the variables within a term and the sub-terms themselves are typed.

where Γ is a commutative list of assumptions of the form $x : \sigma$ for some type σ such that no variable appears more than once.

It is well known (see for instance [8]) that all the terms of $\Lambda_{\rightarrow}$ are strongly normalizing. Hence, the terms of $\Lambda_{\rightarrow}$ form a subset of the set of strongly normalizing terms of Λ. But, not all of the strongly normalizing terms of Λ have a type in $\Lambda_{\rightarrow}$; for instance, consider the term $\lambda x.x(x)$: it is in normal form, and hence it is trivially strongly normalizing, but it cannot have a type within $\Lambda_{\rightarrow}$ because of the instance of *self-application*. It is clear that a complete solution of the strong normalization problem would be a typing system which allows to build all the strongly normalizing terms of Λ, and only them.

Surprisingly, this typing system exists and can be obtained from $\Lambda_{\rightarrow}$ by adding just one type (see [13] or [20] for a recent new proof). The abstract syntax of the types of this calculus $\Lambda_{\wedge}$ of *intersection types* is the following:

$$\mathsf{Type} := \mathsf{BasTypes} \mid \mathsf{Type} \rightarrow \mathsf{Type} \mid \mathsf{Type} \wedge \mathsf{Type}$$

The intended meaning of the new type $\sigma \wedge \tau$ of $\Lambda_{\wedge}$ is that $\sigma \wedge \tau$ denotes the intersection of the two sets denoted by the type σ and τ respectively. Thus, in order to build the elements for these new types, we add the following rules to the previous ones:

$$\text{(intersection introduction)} \qquad \frac{\Gamma \vdash c : \sigma \qquad \Gamma \vdash c : \tau}{\Gamma \vdash c : \sigma \wedge \tau}$$

$$\text{(intersection elimination)} \qquad \frac{\Gamma \vdash c : \sigma \wedge \tau}{\Gamma \vdash c : \sigma} \qquad \frac{\Gamma \vdash c : \sigma \wedge \tau}{\Gamma \vdash c : \tau}$$

An interesting problem when studying a typed lambda calculus is the problem of inhabitation of a type, namely, given a type, is it possible to know whether there is some lambda term which can be typed by such a type?

One possible strategy in order to answer to this question is to use the Curry-Howard correspondence between a typed lambda calculus and a logical system. For instance, if we strip all the lambda terms in the rules of $\Lambda_{\rightarrow}$ we obtain the rules for the implication fragment of the usual intuitionistic propositional calculus. Moreover, it is easy to see that a type of $\Lambda_{\rightarrow}$ is inhabited if and only if the corresponding propositional formula can be proved.

On the other hand, if we strip all the terms in the rules of $\Lambda_{\wedge}$, we obtain an almost standard sequent calculus PC^* for the fragment of the intuitionistic propositional logic containing only implication and conjunction. However, it is necessary to stress that the theorems of PC^* do *not* correspond to the non-empty types of $\Lambda_{\wedge}$; consider for instance the formula $(\alpha \rightarrow \alpha) \wedge (\alpha \rightarrow (\beta \rightarrow \alpha))$: it is provable in PC^* whereas this type is not inhabited in $\Lambda_{\wedge}$.

Some propositional calculi have been proposed to solve this problem (see [14] and [4]).

A step toward a better comprehension of the properties of the intersection type system $\Lambda_\wedge$ can be found in [3], where the following axioms and rules have been proposed for a subtyping relation $\leq_\wedge$ among types of $\Lambda_\wedge$:

Axioms

$$\begin{array}{lll} \alpha \leq_\wedge \omega & \omega \leq_\wedge \omega \to \omega & \\ \alpha \leq_\wedge \alpha \wedge \alpha & \alpha \wedge \beta \leq_\wedge \alpha & \alpha \wedge \beta \leq_\wedge \beta \\ (\alpha \to \beta) \wedge (\alpha \to \gamma) \leq_\wedge (\alpha \to (\beta \wedge \gamma)) & & \end{array}$$

Rules

$$\frac{\alpha \leq_\wedge \beta \qquad \beta \leq_\wedge \gamma}{\alpha \leq_\wedge \gamma} \quad \frac{\alpha_1 \leq_\wedge \beta_1 \qquad \alpha_2 \leq_\wedge \beta_2}{\alpha_1 \wedge \alpha_2 \leq_\wedge \beta_1 \wedge \beta_2} \quad \frac{\alpha_1 \leq_\wedge \alpha_2 \qquad \beta_2 \leq_\wedge \beta_1}{\alpha_2 \to \beta_2 \leq_\wedge \alpha_1 \to \beta_1}$$

Note that the new type ω has been added to the collection of types of $\Lambda_\wedge$; its purpose is to denote the set of all the strongly normalizing lambda terms.

The axioms and the rules above set natural requirements for a subtyping relation among types of $\Lambda_\wedge$; for instance the axiom

$$(\alpha \to \beta) \wedge (\alpha \to \gamma) \leq_\wedge (\alpha \to (\beta \wedge \gamma))$$

states that all the lambda terms to which can be assigned both the type $\alpha \to \beta$ and $\alpha \to \gamma$ can also be typed by $\alpha \to (\beta \wedge \gamma)$. And indeed in $\Lambda_\wedge$ if it holds $\Gamma \vdash \lambda x.c : (\alpha \to \beta) \wedge (\alpha \to \gamma)$ then also $\Gamma \vdash \lambda x.c : \alpha \to (\beta \wedge \gamma)$ can be proved. The subtyping relation $\leq_\wedge$ has been introduced with the intention to extend such a property to any term and not only to those with a particolar shape. To this aim the following assignement rule has been added to the rules of $\Lambda_\wedge$ (see [3]) to obtain a new typing system that we will call *extended intersection type system* and that we will indicate by $\vdash_{\leq_\wedge}$:

$$\frac{\Gamma \vdash_{\leq_\wedge} c : \alpha \qquad \alpha \leq_\wedge \beta}{\Gamma \vdash_{\leq_\wedge} c : \beta}$$

It allows to prove the following theorem.

THEOREM 1.1. *Suppose α and β are two types of $\Lambda_\wedge$. Then $\alpha \leq_\wedge \beta$ if and only if, for any base Γ and any lambda term t, $\Gamma \vdash_{\leq_\wedge} t : \alpha$ yields $\Gamma \vdash_{\leq_\wedge} t : \beta$.*

So, a logical system exists for the subtyping relation among types of $\Lambda_\wedge$, but it is far from enjoying a good presentation. Our purpose here is to introduce a modal calculus which gives a complete interpretation for $\leq_\wedge$ and which is nevertheless a standard logical system.

1.1. A semantics for the subtype relation

In order to disclose the correct idea for the definition of the modal logic that we are looking for, it is useful to recall the notion of filter model for the subtype relation [3].

Let M be a set and R be a three places relation over M and consider any map $\nu : \mathsf{BasTypes} \longrightarrow \mathcal{P}(M)$ from the set of basic types into the set of the subsets of M. Then we can define a forcing relation $\Vdash^\nu$ between elements of M and types by setting

$$
\begin{array}{lll}
x \Vdash^\nu \alpha & \text{iff} & x \in \nu(\alpha),\ \text{for any basic type } \alpha \\
x \Vdash^\nu \omega & \text{iff} & \mathsf{true} \\
x \Vdash^\nu \alpha \wedge \beta & \text{iff} & x \Vdash^\nu \alpha \text{ and } x \Vdash^\nu \beta \\
x \Vdash^\nu \alpha \rightarrow \beta & \text{iff} & (\forall y \in M)(\forall z \in M) R(x,y,z) \Rightarrow (y \Vdash^\nu \alpha \Rightarrow z \Vdash^\nu \beta)
\end{array}
$$

This forcing relation can be used to define an interpretation of the relation $\leq_\wedge$ in a model (M, R, ν). In fact, we can set

$$(M, R, \nu) \models \alpha \leq_\wedge \beta \text{ iff } (\forall x \in M)\ x \Vdash^\nu \alpha \Rightarrow x \Vdash^\nu \beta$$

and this interpretation can be generalized to any structure (M, R) by setting

$$
\begin{array}{ll}
(M, R) \models \alpha \leq_\wedge \beta \ \text{ iff } & (M, R, \nu) \models \alpha \leq_\wedge \beta, \\
 & \text{for any map } \nu : \mathsf{BasTypes} \longrightarrow \mathcal{P}(M)
\end{array}
$$

Thus we arrived at a semantics for the subtype relation, that is,

$$\alpha \models \beta \text{ iff } (M, R) \models \alpha \leq_\wedge \beta,\ \text{for any structure } (M, R)$$

It is now possible to state the following theorem of validity and completeness.

THEOREM 1.2. $\alpha \models \beta$ *if and only if* $\alpha \leq_\wedge \beta$

The proof of this theorem can be found in [3, 11], but it is convenient to recall here the construction of the filter model which is used to prove completeness since it provides the main ideas for the proof of completeness of our modal logic (see section 2).

We call filter of $\Lambda_\wedge$ any non-empty subset F of $\Lambda_\wedge$ closed under $\wedge$, that is, for all $\alpha, \beta \in \Lambda_\wedge$, if $\alpha, \beta \in F$ then $\alpha \wedge \beta \in F$, and upward closed, that is, for all $\alpha, \beta \in \Lambda_\wedge$, if $\alpha \in F$ and $\alpha \leq_\wedge \beta$ then $\beta \in F$.

In the following we will use the following filter construction.

DEFINITION 1.1. Let α be any type. Then

$$\uparrow\!\alpha \equiv \{\beta \in \Lambda_\wedge |\ \alpha \leq_\wedge \beta\}$$

will be called the *filter generated* by α.

It is easy to verify that, for any type α, $\uparrow\!\alpha$ is a filter.

Consider now the set

$$M \equiv \{F \mid F \text{ is a filter of } \Lambda_\wedge\}$$

and define a three place relation R on its elements by setting

$$R(F, G, H) \equiv (\forall\alpha)(\forall\beta)\ (\alpha \to \beta \in F) \Rightarrow (\alpha \in G \Rightarrow \beta \in H)$$

Finally consider the interpretation map ϕ defined by setting

$$\phi(\alpha) = \{F \mid \alpha \in F\}$$

for any basic type α and extend it to a forcing relation $\Vdash^\phi$. It is not difficult to prove that (M, R, ϕ) is a model for $\leq_\wedge$. Moreover, it is possible to prove by induction on type complexity the following lemma.

LEMMA 1.1. *Let α be any type and F be any filter of $\Lambda_\wedge$. Then $F \Vdash^\phi \alpha$ if and only if $\alpha \in F$.*

And this lemma immediately yields the completeness theorem 1.2 since supposing $\alpha \models \beta$ we obtain $(M, R, \phi) \models \alpha \leq_\wedge \beta$ and hence for any filter $F \in M$, if $F \Vdash^\phi \alpha$ then $F \Vdash^\phi \beta$. But, after lemma 1.1, this means that if $\alpha \in F$ then $\beta \in F$. Let us consider now the filter $\uparrow\alpha$; it clearly contains α and hence $\beta \in\uparrow\alpha$, that is $\alpha \leq_\wedge \beta$.

The semantics we considered here is clearly recalling a sort of non-standard Kripke semantics for a modal logic: the idea to define a modal interpretation for the connective $\to$ started here.

2. THE TWO-PLACES MODAL LOGIC BK

Consider the propositional modal language whose formula are inductively defined as follows

1. Any propositional variable is a formula;
2. $\perp$ and $\top$ are formulas;
3. If α and β are formulas then also $\alpha \wedge \beta$, $\alpha \vee \beta$, $\neg\alpha$, $\alpha \to \beta$ are formulas;
4. If α and β are formulas then $\Box(\alpha, \beta)$ is a formula.

We can define a *kripke-like* semantics for the formulas of this language as follows. Let A be a set and R be a ternary relation over A and suppose that v is a map of the propositional variables into subsets of A. Then, supposing $x \in A$ and

p is a propositional variable, set

$$
\begin{array}{lll}
x \Vdash^{\mathsf{v}} p & \text{iff} & x \in \mathsf{v}(p) \\
x \Vdash^{\mathsf{v}} \bot & \text{iff} & \mathsf{falsum} \\
x \Vdash^{\mathsf{v}} \top & \text{iff} & \mathsf{true} \\
x \Vdash^{\mathsf{v}} \alpha \wedge \beta & \text{iff} & x \Vdash^{\mathsf{v}} \alpha \text{ and } x \Vdash^{\mathsf{v}} \beta \\
x \Vdash^{\mathsf{v}} \alpha \vee \beta & \text{iff} & x \Vdash^{\mathsf{v}} \alpha \text{ or } x \Vdash^{\mathsf{v}} \beta \\
x \Vdash^{\mathsf{v}} \neg\alpha & \text{iff} & x \nVdash^{\mathsf{v}} \alpha \\
x \Vdash^{\mathsf{v}} \alpha \rightarrow \beta & \text{iff} & x \Vdash^{\mathsf{v}} \alpha \text{ yields } x \Vdash^{\mathsf{v}} \beta \\
x \Vdash^{\mathsf{v}} \Box(\alpha, \beta) & \text{iff} & (\forall y \in A)(\forall z \in A)\ R(x, y, z) \rightarrow (y \Vdash^{\mathsf{v}} \alpha \rightarrow z \Vdash^{\mathsf{v}} \beta)
\end{array}
$$

A formula α is true in the model (A, R, v) if, for every element $x \in A$, $x \Vdash^{\mathsf{v}} \alpha$ holds, and it is true in the frame (A, R) if, for every valuation v, it is true in the model (A, R, v). Finally, a formula is valid if it is true in every frame.

It is interesting to note that what we defined is a generalization of the usual modal situation. In fact, we can define a standard modality by setting $\Box(\beta) \equiv \Box(\top, \beta)$ and then we obtain the usual definition for a forcing relation by setting $\overline{R}(x, z) \equiv (\exists y \in A)\ R(x, y, z)$. Since no extra condition is required on the relation R, the models that we defined directly generalize the situation for the modal logic K. This is the reason why we called BK this *binary* modal logic.

Consider now any sequent calculus for a classical propositional calculus and add it the following single modal rule

$$
\Box\text{-rule} \quad \frac{\alpha \vdash \alpha_1, \ldots, \alpha_n \qquad \beta_1, \ldots, \beta_m \vdash \beta}{\{\Box(\alpha_i, \beta_j) |\ i = 1 \ldots n, j = 1 \ldots m\} \vdash \Box(\alpha, \beta)} \quad n \geq 0, m \geq 0
$$

This rule is valid in any frame. In fact, let us suppose that its conclusion is not valid in some frame (A, R), that is, let us suppose that there exists a point $x \in A$ and a valuation v such that $x \Vdash^{\mathsf{v}} \neg\Box(\alpha, \beta)$ whereas for all $i = 1 \ldots n$ and $j = 1 \ldots m$, $x \Vdash^{\mathsf{v}} \Box(\alpha_i, \beta_j)$. Then there must be two points $y, z \in A$ such that $R(x, y, z)$ holds and $y \Vdash^{\mathsf{v}} \alpha$ and $z \Vdash^{\mathsf{v}} \neg\beta$. Hence, by the left premise, we obtain that there must be some index i such that $y \Vdash^{\mathsf{v}} \alpha_i$ and thus, for any $j = 1 \ldots m$, $z \Vdash^{\mathsf{v}} \beta_j$, since $x \Vdash^{\mathsf{v}} \Box(\alpha_i, \beta_j)$. But then the right premise forces $z \Vdash^{\mathsf{v}} \beta$, contradiction.

2.1. The completeness theorem

We want now prove that the $\Box$-*rule* is sufficient to prove any valid sequent. To this aim it is convenient to consider two of its instances, which are indeed sufficient to obtain the result. The first one is obtained for $n = 1$ and $\alpha_1 \equiv \alpha$ and the second one for $m = 1$ and $\beta_1 = \beta$.

$$
\begin{array}{lll}
\Box\text{-monotonicity} & \dfrac{\beta_1, \ldots, \beta_m \vdash \beta}{\Box(\alpha, \beta_1), \ldots, \Box(\alpha, \beta_m) \vdash \Box(\alpha, \beta)} & m \geq 0 \\
\Box\text{-antimonotonicity} & \dfrac{\alpha \vdash \alpha_1, \ldots, \alpha_n}{\Box(\alpha_1, \beta), \ldots, \Box(\alpha_n, \beta) \vdash \Box(\alpha, \beta)} & n \geq 0
\end{array}
$$

Note that setting $n = 0$ and $\alpha \equiv \bot$ in $\Box$-*antimonotonicity* we obtain that $\Box(\bot, \beta)$ is provable and setting $m = 0$ and $\beta \equiv \top$ in $\Box$-*monotonicity* we obtain that $\Box(\alpha, \top)$ is provable.

Moreover, the $\Box$-*rule* is sufficient to prove that the binary modal operator is an operation in the Lindenbaum algebra $\mathcal{L}_{BK}$ of BK. In fact, we can prove the following theorem.

THEOREM 2.1. *Let* $\vdash \alpha_1 \leftrightarrow \alpha_2$ *and* $\vdash \beta_1 \leftrightarrow \beta_2$. *Then* $\vdash \Box(\alpha_1, \beta_1) \leftrightarrow \Box(\alpha_2, \beta_2)$.

Proof. It is sufficient to show that if $\alpha_2 \vdash \alpha_1$ and $\beta_1 \vdash \beta_2$ hold then also $\Box(\alpha_1, \beta_1) \vdash \Box(\alpha_2, \beta_2)$ holds, which is immediate by $\Box$-*rule*. ■

It is worth noting that the proof of this theorem shows that the modality that we are considering behaves like an implication, even if one should be aware that the usual rule of implication introduction is not valid for such a modality, that is, $\alpha \vdash \beta$ does not yield $\vdash \Box(\alpha, \beta)$.

We can now prove the completeness theorem.

THEOREM 2.2. *The sequent* $\alpha_1, \ldots, \alpha_n \vdash \beta_1, \ldots, \beta_m$ *is provable if and only if it is valid in any frame.*

We already proved that all the rules are valid. To prove that they are also sufficient we will adapt to BK the standard approach. First note that the Lindenbaum algebra $\mathcal{L}_{BK}$ is a boolean algebra. Consider now the set $\mathcal{U}$ of ultrafilters of $\mathcal{L}_{BK}$ and define a ternary relation R over $\mathcal{U}$ by setting

$$R(F, G, H) \equiv (\forall \gamma, \delta)\, \Box(\gamma, \delta) \in F \to ((\gamma \in G) \to (\delta \in H))$$

Let us now suppose that the sequent $\alpha_1, \ldots, \alpha_n \vdash \beta_1, \ldots, \beta_m$ is not provable; then the formula $(\alpha_1 \wedge \ldots \wedge \alpha_n) \to (\beta_1 \vee \ldots \vee \beta_m)$ is not equal to the top element of $\mathcal{L}_{BK}$ and hence its negation is different from the bottom element of $\mathcal{L}_{BK}$; hence there exists an ultrafilter containing it. In order to conclude it is sufficient to define a valuation V of the propositional variables into the set of the subsets of $\mathcal{U}$ by setting

$$\mathsf{V}(p) \equiv \{F \in \mathcal{U} \mid p \in F\}$$

since this position yields that, for any formula α, $F \Vdash^{\mathsf{V}} \alpha$ if and only if $\alpha \in F$. In fact, let us argue according to the complexity of the formula α. If α is the propositional variable p then by definition $F \Vdash^{\mathsf{V}} p$ if and only if $F \in \mathsf{V}(p)$ if and only if $p \in F$. If $\alpha \equiv \bot$ then the result is immediate since F is a proper filter. If $\alpha \equiv \top$ or $\alpha \equiv \alpha_1 \wedge \alpha_2$ the result follows by induction from the fact that F is a filter of a boolean algebra. If $\alpha \equiv \alpha_1 \vee \alpha_2$ then the result follows by induction from the fact that F is a prime filter. If $\alpha \equiv \neg\alpha_1$ then the result follows by induction and the fact that F is an ultrafilter. Hence also the case

$\alpha \equiv \alpha_1 \to \alpha_2$ is immediate since $\alpha_1 \to \alpha_2$ is logically equivalent to $\neg\alpha_1 \vee \alpha_2$ and filter are closed under logical equivalence. Finally, if $\alpha \equiv \Box(\alpha_1, \alpha_2)$ then we can immediately prove that $\Box(\alpha_1, \alpha_2) \in F$ yields $F \Vdash^{\mathsf{V}} \Box(\alpha_1, \alpha_2)$. In fact, let us suppose that $G, H \in \mathcal{U}$ and $R(F, G, H)$ and $G \Vdash^{\mathsf{V}} \alpha_1$ hold. Then $\alpha_1 \in G$ by inductive hypothesis and hence $\Box(\alpha_1, \alpha_2) \in F$ and $R(F, G, H)$ yields $\alpha_2 \in H$. Then $H \Vdash^{\mathsf{V}} \alpha_2$ by inductive hypothesis and hence $F \Vdash^{\mathsf{V}} \Box(\alpha_1, \alpha_2)$ by definition. The proof that $F \Vdash^{\mathsf{V}} \Box(\alpha_1, \alpha_2)$ yields $\Box(\alpha_1, \alpha_2) \in F$ is more complex. In fact, it will be a proof by absurd, that is, we will assume that $\Box(\alpha_1, \alpha_2) \notin F$ and we will prove that there exist two ultrafilters G and H such that $R(F, G, H)$, $G \Vdash^{\mathsf{V}} \alpha_1$ and $H \Vdash^{\mathsf{V}} \neg\alpha_2$, that is, $F \nVdash^{\mathsf{V}} \Box(\alpha_1, \alpha_2)$.

The idea is to build the ultrafilter G with a continuous attention for the possibility to build H. To this aim let us consider the following inductive definition of a sequence $(Y_i)_{i \in \omega}$ of filters . Let $(\phi_i)_{i \in \omega}$ be any surjective numbering of the elements of $\mathcal{L}_{BK}$ and set

$$
\begin{array}{lll}
Y_0 & = & \uparrow\{\alpha_1\} \\
Y_{i+1} & = & \left\{ \begin{array}{ll} \uparrow(Y_i \cup \{\phi_i\}) & \text{if } \uparrow(Y_i \cup \{\phi_i\}) \text{ is } \langle F, \neg\alpha_2\rangle\text{-consistent} \\ \uparrow(Y_i \cup \{\neg\phi_i\}) & \text{otherwise} \end{array} \right.
\end{array}
$$

where we write $\uparrow A$ to mean the minimal filter of $\mathcal{L}_{BK}$ which contains the subset A, that is, $\uparrow A \equiv \{\gamma \in \mathcal{L}_{BK} \mid (\exists\alpha_1, \ldots, \alpha_n \in A)\ \alpha_1 \wedge \ldots \wedge \alpha_n \vdash \gamma\}$, and we say that a set of formulas A is $\langle F, \neg\alpha_2\rangle$-consistent to mean that the set

$$\{\delta \mid (\exists\gamma \in A)\ \Box(\gamma, \delta) \in F\} \cup \{\neg\alpha_2\}$$

is consistent.

LEMMA 2.1. *For any $i \geq 0$, the filter Y_i is generated by one formula, that is, there exists a formula ψ_i such that $Y_i = \uparrow\{\psi_i\}$.*

Proof. By induction. By definition, Y_0 is generated by α_1 and, supposing that Y_i is generated by ψ_i, then $Y_{i+1} = \uparrow\{\psi_i \wedge \phi_i\}$ or $Y_{i+1} = \uparrow\{\psi_i \wedge \neg\phi_i\}$ according to the clause which applies in the definition of Y_{i+1}. In fact, it is immediate to verify that $\uparrow(Y_i \cup \{\gamma\}) = \uparrow\{\psi_i \wedge \gamma\}$ because $\delta \in \uparrow(Y_i \cup \{\gamma\})$ means that there exist $\gamma_1, \ldots, \gamma_n \in Y_i$ such that $\gamma_1 \wedge \ldots \wedge \gamma_n \wedge \gamma \vdash \delta$ and hence $\psi_i \wedge \gamma \vdash \delta$ because, for each $1 \leq k \leq n$, $\psi_i \vdash \gamma_k$; in the other direction the result is an immediate consequence of the fact that ψ_i is an element of $Y_i = \uparrow\{\psi_i\}$. ■

LEMMA 2.2. *For any $i \geq 0$, the filter Y_i is $\langle F, \neg\alpha_2\rangle$-consistent.*

Proof. By induction. Let us suppose that Y_0, that is, $\uparrow\{\alpha_1\}$, is not $\langle F, \neg\alpha_2\rangle$-consistent; then there exist $\gamma_1, \delta_1, \ldots, \gamma_n, \delta_n$ such that $\delta_1 \wedge \ldots \wedge \delta_n \wedge \neg\alpha_2 \vdash \bot$ and, for any $1 \leq k \leq n$, $\alpha_1 \vdash \gamma_k$ and $\Box(\gamma_k, \delta_k) \in F$. Thus $\delta_1 \wedge \ldots \wedge \delta_n \vdash \alpha_2$ and hence

$$\Box(\alpha_1, \delta_1) \wedge \ldots \wedge \Box(\alpha_1, \delta_n) \vdash \Box(\alpha_1, \alpha_2)$$

by $\Box$-*monotonicity*. But, by using $\Box$-*antimonotonicity*, we obtain that

$$\Box(\gamma_i, \delta_i) \vdash \Box(\alpha_1, \delta_i)$$

and hence $\Box(\alpha_1, \delta_i) \in F$ since F is upward closed. But then we obtain that $\Box(\alpha_1, \alpha_2) \in F$ which is contrary to our assumption.

Suppose now, by inductive hypothesis, that Y_i is $\langle F, \neg\alpha_2\rangle$-consistent and let us assume that both $\uparrow(Y_i \cup \phi_i)$ and $\uparrow(Y_i \cup \neg\phi_i)$ are not $\langle F, \neg\alpha_2\rangle$-consistent. Then there exist $\gamma_1, \delta_1, \ldots, \gamma_n, \delta_n$ and $\gamma'_1, \delta'_1, \ldots, \gamma'_m, \delta'_m$ such that, for any $1 \leq k \leq n$, $\gamma_k \in\uparrow(Y_i \cup \phi_i)$ and $\Box(\gamma_k, \delta_k) \in F$ and, for any $1 \leq h \leq m$, $\gamma'_h \in\uparrow(Y_i \cup \neg\phi_i)$ and $\Box(\gamma'_h, \delta'_h) \in F$. Moreover $\delta_1 \wedge \ldots \wedge \delta_n \vdash \alpha_2$ and $\delta'_1 \wedge \ldots \wedge \delta'_m \vdash \alpha_2$. But, after the previous lemma 2.1, we know that $Y_i \equiv\uparrow\{\psi_i\}$ for some formula ψ_i. Hence, for each $1 \leq k \leq n$, $\psi_i \wedge \phi_i \vdash \gamma_k$ and, for each $1 \leq h \leq m$, $\psi_i \wedge \neg\phi_i \vdash \gamma'_k$. Then, by $\Box$-*antimonotonicity*, for each $1 \leq k \leq n$, $\Box(\gamma_k, \delta_k) \vdash \Box(\psi_i \wedge \phi_i, \delta_k)$ and hence $\Box(\psi_i \wedge \phi_i, \delta_k) \in F$ and, for each $1 \leq h \leq m$, $\Box(\gamma'_h, \delta'_h) \vdash \Box(\psi_i \wedge \neg\phi_i, \delta'_h)$ and hence $\Box(\psi_i \wedge \neg\phi_i, \delta'_h) \in F$. But by $\Box$-*monotonicity*,

$$\Box(\psi_i \wedge \phi_i, \delta_1) \wedge \ldots \wedge \Box(\psi_i \wedge \phi_i, \delta_n) \vdash \Box(\psi_i \wedge \phi_i, \alpha_2)$$

and

$$\Box(\psi_i \wedge \neg\phi_i, \delta'_1) \wedge \ldots \wedge \Box(\psi_i \wedge \neg\phi_i, \delta'_m) \vdash \Box(\psi_i \wedge \neg\phi_i, \alpha_2)$$

hold and hence both $\Box(\psi_i \wedge \phi_i, \alpha_2) \in F$ and $\Box(\psi_i \wedge \neg\phi_i, \alpha_2) \in F$. We can now conclude immediately if we observe that $\psi_i \vdash (\psi_i \wedge \phi_i) \vee (\psi_i \wedge \neg\phi_i)$ is a tautology and then, by $\Box$-*antimonotonicity* we can infer that

$$\Box(\psi_i \wedge \phi_i, \alpha_2) \wedge \Box(\psi_i \wedge \neg\phi_i, \alpha_2) \vdash \Box(\psi_i, \alpha_2)$$

and hence $\Box(\psi_i, \alpha_2) \in F$, that is, Y_i is not $\langle F, \neg\alpha_2\rangle$-consistent, which is contrary to the inductive hypothesis. ■

Now let us set

$$G \equiv \bigcup_{i\in\omega} Y_i$$

and we can prove the following lemma.

LEMMA 2.3. *G is a $\langle F, \neg\alpha_2\rangle$-consistent ultrafilter.*

Proof. G is a filter because $\top \in G$ since $\top \in Y_0 \equiv\uparrow\{\alpha_1\}$ and, supposing $\gamma_1, \gamma_2 \in G$, there is an index i such that $\gamma_1, \gamma_2 \in Y_i$, that is, $\psi_i \vdash \gamma_1$ and $\psi_i \vdash \gamma_2$, because for any i, $Y_i \subseteq Y_{i+1}$ obviously holds; hence $\psi_i \vdash \gamma_1 \wedge \gamma_2$, that is, $\gamma_1 \wedge \gamma_2 \in Y_i$, and hence $\gamma_1 \wedge \gamma_2 \in G$; finally, if $\gamma_1 \in G$ and $\gamma_1 \vdash \gamma_2$ then there is an index i such that $\gamma_1 \in Y_i$, that is, $\psi_i \vdash \gamma_1$, and hence $\psi_i \vdash \gamma_2$, that is, $\gamma_2 \in Y_i$, so that $\gamma_2 \in G$.

Moreover, if G would not be $\langle F, \neg\alpha_2\rangle$-consistent then there would be γ_1, δ_1, $\ldots$, γ_n, δ_n such that $\gamma_1, \ldots, \gamma_n \in G$, $\Box(\gamma_1, \delta_1) \in F$, $\ldots$, $\Box(\gamma_n, \delta_n) \in F$ and $\delta_1 \wedge \ldots \wedge \delta_n \vdash \alpha_2$, but then there would exist an index i such that $\gamma_1, \ldots, \gamma_n \in Y_i$, that is Y_i would not be $\langle F, \neg\alpha_2\rangle$-consistent, contrary to lemma 2.2.

To prove that G is an ultrafilter we have only to prove it is a complete consistent filter. Since any formula γ appears in the sequence $(\phi_i)_{i\in\omega}$, that is, $\gamma \equiv \phi_i$ for some $i \in \omega$, we obtain that $\gamma \in Y_{i+1}$ or $\neg\gamma \in Y_{i+1}$, and thus $\gamma \in G$ or $\neg\gamma \in G$, that is, G is complete. Finally consistency is a consequence of the fact that G is $\langle F, \neg\alpha_2\rangle$-consistent. In fact, if G is not consistent then $\perp \in G$ and hence $\perp \in \{\delta |\ (\exists\gamma \in A)\ \Box(\gamma, \delta) \in F\} \cup \{\neg\alpha_2\}$, because $\Box(\perp, \perp)$ is provable and hence it belongs to every filter. ∎

In order to build the ultrafilter H, let us consider the set

$$Z \equiv \{\delta |\ (\exists\gamma \in G)\ \Box(\gamma, \delta) \in F\} \cup \{\neg\alpha_2\}$$

The set Z is consistent by definition since G is $\langle F, \neg\alpha_2\rangle$-consistent; then Z can be extended to a proper ultrafilter H in the usual way. Now it is immediate to prove that $R(F, G, H)$, that is, for all γ and δ, if $\Box(\gamma, \delta) \in F$ and $\gamma \in G$ then $\delta \in H$; moreover $\alpha_1 \in G$ by construction and $\neg\alpha_2 \in H$ because $\neg\alpha_2 \in Z$. We have thus completed the proof of theorem 2.2.

2.2. Cut-elimination

In the previous section we proved that $\Box$-*rule* is valid with respect to the Kripke models that we proposed and sufficient to obtain a completeness proof. On the other hand, to investigate the proof theorethical properties of a logical system it is often convenient to have some kind of cut-elimination procedure for its sequent calculus. In fact, we are able to define such a procedure for a version of the sequent calculus for BK obtained by a slight modification of the modal rule.

$$(\Box\text{-gen-rule})\quad \frac{\alpha \vdash \bigwedge_{i=1\ldots n} \bigvee_{j=1\ldots m_i} \gamma_{ij} \qquad \bigwedge_{i=1\ldots n} \bigvee_{j=1\ldots m_i} \delta_{ij} \vdash \beta}{\Box(\gamma_{11}, \delta_{11}), \ldots, \Box(\gamma_{nm_n}, \delta_{nm_n}) \vdash \Box(\alpha, \beta)}$$

with the obvious meaning of the generalized connectives. Note that distributivity of $\wedge$ over $\vee$ allows to present $\Box$-*gen-rule* like a more standard rule, provided we use sequents with many premises instead of generalized quantifiers, that is,

$$(\Box\text{-gen-rule})\quad \frac{\begin{array}{cc} & \delta_{11}, \delta_{21}, \ldots, \delta_{n1} \vdash \beta \\ & \delta_{12}, \delta_{21}, \ldots, \delta_{n1} \vdash \beta \\ \alpha \vdash \gamma_{11}, \ldots, \gamma_{1m_1} & \vdots \\ \vdots & \delta_{1m_1}, \delta_{21}, \ldots, \delta_{n1} \vdash \beta \\ \alpha \vdash \gamma_{n1}, \ldots, \gamma_{nm_n} & \vdots \\ & \delta_{1m_1}, \delta_{2m_2}, \ldots, \delta_{nm_n} \vdash \beta \end{array}}{\Box(\gamma_{11}, \delta_{11}), \ldots, \Box(\gamma_{nm_n}, \delta_{nm_n}) \vdash \Box(\alpha, \beta)}$$

It is easy to check that $\Box$-*gen-rule* is valid in any of the considered Kripke model. In fact, let us suppose that there is a point x in a model such that $x \Vdash \neg\Box(\alpha, \beta)$ and $x \Vdash \Box(\gamma_{ij}, \delta_{ij})$ for any $1 \leq i \leq n$ and $1 \leq j \leq m_i$; hence there must exist in the model two points y and z, in relation with x, such that $y \Vdash \alpha$ and $z \Vdash \neg\beta$; then $y \Vdash \bigwedge_{i=1\ldots n} \bigvee_{j=1\ldots m_i} \gamma_{ij}$, and hence for all $i = 1 \ldots n$, there is at least one $1 \leq j \leq m_i$ such that $y \Vdash \gamma_{ij}$ holds; hence $z \Vdash \delta_{ij}$, because $x \Vdash \Box(\gamma_{ij}, \delta_{ij})$, and so $z \Vdash \bigwedge_{i=1\ldots n} \bigvee_{j=1\ldots m_i} \delta_{ij}$ which yields $z \Vdash \beta$; contradiction.

It is worth noting that $\Box$-*monotonicity* and $\Box$-*antimonotonicity* are special instances of $\Box$-*gen-rule*. In fact, let us put $m_i = 1$ for each $1 \leq i \leq n$ and $\gamma_{ij} \equiv \alpha$ in the $\Box$-*gen-rule* rule, then we obtain

$$\frac{\alpha \vdash \alpha \wedge \ldots \wedge \alpha \quad \delta_1 \wedge \ldots \wedge \delta_n \vdash \beta}{\Box(\alpha, \delta_1), \ldots, \Box(\alpha, \delta_n) \vdash \Box(\alpha, \beta)}$$

which is equivalent to $\Box$-*monotonicity*. And if we put $n = 1$, $m_1 = m$ and $\delta_{1j} \equiv \beta$, then we obtain

$$\frac{\alpha \vdash \gamma_1 \vee \ldots \vee \gamma_m \quad \beta \vee \ldots \vee \beta \vdash \beta}{\Box(\gamma_1, \beta), \ldots, \Box(\gamma_m, \beta) \vdash \Box(\alpha, \beta)}$$

which is equivalent to $\Box$-*antimonotonicity*.

Thus, after the completeness theorem 2.2, $\Box$-*gen-rule* is equivalent to $\Box$-*monotonicity* and $\Box$-*antimonotonicity* together. But our interest in this rule is in that by using it we do not need to use the cut-rule. The proof of cut-eliminability is almost standard, that is, it is by principal induction on the complexity of the cut-formula and secondary induction on the length of the thread of the cut-formula. The reductions to lower the length of the threads and those for lowering the complexity of the cut-formula in the non-modal cases are standard. Thus, we consider here only the case the cut-formula is $\Box(\alpha, \beta)$ and a modal rule is applied, that is,

$$\frac{\dfrac{\alpha \vdash \bigwedge_i \bigvee_{j_i} \gamma_{ij_i} \quad \bigwedge_i \bigvee_{j_i} \delta_{ij_i} \vdash \beta}{\{\Box(\gamma_{ij_i}, \delta_{ij_i})\}_{i,j_i} \vdash \Box(\alpha, \beta)} \quad \dfrac{\phi \vdash \bigwedge_h \bigvee_{k_h} \phi_{hk_h} \quad \bigwedge_h \bigvee_{k_h} \psi_{hk_h} \vdash \psi}{\{\Box(\phi_{hk_h}, \psi_{hk_h})\}_{h,k_h} \vdash \Box(\phi, \psi)}}{\{\Box(\gamma_{ij_i}, \delta_{ij_i})\}_{i,j_i} \cup (\{\Box(\phi_{hk_h}, \psi_{hk_h})\}_{h,k_h} \setminus \Box(\alpha, \beta)) \vdash \Box(\phi, \psi)}$$

In this case, $\alpha \vdash A_1 \wedge \ldots \wedge A_n$ and $\phi \vdash B_1 \wedge \ldots \wedge (B_h \vee \alpha) \wedge \ldots \wedge B_m$ where $A_1 \equiv \bigvee_{j_1} \gamma_{1j_1}, \ldots, A_n \equiv \bigvee_{j_n} \gamma_{nj_n}$ and $B_1 \equiv \bigvee_{k_1} \phi_{1k_1}, \ldots, B_m \equiv \bigvee_{k_m} \phi_{mk_m}$. Hence, $\phi \vdash B_1; \ldots; \phi \vdash B_h, \alpha; \ldots; \phi \vdash B_m$. Then, by using a cut on the formula α, we obtain that $\phi \vdash B_h, A_1 \wedge \ldots \wedge A_n$ and hence we can construct, by using no cut, a proof of

$$\phi \vdash B_1 \wedge \ldots \wedge (B_h \vee A_1) \wedge \ldots \wedge (B_h \vee A_n) \wedge \ldots \wedge B_m$$

In a similar way, from $A'_1 \wedge \ldots \wedge A'_n \vdash \beta$ and $B'_1 \wedge \ldots \wedge (B'_h \vee \beta) \wedge \ldots \wedge B'_m \vdash \psi$, where $A'_1 \equiv \bigvee_{j_1} \delta_{1j_1}, \ldots, A'_n \equiv \bigvee_{j_n} \delta_{nj_n}$ and $B'_1 \equiv \bigvee_{k_1} \psi_{1k_1}, \ldots, B'_m \equiv \bigvee_{k_m} \psi_{mk_m}$, we obtain both that $B'_1 \wedge \ldots \wedge B'_h \wedge \ldots \wedge B'_m \vdash \psi$ and that $B'_1 \wedge \ldots \wedge \beta \wedge \ldots \wedge B'_m \vdash \psi$. Hence, by using a cut on β, we obtain $B'_1 \wedge \ldots \wedge A'_1 \wedge \ldots \wedge A'_n \wedge \ldots \wedge B'_m \vdash \psi$. Thus, by using no cut, we can construct also a proof of

$$B'_1 \wedge \ldots \wedge (B'_h \vee A'_1) \wedge \ldots \wedge (B'_h \vee A'_n) \wedge \ldots \wedge B'_m \vdash \psi$$

Then we can conclude; in fact, by using an instance of $\Box$-*gen-rule* we obtain the sequent in the conclusion of the application of the cut-rule, except for the non essential repetition of some of the boxed assumptions.

2.3. Decidability and finite model property

Nice consequences of the theorem of cut-elimination that we proved in the previous section 2.2 are decidability of BK and the finite model property. These results are an immediate consequence of the fact that a always terminating procedure for looking for the derivability of any sequent which does not use the cut-rule can be provided; and such a procedure is correct, that is, when it fails we can use the proof tentative to build a finite counter-model for the non provable sequent.

The proof of this statement follows the general ideas of a cut-redundancy proof (see for instance [16]) and we have only to add a special treatment for the modal case like it has been done in [18] for the modal logic K. For this reason in this section we assume to work with a set of double-sound rules for the propositional connectives. We mean that a rule is double-sound when the conclusion holds if and only if all of the premises do. For instance a double-sound rule for the introduction of the connective $\wedge$ is

$$\frac{\Gamma \vdash \alpha, \Delta \qquad \Gamma \vdash \beta, \Delta}{\Gamma \vdash \alpha \wedge \beta, \Delta}$$

because $\Gamma \vdash \alpha \wedge \beta, \Delta$ does not hold if either $\Gamma \vdash \alpha, \Delta$ or $\Gamma \vdash \beta, \Delta$ does not hold. It is well known that a complete set of double-sound rules can be provided for the classical propositional calculus (see for instance [18]).

The decision strategy for the non-modal case is simply to apply any applicable propositional rule. Since the premise(s) of each propositional rule is (are) strictly simpler than the conclusion, this search procedure is going to arrive in a finite number of steps at an axiom or at a sequent of the following shape:

$$p_1, \ldots, p_r, \Box(\alpha_1, \beta_1), \ldots, \Box(\alpha_n, \beta_n) \vdash \Box(\phi_1, \psi_1), \ldots, \Box(\phi_m, \psi_m), q_1, \ldots, q_s \tag{1}$$

where $p_1, \ldots, p_r$ and $q_1, \ldots, q_s$ are propositional variables. If all of the leaves of the search tree we arrived at in this way are axioms then our search procedure stops with a success. On the other hand, let us suppose that we did not arrive at an axiom. To begin with a simple case let us first consider the case that the sequent 1 that we are examining is

$$p_1, \ldots, p_r, \Box(\alpha_1, \beta_1), \ldots, \Box(\alpha_n, \beta_n) \vdash q_1, \ldots, q_s$$

that is, the case $m = 0$ and $\{p_1, \ldots, p_r\} \cap \{q_1, \ldots, q_s\} = \emptyset$. In this case the sequent can easily be falsified in the finite model $(\{*\}, R, \nu)$ defined on the one element set $\{*\}$ by setting $R = \emptyset$ and $\nu(p) = \{*\}$ if and only if $p \in \{p_1, \ldots, p_r\}$.

On the other hand, that is, when we consider the case $m \geq 1$, the cut-elimination theorem suggests that the sequent 1, provided it is not already an axiom, can only be obtained by weakening from:

$$\Box(\alpha_1, \beta_1), \cdots, \Box(\alpha_n, \beta_n) \vdash \Box(\phi_h, \psi_h) \tag{2}$$

for some $1 \leq h \leq m$. Indeed, if we will be able to find a suitable index h and prove the corresponding sequent 2, then we will eventually obtain a proof of the sequent 1 by using some instances of weakening. Of course, the problem will be in proving that if, for no sequent like 2, for $1 \leq h \leq m$, a proof tentative is successful then the sequent 1 is not valid and it can be falsified by using some finite counter-model.

In general, a cut free proof of the sequent 2 should be obtained by an application of the $\Box$-*gen-rule* and the left premise of such a rule should have the following shape:

$$\phi_h \vdash \bigwedge_{A \in \mathcal{G}} \bigvee_{j \in A} \alpha_j \tag{3}$$

where $\mathcal{G}$ is a suitable partition of the set $\{1, \ldots, n\}$. But the sequent 3 is provable if and only if, for any $A \in \mathcal{G}$, $\phi_h \vdash \bigvee_{j \in A} \alpha_j$. So, in the search for the left premise of the required $\Box$-*gen-rule* we can consider only the following set of set of indexes:

$$\mathcal{H} = \{A \subseteq \{1, \ldots, n\} |\ \phi_h \vdash \bigvee_{j \in A} \alpha_j\}$$

It is clear that all of the sequents $\phi_h \vdash \bigvee_{j \in A} \alpha_j$ are simpler than the sequent 2 and thus we can assume to be able to decide on membership to $\mathcal{H}$.

Let us note that supposing $\mathcal{H}$ is empty, that is, supposing there is no subset $A \subseteq \{1,\ldots,n\}$ such that $\phi_h \vdash \bigvee_{j\in A}\alpha_j$, yields in particular that $\phi_h \not\vdash \alpha_1,\ldots,\alpha_n$ and hence, by inductive hypothesis, a finite model (M'_h, R'_h, ν'_h) can be built which contains a point y_h such that $y_h \Vdash \phi_h$ and, for all $1 \leq j \leq n$, $y_h \Vdash \neg\alpha_j$. Let us now observe that $\not\vdash \psi_h$, otherwise the sequent 2 is obviously provable by an instance of $\Box$-*gen-rule* with premises $\phi_h \vdash \bigwedge_{i=1\ldots 0, j=1\ldots m_i}\alpha_{ij}$ and $\bigwedge_{i=1\ldots 0, j=1\ldots m_i}\beta_{ij} \vdash \psi_h$. Then a finite model (M''_h, R''_h, ν''_h) can be built which contains a point z_h such that $z_h \Vdash \neg\psi_h$. Thus a finite model (M_h, R_h, ν_h) which falsifies the sequent 2 can be built by adding a new point x_h to M'_h and M''_h, in order to obtain $M_h = \{x\} \cup M'_h \cup M''_h$, and setting $R_h = \{\langle x,y,z\rangle\} \cup R'_h \cup R''_h$ and $\nu_h = \nu'_h \cup \nu''_h$.

So, let us continue under the assumption that $\mathcal{H} \neq \emptyset$. We will use in the sequel the fact that in this case $\{1,\ldots,n\} \in \mathcal{H}$. If $\mathcal{H} \neq \emptyset$, then if we would be able to find a subset $\mathcal{G}$ of $\mathcal{H}$ such that

$$\bigwedge_{A\in\mathcal{G}}\bigvee_{j\in A}\beta_j \vdash \psi_h \tag{4}$$

then we would have found the required instance of $\Box$-*gen-rule*. And all of the sequents 4 can be assumed to be decidable since they also are simpler than the sequent 2.

It can be useful to note that the elements of the set $\mathcal{H}$ are *not* a partition of the set $\{1,\ldots,n\}$, but this is not a real problem. In fact, the only difference in the conclusion of the considered $\Box$-*gen-rule* with respect to the sequent 2 is the possible repetition of some of the boxed-formulas on the left hand side.

In order to conclude the proof of decidability of BK, we have to show that if no subset $\mathcal{G}$ of $\mathcal{H}$ can be found such that the sequent 4 holds, then a finite counter-model for the sequent 2 can be built. To this aim, we need some preliminary lemmas. Let us consider the set $\mathcal{F}$ of all the functions $\phi : \mathcal{H} \longrightarrow \{1,\ldots,n\}$ such that $\phi(A) \in A$.

LEMMA 2.4. *Suppose no subset $\mathcal{G}$ of $\mathcal{H}$ can be found such that the sequent 4 holds. Then there is a function $\phi^* \in \mathcal{F}$ such that*

$$\bigwedge_{A\in\mathcal{H}}\beta_{\phi^*(A)} \not\vdash \psi_h \tag{5}$$

Proof. If no subset $\mathcal{G}$ of $\mathcal{H}$ can be found which satisfies the condition in the hypothesis, then in particular, namely, for $\mathcal{G} \equiv \mathcal{H}$, we have that

$$\bigwedge_{A\in\mathcal{H}}\bigvee_{j\in A}\beta_j \not\vdash \psi_h$$

Then, by distributivity we obtain

$$\bigvee_{\phi \in \mathcal{F}} \bigwedge_{A \in \mathcal{H}} \beta_{\phi(A)} \nvdash \psi_h$$

Hence the result is immediate. ■

The function ϕ^* that we pointed out in the previous lemma is useful for finding a suitable subset of indexes $B = \bigcup_{A \in \mathcal{H}} \phi^*(A)$ of the set $\{1, \ldots, n\}$ such that, by induction on the complexity of the considered sequent, a finite model (M'_h, R'_h, ν'_h) can be built which contains a point z_h such that, for any β_i with $i \in B$, $z_h \Vdash \beta_i$ whereas $z_h \Vdash \neg\psi_h$.

Note that to build a finite counter-model for the sequent 2 when $B = \{1, \ldots, n\}$ we need only to build a finite model (M''_h, R''_h, ν''_h) which contains a point y_h such that $y_h \Vdash \phi_h$. Since the sequent 2 is clearly provable if $\vdash \neg\phi_h$, and hence our proof search would have stoped with a proof in this case, we can suppose, by inductive hypothesis, to know how to build such a model.

The next lemma will show how to proceed in building the finite counter-model for the sequent 2 when the set of indexes B is not $\{1, \ldots, n\}$.

LEMMA 2.5. *Suppose* $B \neq \{1, \ldots, n\}$ *and set* $C = \{1, \cdots, n\} \setminus B$. *Then*

$$\phi_h \nvdash \bigvee_{j \in C} \alpha_j \tag{6}$$

Proof. Suppose the sequent $\phi_h \vdash \bigvee_{j \in C} \alpha_j$ is provable. Then $C \in \mathcal{H}$. Consider now the function ϕ^* that we pointed out in the previous lemma 2.4. Then, we get that $\phi^*(C) \in C$ since $\phi^* \in \mathcal{F}$ whereas the very definition of C yields that $\phi^*(C) \notin C$. Contradiction. ■

Thus, by inductive hypothesis, we can build a finite model (M''_h, R''_h, ν''_h) such that there is a point y_h such that for any α_j, with $j \in \{1, \ldots, n\} \setminus B$, $y_h \Vdash \neg\alpha_j$ and $y_h \Vdash \phi_h$.

In order to build a finite counter-model (M_h, R_h, ν_h) for the sequent 2 we can now put together the two models we built and add them a new point x_h, that is,

$$M_h = \{x_h\} \cup M'_h \cup M''_h$$

and define the relation R_h by setting

$$R_h \equiv \{< x_h, y_h, z_h >\} \cup R'_h \cup R''_h$$

and the interpretation ν_h by setting, for any $w \in M_h$ and any propositional variable p,

$$w \in \nu_h(p) \text{ if and only if } w \in \nu'_h(p) \text{ or } w \in \nu''_h(p)$$

Let us go back now to the problem of the proof of the sequent 1 and let us suppose that for no $1 \leq h \leq m$, the corresponding sequent 2 is provable, otherwise we would have the required proof of the sequent 1. Then, for each $1 \leq h \leq m$, we can construct as above the finite models (M'_h, R'_h, ν'_h) and (M''_h, R''_h, ν''_h) with suitable points y_h and z_h. Then in order to build a finite counter-model (M, R, ν) for the sequent 1, it is sufficient to put all of these models together, that is, we have to add a new point x and connect it with all the couple (y_h, z_h). So,

$$\begin{array}{lcl} M & \equiv & \{x\} \cup M'_1 \cup M''_1 \cup \ldots \cup M'_m \cup M''_m \\ R & \equiv & \{\langle x, y_1, z_1 \rangle, \ldots, \langle x, y_m, z_m \rangle\} \cup R'_1 \cup R''_1 \cup \ldots \cup R'_m \cup R''_m \\ \nu(p) & = & \begin{cases} \nu'_1(p) \cup \nu''_1(p) \cup \ldots \nu'_m(p) \cup \nu''_m(p) \cup \{x\} & \text{if } p \in \{p_1, \ldots, p_r\} \\ \nu'_1(p) \cup \nu''_1(p) \cup \ldots \nu'_m(p) \cup \nu''_m(p) & \text{otherwise} \end{cases} \end{array}$$

It is now obvious that the point x falsifies the sequent 1. In fact, for each $p \in \{p_1, \ldots, p_r\}$, $x \Vdash p$ holds by definition of the valuation ν and, for $1 \leq i \leq n$, $x \Vdash \Box(\alpha_i, \beta_i)$ since, for each $1 \leq h \leq m$ and for each y_h and z_h, if $y_h \Vdash \alpha_i$ then $z_h \Vdash \beta_i$. Finally, for no $q \in \{q_1, \ldots, q_s\}$, $x \Vdash q$, again by definition of the valuation ν, and, for each $1 \leq h \leq m$, there are suitable points y_h and z_h in M such that $R(x, y_h, z_h)$ holds and $y_h \Vdash \phi_h$ and $z_h \Vdash \neg\psi_h$ and hence $x \Vdash \neg\Box(\phi_h, \psi_h)$.

3. RELATIONS BETWEEN BK AND $\leq_\wedge$

In this section we will prove that BK furnishes a complete interpretation of the subtype relation between types of $\Lambda_\wedge$. Let us begin by introducing the obvious interpretation of types into modal formulas. To simplify the notation we assume to use the same notation for variables for types of $\Lambda_\wedge$ and propositional variables of BK.

Interpretation

$$\begin{array}{lcl} \mathcal{I}(\alpha) & = & \alpha \text{ for every type variable } \alpha \\ \mathcal{I}(\omega) & = & \top \\ \mathcal{I}(\alpha \wedge \beta) & = & \mathcal{I}(\alpha) \wedge \mathcal{I}(\beta) \\ \mathcal{I}(\alpha \to \beta) & = & \Box(\mathcal{I}(\alpha), \mathcal{I}(\beta)) \end{array}$$

It is immediate to prove the validity of this interpretation.

THEOREM 3.1. *Let α and β be two types of $\Lambda_\wedge$. Then, if $\alpha \leq_\wedge \beta$ then $\mathcal{I}(\alpha) \vdash \mathcal{I}(\beta)$.*

Proof. The proof is almost straghtforward by induction on the length of the proof of $\alpha \leq_\wedge \beta$. The result is almost immediate for the axioms and concerning the rules almost no proof is necessary since the first and the second rule are valid in both the deductive systems and we already proved the third one within the proof of theorem 2.1. ■

To prove the other implication, namely, if $\alpha \not\leq_\wedge \beta$ then $\mathcal{I}(\alpha) \not\Vdash \mathcal{I}(\beta)$, even if a straight proof is possible which uses the Kripke-like semantics of the previous sections, we think that it is more interesting to pass throught the definition of a different semantics both for the subtyping relation and BK.

Let us recall the definition of *applicative structure* and *combinatory algebra* (see [2]).

DEFINITION 3.1. Let A be a set and $\cdot$ be a (partial) operation over A. Then $\mathcal{A} \equiv (A, \cdot)$ is an *applicative structure*. An applicative structure is a *combinatory algebra* if the operation $\cdot$ is total and there are in A two elements K and S such that, for any $x, y, z \in A$, $K \cdot x \cdot y = x$ and $S \cdot x \cdot y \cdot z = x \cdot y \cdot (x \cdot z)$.

It is well known that any combinatory algebra is functionally complete, that is, for any term $t[x_1 \dots x_n, x]$ over A there exists a term $f[x_1 \dots x_n]$ such that $f[x_1 \dots x_n] \cdot x = t[x_1 \dots x_n, x]$. We will indicate the term $f[x_1 \dots x_n]$ by $\Delta x.t$.

Functional completeness can be used to show that, given any combinatory algebra $\mathcal{A}$, any map $\eta : \mathsf{Var} \to A$ can be extended to an interpretation of the terms of the pure lambda calculus Λ into A, which respects β-reduction. In fact we can first define an interpretation $(-)^*$ of λ-terms into terms over A by setting

$$\begin{array}{lcl} (x)^* & = & x \\ (t(u))^* & = & (t)^* \cdot (u)^* \\ (\lambda x.t)^* & = & \Delta x.(t)^* \end{array}$$

and then instantiate such terms by extending η to all terms by setting $\eta(a) \equiv a$, for all element $a \in A$, and $\eta(a \cdot b) \equiv \eta(a) \cdot \eta(b)$.

We will write $(\mathcal{A}, \eta)$ to mean the λ-model built over the combinatory algebra $\mathcal{A}$ by using the interpretation η.

Given any combinatory algebra $\mathcal{A}$, the types of $\Lambda_\wedge$ can be interpreted into subsets of A by extending any map $\nu : \mathsf{BasicType} \to \mathcal{P}(A)$ by setting

$$\begin{array}{rcl} \nu(\omega) & \equiv & A \\ \nu(\alpha \wedge \beta) & \equiv & \nu(\alpha) \cap \nu(\beta) \\ \nu(\alpha \to \beta) & \equiv & \{x \in A |\ (\forall y \in \nu(\alpha))\ x \cdot y \in \nu(\beta)\} \end{array}$$

It is easy to check the following theorem.

THEOREM 3.2. *Let α and β be two types of $\Lambda_\wedge$. Then $\alpha \leq_\wedge \beta$ if and only if for any combinatory algebra $\mathcal{A}$ and any interpretation ν of the types of $\Lambda_\wedge$, $\nu(\alpha) \subseteq \nu(\beta)$.*

A λ-model $(\mathcal{A}, \eta)$ can be combined with an interpretation ν for the types of $\Lambda_\wedge$ in order to obtain a model for the extended intersection type system that was introduced in the end of section 1.

DEFINITION 3.2. Let $(\mathcal{A}, \eta)$ be a λ-model and ν be an interpretation of the types of $\Lambda_\wedge$ into subsets of A. Then a $\Lambda_{\leq_\wedge}$-*model* $(\mathcal{A}, \eta, \nu)$ is defined by setting

$$
\begin{array}{lll}
(\mathcal{A}, \eta, \nu) \vDash c : \alpha & \text{iff} & \eta(c) \in \nu(\alpha) \\
(\mathcal{A}, \eta, \nu) \vDash \Gamma & \text{iff} & \text{for all } (x : \alpha) \in \Gamma, \eta(x) \in \nu(\alpha)
\end{array}
$$

Then

$$\Gamma \vDash c : \alpha \text{ iff for all } \Lambda_{\leq_\wedge}\text{-model } (\mathcal{A}, \eta, \nu) \text{ if } (\mathcal{A}, \eta, \nu) \Vdash \Gamma \text{ then } (\mathcal{A}, \eta, \nu) \Vdash c : \alpha$$

It is now possible to prove the following theorem (see [3]).

THEOREM 3.3. *$\Lambda_{\leq_\wedge}$-models are a valid and complete semantics for the extended intersection type system, that is,*

$$\Gamma \vdash_{\leq_\wedge} c : \alpha \textit{ if and only if } \Gamma \vDash c : \alpha$$

In a similar way, applicative structures can be used to give an interpretation to the formulas of BK. We need only to extend to all of the classical connectives what we already did for $\wedge$.

DEFINITION 3.3. Let $\mathcal{A}$ be an applicative structure. Then, any map ν from the propositional variables of BK into subsets of A can be inductively extended to an interpretation of the formulas of BK by setting:

$$
\begin{array}{lll}
\nu(\top) & = & A \\
\nu(\bot) & = & \emptyset \\
\nu(\alpha \wedge \beta) & = & \nu(\alpha) \cap \nu(\beta) \\
\nu(\alpha \vee \beta) & = & \nu(\alpha) \cup \nu(\beta) \\
\nu(\neg\alpha) & = & A \setminus \nu(\alpha) \\
\nu(\alpha \to \beta) & = & (A \setminus \nu(\alpha)) \cup \nu(\beta) \\
\nu(\Box(\alpha, \beta)) & = & \{x \in A | \text{ for all } y \in \nu(\alpha), \text{ if } x \cdot y \text{ is defined then } x \cdot y \in \nu(\beta)\}
\end{array}
$$

It is not difficult to use the proof of decidability in the previous section in order to obtain a proof of validity and completeness for the semantics of the applicative structures.

THEOREM 3.4. *Let $\Gamma \vdash \Delta$ be any sequent of* BK. *Then $\Gamma \vdash \Delta$ is provable if and only if, for every applicative structure $\mathcal{A}$ and every valuation ν of the formulas of* BK *into A, $\bigcap_{\alpha \in \Gamma} \nu(\alpha) \subseteq \bigcup_{\beta \in \Delta} \nu(\beta)$.*

Proof. We will show the proof of the validity for the only rule whose proof is not completely straightforward, that is, the $\Box$-rule. Let us suppose that $\nu(\alpha) \subseteq$

$\bigcup_{i=1\dots n}\nu(\alpha_i)$ and that $\bigcap_{j=1\dots m}\nu(\beta_j) \subseteq \nu(\beta)$. Then, we have to show that $\bigcap_{i=1\dots n, j=1\dots m}\nu(\Box(\alpha_i,\beta_j)) \subseteq \nu(\Box(\alpha,\beta))$. To this aim, let us suppose that $x \in \bigcap_{i=1\dots n, j=1\dots m}\nu(\Box(\alpha_i,\beta_j))$, that is, for any $i = 1\dots n$, $j = 1\dots m$ and $y \in \alpha_i$, if $x \cdot y$ is defined then it belongs to β_j, and let us assume that $y \in \nu(\alpha)$. Then $\nu(\alpha) \subseteq \bigcup_{i=1\dots n}\nu(\alpha_i)$ yields that there exists $1 \leq i \leq n$ such that $y \in \nu(\alpha_i)$ and hence we obtain that for any $j = 1\dots m$, provided $x \cdot y$ is defined, $x \cdot y \in \nu(\beta_j)$. Hence, provided $x \cdot y$ is defined, $\bigcap_{j=1\dots m}\nu(\beta_j) \subseteq \nu(\beta)$ yields that $x \cdot y \in \nu(\beta)$.

Let us consider now completeness. First observe that the finite counter-model (M, R, ν) that we provided in section 2.3 for any non provable sequent was built in such a way that for every x and y in M there exists at most one element z in M such that $R(x, y, z)$ holds. Then, supposing $\Gamma \nvdash \Delta$, define an applicative structure $\mathcal{A}$ by setting $A \equiv M$ and $x \cdot y = z$ if and only if $R(x, y, z)$. Then it is easy to check that for the interpretation defined by setting, for any propositional variable p, $\nu(p) \equiv \{x \mid x \Vdash p\}$, $\bigcap_{\alpha\in\Gamma}\nu(\alpha) \subseteq \bigcup_{\beta\in\Delta}\nu(\beta)$ holds. ■

It is now possible to estabilish the conservativity theorem that we were looking for.

COROLLARY 3.1. *Suppose* $\mathcal{I}(\alpha) \vdash \mathcal{I}(\beta)$. *Then* $\alpha \leq_\wedge \beta$.

Proof. Let us suppose that $\alpha \not\leq_\wedge \beta$. Then there exists a λ-model $(\mathcal{A}, \nu)$ such that $\nu(\alpha) \not\subseteq \nu(\beta)$. It is easy to check that ν can be extended to a valuation of the formulas of BK such that $\nu(\mathcal{I}(\alpha)) \not\subseteq \nu(\mathcal{I}(\beta))$. Hence the *validity* part of theorem 3.4 shows that $\mathcal{I}(\alpha) \nvdash \mathcal{I}(\beta)$. ■

4. CONCLUSION AND OPEN PROBLEMS

The new semantics that we proposed for BK seems to suggest that this logic extands in a natural way the subtyping relation $\leq_\wedge$. Unfortunately the completeness theorem 3.4 holds only if we let the algebra $\mathcal{A}$ vary over all the applicative structures and not only over the combinatory algebras; indeed the algebras used in the proof of the completeness theorem 3.4 is far from being a combinatory algebra. Thus, a question naturally arises:"Is it possible to define a complete sequent calculus for the sub-logic of BK defined by the set of formulas which are valid in all the combinatory algebras?".

An answer to this question would be a great progress in the direction towards the setting of a type system for the λ-calculus which extends intersection types with a $\vee$ constructor, that is, the syntactical counterpart of the operation of union, and also with a $\neg$ constructor, that is, the counterpart of the operation of complement. In fact, the subtyping relation for this type system would extend with negation the subtyping relation Ξ proposed in [1].

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