Exponentiation of unary topologies over inductively generated formal topologies

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Abstract

We prove that unary formal topologies are exponentiable in the category of inductively generated formal topologies. From an impredicative point of view, this means that algebraic dcpos with a bottom element are exponentiable in the category of open locales.

Contents

1 Introduction

Formal topology is nowadays recognized like one of the main approaches to the development of constructive topology, where by constructive we mean both intuitionistic and predicative. Many results of classical and impredicative topology have been already studied, and found their place in a predicative framework, by using formal topology (see [Sam03] for an updated overview on formal topology). Moreover, the category FTop of formal topologies and continuous relations is a predicative presentation of the category OpLoc of open locales (see [JT84]) and the category FTop^- of formal topologies without positivity predicate is a predicative presentation of the category Loc of locales (see [Joh82]).

In this paper, we begin to study a full sub-category of FTop, that is, the category $\mathsf{FTop_i}$ of the inductively generated formal topologies (see [CSSV03]). We consider such a category instead of FTop because it is predicatively known to be cartesian while FTop is not, even if these categories are equivalent from an impredicative point of view. In particular, we show that unary topologies (see section 2.7) are exponentiable within $\mathsf{FTop_i}$. Our proof is intuitionistically valid but not yet entirely predicative since the co-inductive definition of the positivity predicate for the exponent topology that we propose is, at present, justified only by using Tarski fixed-point theorem. However, as a consequence of our result, one gets an entirely predicative proof that unary formal topologies are exponentiable in the category $\mathsf{FTop_i}^-$ of inductively generated formal topologies without the positivity predicate. Since, from an impredicative point of view, unary topologies essentially correspond to algebraic dcpos with a bottom element, our result states that algebraic dcpos are exponentiable in OpLoc.

The question of characterizing exponentiable topologies has a long history in the development of topology. It is well known that the category Top of topological spaces and continuous functions is not cartesian closed. In fact, the topological spaces that can be exponentiated in Top are only those whose frames of open sets are locally compact locales (for an overview on the topic see [EHar]). This result was reproduced by Hyland in the context of the intuitionistic but impredicative theory of locales by showing that in Loc only the locally compact locales can be exponentiated [Hyl81]. Later, his proof of exponentiation was adapted to the language of formal topology, but still working within an impredicative setting (see [Sig95]). More recently, Vickers reproduced most of Hyland's results by using geometric reasoning (see [Vic01]).

We think that our main contribution in proving exponentiation of unary topologies in $\mathsf{FTop_i}$ is a detailed analysis of the conditions characterizing continuous relations between a unary formal topology and an inductively generated one. In fact, after this analysis, the axioms defining the cover on the exponent topology and its corresponding positivity predicate emerge naturally.

In order to obtain a self contained paper we decided to begin our presentation with a section containing the main definitions and results on formal topologies and their maps. Indeed, it is still difficult to find a complete introduction to the topic and some of the definitions and results appear here for the first time.

2 Formal topologies and their morphisms

In this section the basic definitions of formal topology will be quickly recalled. The reader interested to have more details on formal topology and a deeper analysis of the foundational motivations for the formal development of topology within Martin-Löf's constructive type theory [NPS90, Mar84] is invited to look, for instance, at the updated overview in [Sam03].

2.1 Concrete topological spaces

We start by recalling how to describe predicatively a topological space. Let X be a set. Then $(X, \Omega(X))$ is a topological space if $\Omega(X)$ is a subset of $\mathcal{P}(X)$ which contains $\emptyset$ and X and is closed under finite intersection and under arbitrary union. The quantification implicitly used in this last condition is of the third order, since it says that, for all $F \subseteq \Omega(X)$, $\bigcup F \in \Omega(X)$. We can "go down" one step by thinking of $\Omega(X)$ as a family of subsets indexed by a set S through a map $\mathsf{ext} : S \to \mathcal{P}(X)$. Indeed, we can now quantify on S rather than on $\Omega(X)$. But, we have to say that, for all $U \in \mathcal{P}(S)$ there exists $c \in S$ such that $\cup_{a \varepsilon U} \mathsf{ext}(a) = \mathsf{ext}(c)$, which is still impredicative[1]. We can "go down" another step by defining opens to be of the form $\mathsf{Ext}(U) \equiv \cup_{a\varepsilon U} \mathsf{ext}(a)$ for an arbitrary subset U of S. In this way $\emptyset$ is open, because $\mathsf{Ext}(\emptyset) = \emptyset$, and closure under union is automatic, because obviously $\cup_{i \in I} \mathsf{Ext}(U_i) = \mathsf{Ext}(\cup_{i\in I} U_i)$. So, all we

[1] All the set-theoretical notions that we use conform to the subset theory for Martin-Löf's type theory as presented in [SV97]. In particular, we use the symbol $\in$ for the membership relation between an element and a set or a collection and ε for the membership relation between an element and a subset, which is never a set but a propositional function, so that $a \varepsilon U$ holds if and only if $U(a)$ holds.

have to do is to require that $\mathsf{Ext}(S)$ is the whole X and closure under finite intersections, that is,

$$(*) \quad (\forall a, b \in S)(\forall x \in X)\ (x \varepsilon \mathsf{ext}(a) \cap \mathsf{ext}(b) \rightarrow \\ (\exists c \in S)\ (x \varepsilon \mathsf{ext}(c)\ \&\ \mathsf{ext}(c) \subseteq \mathsf{ext}(a) \cap \mathsf{ext}(b)))$$

It is not difficult to realize that this amounts to the standard definition saying that $\{\mathsf{ext}(a) \subseteq X \mid a \in S\}$ is a base (see for instance [Eng77]).

We can make $(*)$ a bit shorter by introducing the abbreviation

$$a \downarrow b \equiv \{c : S \mid \mathsf{ext}(c) \subseteq \mathsf{ext}(a)\ \&\ \mathsf{ext}(c) \subseteq \mathsf{ext}(b)\}$$

so that it becomes $(\forall a, b \in S)\ \mathsf{ext}(a) \cap \mathsf{ext}(b) \subseteq \mathsf{Ext}(a \downarrow b)$. Now, note that $c \varepsilon a \downarrow b$ implies that $\mathsf{ext}(c) \subseteq \mathsf{ext}(a) \cap \mathsf{ext}(b)$, so that $\mathsf{Ext}(a \downarrow b) \equiv \cup_{c \varepsilon a \downarrow b} \mathsf{ext}(c) \subseteq \mathsf{ext}(a) \cap \mathsf{ext}(b)$. Thus we arrived at the definition of concrete topological space.

Definition 2.1 (Concrete topological space) *A* concrete topological space *is a triple* $\mathcal{X} \equiv (X, S, \mathsf{ext})$ *where* X *and* S *are sets and* ext *is a map from* S *to* $\mathcal{P}(X)$ *satisfying:*

$$(B_1) \quad X = \mathsf{Ext}(S)$$

$$(B_2) \quad (\forall a, b \in S)\ \mathsf{ext}(a) \cap \mathsf{ext}(b) = \mathsf{Ext}(a \downarrow b)$$

2.2 Formal topologies

The notion of formal topology arises by describing, as well as possible, the structure induced by a concrete topological space (X, S, ext) on the set S, and then by taking the result as an axiomatic definition. The reason for such a move is that the definition of concrete topological space is too restrictive given that in the most interesting cases of topological space we do not have, from a constructive point of view, a *set* of points to start with[2].

Since the elements in S are *names* for the basic opens of the topology on X, and any open set is the union of basic opens, we can specify an open set A by using the subset U_A of all the (names of the) basic opens which are used to form it, that is, $A = \mathsf{Ext}(U_A)$. However, it is clear that in general it is well possible that two different subsets of S have the same extension. Thus, we don't have a bijective correspondence between concrete opens and subsets of S and we need to introduce an equivalence relation if we want to obtain it. What we need is a relation which identifies the subsets U and V when $\mathsf{Ext}(U) = \mathsf{Ext}(V)$. The following lemma gives the correct hint.

Lemma 2.2 *Let* U *and* V *be subsets of* S. *Then* $\mathsf{Ext}(U) = \mathsf{Ext}(V)$ *if and only if, for all* $a \in S$, $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U) \leftrightarrow \mathsf{ext}(a) \subseteq \mathsf{Ext}(V)$.

[2] Here we commit ourselves to Martin-Löf's constructive set theory; hence we distinguish between sets, which can be inductively generated, and collections.

Thus, in order to define the equivalence relation among subsets of S that we are looking for, we need to introduce a new proposition $a \lhd U$ between an element a and a subset U of S whose intended meaning is that $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$. Indeed, provided we are able to formalize such a relation with no reference to the elements of X, we can define the equivalence relation over the subsets of S that we are looking for by putting

$$U =_{\lhd} V \equiv (\forall a \in S)\ a \lhd U \leftrightarrow a \lhd V$$

Now, we can simply state that a *formal open* is the "fullest" among the subsets which have the same extension, that is, for any subset U, we choose

$$\lhd(U) \equiv \{a \in S \mid a \lhd U\}$$

In fact, it is possible prove that $\lhd(U) =_{\lhd} U$ by using the following valid conditions on $\lhd$:

$$\text{(reflexivity)}\ \frac{a \varepsilon U}{a \lhd U} \qquad \text{(transitivity)}\ \frac{a \lhd U \qquad U \lhd V}{a \lhd V}$$

where $U \lhd V$ is a short-hand for a derivation of $u \lhd V$ under the assumption that $u \varepsilon U$.

Thus, we found a relation, that is, $\lhd$, and two conditions over it, that is, *reflexivity* and *transitivity*, which allow to deal with concrete open subsets by using only the subsets of S. But these conditions are not sufficient to describe completely the concrete situation; for instance there is no condition which describe formally the conditions (B_1) and (B_2).

Now, (B_1) states that, for any $x \in X$, there exists an element $a \in S$ such that $x \varepsilon \mathsf{ext}(a)$. The easiest way to meet such a condition is to require that there is an element $\top$ in S such that $X = \mathsf{ext}(\top)$. It is clear that such a condition is not necessary, but, on the other hand, it does introduce no real constrain on any concrete topological space. We can now formulate (B_1) within the formal side by requiring that, for any $a \in S$,

$$a \lhd \{\top\}$$

To formulate (B_2) within the formal side, we can use the fact that

$$\mathsf{Ext}(U) \cap \mathsf{Ext}(V) \subseteq \mathsf{Ext}(U \downarrow V)$$

where $U \downarrow V \equiv \{a \in S \mid ((\exists u \varepsilon U)\ \mathsf{ext}(a) \subseteq \mathsf{ext}(u))\ \&\ ((\exists v \varepsilon V)\ \mathsf{ext}(a) \subseteq \mathsf{ext}(v))\}$. Now, let us suppose $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U)$ and $\mathsf{ext}(a) \subseteq \mathsf{Ext}(V)$, then we immediately obtain $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U) \cap \mathsf{Ext}(V)$ and hence $\mathsf{ext}(a) \subseteq \mathsf{Ext}(U \downarrow V)$. Its formal counterpart is

$$(\downarrow\text{-right})\qquad \frac{a \lhd U \qquad a \lhd V}{a \lhd U \downarrow V}$$

where $U \downarrow V \equiv \{c \in S \mid (\exists u \varepsilon U)\ c \lhd \{u\}\ \&\ (\exists v \varepsilon V)\ c \lhd \{v\}\}$.

To express constructively the fact that a basic open subset is inhabited it is convenient[3] to introduce also a second primitive predicate, called $\mathsf{Pos}(-)$, on the elements of S. Its intended meaning is that, for any $a \in S$, $\mathsf{Pos}(a)$ holds if and only if there exists $x \in X$ such that $x \varepsilon \mathsf{ext}(a)$. We require the following conditions on this predicate.

$$\text{(monotonicity)} \quad \frac{\mathsf{Pos}(a) \qquad a \lhd U}{(\exists u \varepsilon U)\ \mathsf{Pos}(u)}$$

$$\text{(positivity)} \quad \frac{\mathsf{Pos}(a) \to a \lhd U}{a \lhd U}$$

While the meaning of *monotonicity* is obvious and the proof of its validity in any concrete topological space is immediate, *positivity* may require some explanation. It states two things in one condition: first, that a not-inhabited basic open subset is covered by any subset, second that proof by cases on the positivity of a are valid when the conclusion is $a \lhd U$ (see [SVV96]). The proof of validity of *positivity* is straightforward and it uses only intuitionistic logic.

It is worth noting that, provided that there exists a positive element $a \in S$, *monotonicity* yields $\mathsf{Pos}(\top)$ since $a \lhd \{\top\}$.

We thus arrived at the main definition.

Definition 2.3 (Formal topology) *A* formal topology *is a structure* $\mathcal{S} \equiv (S, \top, \lhd, \mathsf{Pos})$ *where S is a set, $\top$ is a distinguished element of S, $\lhd$ is an infinitary relation, called* cover *relation, between elements and subsets of S satisfying the following conditions:*

$$\text{(top-element)} \quad a \lhd \{\top\}$$

$$\text{(reflexivity)} \quad \frac{a \varepsilon U}{a \lhd U}$$

$$\text{(transitivity)} \quad \frac{a \lhd U \qquad U \lhd V}{a \lhd V}$$

$$(\downarrow\text{-right}) \quad \frac{a \lhd U \qquad a \lhd V}{a \lhd U \downarrow V}$$

and Pos *is a predicate over S satisfying the following conditions:*

$$\text{(monotonicity)} \quad \frac{\mathsf{Pos}(a) \qquad a \lhd U}{(\exists u \varepsilon U)\ \mathsf{Pos}(u)}$$

$$\text{(positivity)} \quad \frac{\mathsf{Pos}(a) \to a \lhd U}{a \lhd U}$$

It is useful to recall the following equivalent formulations of the *positivity* condition that we will often use in the next sections (see [Sam87]). To state them, given any predicate $\mathsf{Pos}(-)$ over elements of S and any subset U of S, we write U^+ to mean the subset $\{x \in S \mid x \varepsilon U \ \&\ \mathsf{Pos}(x)\}$.

[3] All what appears from here on can be developed as well with no reference to the positivity predicate that we introduce now. The reader who prefers to work without it can just skip the parts where it appears, or, better, to specialize all the results to the case of an always true positivity predicate.

Proposition 2.4 *Let S be a set, $\lhd$ be a relation between elements and subsets of S which satisfies* reflexivity *and* transitivity *and* Pos *be a predicate on elements of S. Then, the following conditions are equivalent:*

1. (positivity) *for any $a \in S$ and $U \subseteq S$, $\mathsf{Pos}(a) \rightarrow a \lhd U$ yields $a \lhd U$;*
2. (axiom positivity) *for any $a \in S$, $a \lhd \{a\}^+$;*
3. (cover positivity) *for any $a \in S$ and $U \subseteq S$, $a \lhd U$ yields $a \lhd U^+$.*

The first consequence of the previous proposition is the following theorem which shows that the cover relation uniquely determines the positivity predicate.

Theorem 2.5 *Let $- \lhd -$ be a cover relation over a set S and $\mathsf{Pos}_1(-)$ and $\mathsf{Pos}_2(-)$ be two positivity predicates with respect to such a cover. Then $\mathsf{Pos}_1(-)$ and $\mathsf{Pos}_2(-)$ are equivalent, namely, for any $a \in S$, $\mathsf{Pos}_1(a)$ if and only if $\mathsf{Pos}_2(a)$.*

Proof. By the *positivity axiom*, for every $a \in S$, $a \lhd a^{+_2}$, where a^{+_2} is a shorthand for the subset $\{x \in S \mid x = a \ \& \ \mathsf{Pos}_2(x)\}$. Now, let us assume $\mathsf{Pos}_1(a)$; then, by *monotonicity*, there exists $x \varepsilon a^{+_2}$ such that $\mathsf{Pos}_1(x)$ holds. But $x \varepsilon a^{+_2}$ means both that $x = a$ and $\mathsf{Pos}_2(x)$ hold and hence $\mathsf{Pos}_2(a)$ follows. Thus, by discharging the assumption $\mathsf{Pos}_1(a)$, we proved that $\mathsf{Pos}_1(a)$ yields $\mathsf{Pos}_2(a)$. In a completely analogous way we can prove the other implication.

The definition of formal topology that we recalled here is almost identical to the one in [CSSV03]. In fact, the only difference rests in the presence of the top element and the corresponding axiom. In general many of the results in the paper can be obtained also by using the definition in [CSSV03]. However, the use of the top element seems to be unavoidable when dealing with exponentiation; anyhow we will try to restrict its usage only to the cases where we think that it is convenient or necessary.

On the other hand, the definition that we proposed here differs more deeply from the one in [Sam87]; indeed there the notion of formal topology is introduced starting from a base closed under a monoid operation which is then lifted at the level of subsets in order to describe the intersection between open subsets. In any case, both definitions allow a predicative presentation of frames.

2.3 Formal points

When working in formal topology one is in general interested to those properties of a concrete topological space (X, S, ext) which make no reference to the elements of X. Thus, (s)he is dispensed of the collection X and it is possible to work by using the set S only. But this does not mean that points are out of reach. In fact, a point $x \in X$ can be identified with the filter of the basic opens that, in the concrete case, contain x itself. So, we can associate to any $x \in X$, the following subset of S

$$\alpha_x \equiv \{a \in S \mid x \varepsilon \mathsf{ext}(a)\}$$

Now, note that from a topological point of view we can *see* only those points which can be distinguished by using the open sets and hence we are led to identify the concrete point x with the subset α_x.

If we want to move to the formal side, we have to find those properties which characterize such subsets and are expressible in our language. Here we point out the following ones:

$$\begin{array}{ll}
\text{(point inhabitance)} & (\exists a \in S)\ a \varepsilon \alpha_x \\[1ex]
\text{(point convergence)} & \dfrac{a \varepsilon \alpha_x \quad b \varepsilon \alpha_x}{(\exists c \varepsilon a \downarrow b)\ c \varepsilon \alpha_x} \\[1ex]
\text{(point splitness)} & \dfrac{a \varepsilon \alpha_x \quad \mathsf{ext}(a) \subseteq \mathsf{Ext}(U)}{(\exists u \varepsilon U)\ u \varepsilon \alpha_x} \\[1ex]
\text{(point positivity)} & \dfrac{a \varepsilon \alpha_x}{(\exists x \in X)\ x \varepsilon \mathsf{ext}(a)}
\end{array}$$

In fact, *point inhabitance* is an obvious corollary of the condition B_1, *point convergence* is an immediate consequence of the condition B_2, and *point splitness* and *point positivity* follows by logic. Thus, we are led to the following definition.

Definition 2.6 (Formal point) *Let* $(S, \lhd, \mathsf{Pos})$ *be a formal topology. Then an inhabited subset* α *of* S *is a* formal point *if, for any* $a, b \in S$ *and any* $U \subseteq S$, *it satisfies the following conditions:*

$$\begin{array}{ll}
\text{(point convergence)} & \dfrac{a \varepsilon \alpha \quad b \varepsilon \alpha}{(\exists c \varepsilon a \downarrow b)\ c \varepsilon \alpha} \\[1ex]
\text{(point splitness)} & \dfrac{a \varepsilon \alpha \quad a \lhd U}{(\exists u \varepsilon U)\ u \varepsilon \alpha}
\end{array}$$

As observed by Peter Aczel, we can avoid to require the condition of *point positivity*, namely, that $\mathsf{Pos}(a)$ is a consequence of $a \varepsilon \alpha$, since it can be proved by using *point splitness*. In fact, we know that $a \lhd a^+$ and hence if, for some point α, $a \varepsilon \alpha$ then *point splitness* shows that there exists some element x in a^+ such that $x \varepsilon \alpha$. Then $x = a$ and $\mathsf{Pos}(x)$ hold and hence $\mathsf{Pos}(a)$ follows.

It is worth noting that, by *top-element* and *point splitness*, the requirement that a point is inhabited is equivalent to the following condition:

$$\text{(point inhabitance)} \quad \top \varepsilon \alpha$$

In the following we call $\mathsf{Pt}(S)$ the collection of formal points of the formal topology $\mathcal{S}$. We can give $\mathsf{Pt}(S)$ the structure of a topological space if we mimic the situation of a concrete topological space even if $\mathsf{Pt}(S)$ is a collection and not a set. So, let us set, for any $a \in S$,

$$\mathsf{ext}^{Pt}(a) \equiv \{\alpha \in \mathsf{Pt}(S) \mid a \varepsilon \alpha\}$$

and use the set-indexed family $(\mathsf{ext}^{Pt}(a))_{a \in S}$ as a base for a topology on $\mathsf{Pt}(S)$.

2.4 Continuous relations

In this section we report and explain the conditions defining continuous relations between formal topologies. The notion of continuous relation essentially goes back to the notion of frame morphism in [Sam87]. The explanations motivating the definition of continuous relations are based on the work by Valentini and Virgili [Vir90] in collaboration with Sambin and, later, with Gebellato.

A map between the topological space $\mathcal{X}$ and the topological space $\mathcal{Y}$ is a function $\phi : X \to Y$ such that, for any basic open subset $\mathcal{B}$ in $\mathcal{Y}$, the subset $\phi^{-1}(\mathcal{B}) \equiv \{x \in X \mid \phi(x) \in \mathcal{B}\}$ is an open set of $\mathcal{X}$. If we write this condition for the concrete topological spaces (X, S, ext_1) and (Y, T, ext_2) we obtain that the condition for a function $\phi : X \to Y$ to be continuous becomes

$$(\forall b \in T)(\exists U \subseteq S)\ \phi^{-1}(\mathsf{ext}_2(b)) = \mathsf{Ext}_1(U)$$

There is only one possible constructive meaning for this sentence, that is, there exists a map $\overleftarrow{F}: T \to \mathcal{P}(S)$ such that, for any $b \in T$, $\mathsf{Ext}_1(\overleftarrow{F}(b))$ is equal to $\phi^{-1}(\mathsf{ext}_2(b))$. Since $\mathsf{Ext}_1(\{a \in S \mid \mathsf{ext}_1(a) \subseteq \phi^{-1}(\mathsf{ext}_2(b))\})$ is always contained in $\phi^{-1}(\mathsf{ext}_2(b))$, the continuity requirement rests in the fact that $\phi^{-1}(\mathsf{ext}_2(b))$ is contained in $\mathsf{Ext}_1(\{a \in S \mid \mathsf{ext}_1(a) \subseteq \phi^{-1}(\mathsf{ext}_2(b))\})$. Hence, the best possible definition is to state that $\overleftarrow{F}(b)$ is the subset of all the basic opens $a \in S$ such that $\mathsf{ext}_1(a)$ is contained in $\phi^{-1}(\mathsf{ext}_2(b))$, that is, the image through ϕ of any point in the basic open $\mathsf{ext}_1(a)$ is in the basic open $\mathsf{ext}_2(b)$. Thus, the formal counterpart of a continuous function ϕ between X and Y is a relation F between elements of S and elements of T such that $a\ F\ b$ holds if and only if $a \varepsilon \overleftarrow{F}(b)$. So to find a completely formal characterization of the notion of continuous function between topological spaces we have to express the condition above with no reference to the elements of X and Y.

In solving this problem we will use also an equivalent formulation of continuity, namely, that a function ϕ between the concrete topological spaces (X, S, ext_1) and (Y, T, ext_2) is continuous if and only if,

$$\begin{array}{l}(\forall b \in T)(\forall x \in X)\ \phi(x) \varepsilon \mathsf{ext}_2(b) \to \\ \qquad (\exists a \in S)\ x \varepsilon \mathsf{ext}_1(a)\ \&\ (\forall z \in X)\ z \varepsilon \mathsf{ext}_1(a) \to \phi(z) \varepsilon \mathsf{ext}_2(b)\end{array}$$

that can by simplified in

$$(\forall b \in T)(\forall x \in X)\ \phi(x) \varepsilon \mathsf{ext}_2(b) \to (\exists a \in S)\ x \varepsilon \mathsf{ext}_1(a)\ \&\ aFb$$

provided that $-F-$ is the relation associated to ϕ that we want to characterize.

Now we look for suitable conditions, that do not rely on the presence of the set of concrete points in order to be formulated, and express that the relation F is the formal counterpart of a continuous function. To achieve this result we will proceed as follows. First, we will define a function ϕ_F from $\mathsf{Pt}(S)$ to $\mathsf{Pt}(T)$ associated with the relation F. Then, we will look for the conditions on F which are both expressible in the language of formal topologies and allow to prove that ϕ_F is a continuous function from $\mathsf{Pt}(S)$ to $\mathsf{Pt}(T)$. And finally, we will check the validity of such conditions in every concrete topological space.

So, let us suppose that F is a relation between two formal topologies. Then we want to define a continuous map ϕ_F from $\mathsf{Pt}(S)$ to $\mathsf{Pt}(T)$ such that $a\ F\ b$ holds if and only if, for any formal point $\alpha \in \mathsf{Pt}(S)$, if $\alpha \in \mathsf{ext}_1^{Pt}(a)$ then $\phi_F(\alpha) \in \mathsf{ext}_2^{Pt}(b)$.

An immediate consequence of this requirement is that if aFb and $a\varepsilon\alpha$ then $\phi_F(\alpha) \in \mathsf{ext}_2^{Pt}(b)$. Now, $a\varepsilon\alpha$ means that $\alpha \in \mathsf{ext}_1^{Pt}(a)$ and $\phi_F(\alpha) \in \mathsf{ext}_2^{Pt}(b)$ means that $b\varepsilon\phi_F(\alpha)$. Hence, provided that we write $\overrightarrow{F}(a)$ to mean the subset $\{b \in T \mid aFb\}$, we have that

$$\bigcup_{a\varepsilon\alpha} \overrightarrow{F}(a) \subseteq \phi_F(\alpha)$$

On the other hand, the continuity of ϕ_F means that

$$(\forall b \in T)(\forall \alpha \in \mathsf{Pt}(X))\ \phi_F(\alpha)\varepsilon\mathsf{ext}_2^{Pt}(b) \rightarrow (\exists a \in S)\ \alpha\varepsilon\mathsf{ext}_1^{Pt}(a)\ \&\ aFb$$

and hence if $b\varepsilon\phi_F(\alpha)$ then there exists $a\varepsilon\alpha$ such that aFb, that is,

$$\phi_F(\alpha) \subseteq \bigcup_{a\varepsilon\alpha} \overrightarrow{F}(a)$$

Thus, we are forced to the following definition

$$\phi_F(\alpha) \equiv \bigcup_{a\varepsilon\alpha} \overrightarrow{F}(a)$$

Note that this definition guarantees that, if ϕ_F is a function from $\mathsf{Pt}(S)$ to $\mathsf{Pt}(T)$, then it is continuous. Hence, we only have to look for the conditions which make ϕ_F be a function between formal points, that is, the image $\phi_F(\alpha)$ of a formal point α of S is a formal point of T.

To begin with, we have to prove that $\phi_F(\alpha)$ is inhabited, namely, that there exists $b \in T$ such that, for some $a\varepsilon\alpha$, aFb holds. Now, we know that the point α is inhabited and hence in order to obtain the result it is sufficient to require

$$\text{(function totality)} \quad (\forall a \in S)(\exists b \in T)\ aFb$$

Indeed, suppose $c\varepsilon\alpha$. Then $(\forall a \in S)(\exists b \in T)\ aFb$ yields trivially that there exists $b \in T$ such that cFb. Now, we have to check that *function totality* is valid for any concrete topological space. So, let us assume that (X, S, ext_1) and (Y, T, ext_2) are two concrete topological spaces, ϕ is a continuous map from X to Y and F is a relation between S and T such that aFb if and only in $(\forall x \in X)\ x\varepsilon\mathsf{ext}_1(a) \rightarrow \phi(x)\varepsilon\mathsf{ext}_2(b)$. Then we have to show that, for all $a \in S$, there exists $b \in T$ such that, for all $x \in X$, if $x\varepsilon\mathsf{ext}_1(a)$ then $\phi(x)\varepsilon\mathsf{ext}_2(b)$. Since we assumed that each base contains a top element, the easiest way to satisfy *function totality* is by considering the top element of T. Indeed, for any element $y \in Y$, $y\varepsilon\mathsf{ext}_2(\top_T)$, and hence in particular $\phi(x)\varepsilon\mathsf{ext}_2(\top_T)$. Later we will show an equivalent condition whose justification does not require the presence of the top element.

The second condition that we have to verify is that, supposing $b\varepsilon\phi_F(\alpha)$ and $d\varepsilon\phi_F(\alpha)$, there exists $k\varepsilon b\downarrow d$ such that $k\varepsilon\phi_F(\alpha)$. To obtain this result it is sufficient to require the following two conditions:

$$\text{(function weak-saturation)} \quad \frac{a \lhd_S c \qquad cFb}{aFb}$$

$$\text{(function convergence)} \quad \frac{aFb \qquad aFd}{a \lhd_S F^-(b\downarrow d)}$$

where, for any subset V of T, $F^-(V) \equiv \{c \in S \mid (\exists v\varepsilon V)\ cFv\}$. In fact $b\varepsilon\phi_F(\alpha)$ and $d\varepsilon\phi_F(\alpha)$ yield that there are $a\varepsilon\alpha$ and $c\varepsilon\alpha$ such that aFb and cFd, and hence by *point convergence* there is also $e\varepsilon a\downarrow c$, namely, $e \lhd a$ and $e \lhd c$, such that $e\varepsilon\alpha$. So, by using *function weak-saturation*, we obtain both eFb and eFd, which, by *function convergence*, yield $e \lhd F^-(b\downarrow d)$. Then, by *point splitness*, $(\exists h\varepsilon F^-(b\downarrow d))\ h\varepsilon\alpha$, that is, there exists $k\varepsilon b\downarrow d$ such that $k\varepsilon\phi_F(\alpha)$. Also in this case it is necessary to check that the two required conditions are valid. In fact, it is easy to check that the following generalization of *weak-saturation*

$$\text{(function saturation)} \quad \frac{a \lhd_S W \qquad (\forall w\varepsilon W)\ wFb}{aFb}$$

is an immediate consequence, by intuitionistic logic, of the condition linking F and ϕ_F. Thus, let us prove the validity of *function convergence*. Suppose $x \in X$ and $x\varepsilon\mathsf{ext}_1(a)$, then aFb yields $\phi(x)\varepsilon\mathsf{ext}_2(b)$ and aFd yields $\phi(x)\varepsilon\mathsf{ext}_2(d)$; then, by (B_2), there exists $k\varepsilon b\downarrow d$ such that $\phi(x)\varepsilon\mathsf{ext}_2(k)$. Finally, continuity of ϕ yields that there exists $h \in S$ such that $x\varepsilon\mathsf{ext}_1(h)$ and hFk, that is, $h\varepsilon F^-(b\downarrow d)$.

The third condition for $\phi_F(\alpha)$ being a formal point is that, if $b\varepsilon\phi_F(\alpha)$ and $b \lhd V$, then there exists $v\varepsilon V$ such that $v\varepsilon\phi_F(\alpha)$. The necessary condition is

$$\text{(function continuity)} \quad \frac{aFb \qquad b \lhd_T V}{a \lhd_S F^-(V)}$$

Indeed, $b\varepsilon\phi_F(\alpha)$ yields that there is $a\varepsilon\alpha$ such that aFb and hence *function continuity*, together with *point splitness*, yields that there exists $c\varepsilon F^-(V)$ that is also an element of α, i.e., there is $v\varepsilon V$ such that cFv and $c\varepsilon\alpha$. The proof of validity of this condition is immediate. Indeed, suppose that both aFb and $b \lhd V$ hold. Then, for all $x \in X$, $x\varepsilon\mathsf{ext}_1(a)$ yields $\phi(x)\varepsilon\mathsf{ext}_2(b)$ and, for all $y \in Y$, $y\varepsilon\mathsf{ext}_2(b)$ yields that there exists $v\varepsilon V$ such that $y\varepsilon\mathsf{ext}_2(v)$. Thus, for any $x\varepsilon\mathsf{ext}_1(a)$, there is $v\varepsilon V$ such that $\phi(x)\varepsilon\mathsf{ext}_2(v)$ and hence, by continuity of ϕ, there is $c \in S$ such that $x\varepsilon\mathsf{ext}_1(c)$ and cFv.

So, we have finished to look for the conditions that make ϕ_F a well-defined function between $\mathsf{Pt}(S)$ and $\mathsf{Pt}(T)$. Hence, we can give the following definition of continuous relation between formal topologies.

Definition 2.7 (Continuous relation) *Suppose that $\mathcal{S} = (S, \top_S, \lhd_S, \mathsf{Pos}_S)$ and $\mathcal{T} = (T, \top_T, \lhd_T, \mathsf{Pos}_T)$ are two formal topologies. Then a* continuous relation *between $\mathcal{S}$ and $\mathcal{T}$ is a binary proposition aFb, for $a \in S$ and $b \in T$, such* *that the following conditions are satisfied:*

(function totality)	$(\forall x \in S)(\exists y \in T)\ xFy$
(function convergence)	$\dfrac{aFb \qquad aFd}{a \lhd_S F^-(b \downarrow d)}$
(function saturation)	$\dfrac{a \lhd_S W \quad (\forall w \varepsilon W)\ wFb}{aFb}$
(function continuity)	$\dfrac{aFb \qquad b \lhd_T V}{a \lhd_S F^-(V)}$

Note that the definition of continuous relation above is obtained from the definition of frame morphism expressed in terms of relation in [Sam87] by taking the opposite relation and adapting the condition of *function convergence* to our setting.

An immediate consequence of the definition is the following lemma.

Lemma 2.8 *Let $\mathcal{S}$ and $\mathcal{T}$ be formal topologies and F be a continuous relation between them. Then, if $V \lhd_T W$ then $F^-(V) \lhd_S F^-(W)$.*

Proof. Suppose $x \varepsilon F^-(V)$. Then, there exists $v \varepsilon V$ such that xFv and hence $V \lhd W$ yields immediately $x \lhd F^-(W)$ by *function continuity.*

Let us recall here also the following standard result on relation composition.

Lemma 2.9 *Let $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{U}$ be three formal topologies, and F and G be relations between S and T and T and U respectively. Then, for any $W \subseteq U$,*

$$F^-(G^-(W)) = (G \circ F)^-(W)$$

where by $G \circ F$ we mean the operation of composition between relations, namely, $s\ G \circ F\ u$ holds if and only if there exists $t \in T$ such that $s\ F\ t$ and $t\ G\ u$.

We want to prove now that formal topologies form a category with respect to continuous relations. The main problem is to define a suitable operation of composition between continuous relations. The first and naive idea is of course to define composition of continuous relations as relation composition but unfortunately relation composition of two continuous relations is not continuous because in general it does not satisfy *function saturation.* Indeed, we can prove only the following lemma.

Lemma 2.10 *Given a continuous relation F between S and T and a continuous relation G between T and U, $G \circ F$ satisfies* function totality*,* function convergence *and* function continuity*.*

Proof. Let us check that the various conditions hold.

- (function totality) Let $x \in S$. Then, by *function totality* for F, there exists $y \in T$ such that $x\,F\,y$ holds. Then, by *function totality* for G, there exists $z \in U$ such that $y\,G\,z$ holds. Hence, $x\,G \circ F\,z$ follows.

- (function convergence) Suppose $a\ G \circ F\ c_1$ and $a\ G \circ F\ c_2$. Then, there exist $b_1, b_2 \in T$ such that $a\ F\ b_1\ \&\ b_1\ G\ c_1$ and $a\ F\ b_2\ \&\ b_2\ G\ c_2$. Thus $a \lhd F^-(b_1 \downarrow b_2)$. So $a \lhd (G \circ F)^-(c_1 \downarrow c_2)$ follows by *transitivity* since $F^-(b_1 \downarrow b_2) \lhd (G \circ F)^-(c_1 \downarrow c_2)$. Indeed, $b_1 \downarrow b_2 \lhd G^-(c_1 \downarrow c_2)$ holds because for every $x \varepsilon b_1 \downarrow b_2$ we get $x\ G\ c_1$ and $x\ G\ c_2$ by *weak-saturation* from $b_1\ G\ c_1$ and $b_2\ G\ c_2$. Hence we can obtain $F^-(b_1 \downarrow b_2) \lhd (G \circ F)^-(c_1 \downarrow c_2)$ by applying first lemma 2.8 and then lemma 2.9.

- (function continuity) Suppose $a\ G \circ F\ c$ and $c \lhd W$. Then there exists $y \in T$ such that $a\ F\ y$ and $y\ G\ c$ and hence $y \lhd G^-(W)$ by *continuity* of G. But this yields $a \lhd (G \circ F)^-(W)$ by *continuity* of F since for any $W \subseteq U$, $F^-(G^-(W)) = (G \circ F)^-(W)$ by lemma 2.9.

The following proposition can be used to fix the problem of the missing condition.

Proposition 2.11 *Let $\mathcal{S}$ and $\mathcal{T}$ be two formal topologies and suppose that F is a relation which satisfies all the conditions for a continuous relation except for* saturation *which is replaced by* weak-saturation*. Then*

$$aF^{\lhd}b \equiv a \lhd_S \{c \in S \mid cFb\}$$

is the minimal continuous relation which extends F.

Proof. In the proof we will often use the fact that, for any $W \subseteq T$,

$$F^-(W) \subseteq (F^{\lhd})^-(W)$$

which can be proved as follows. Suppose that aFb, then $aF^{\lhd}b$ follows because aFb yields $a\varepsilon\{c \in S \mid cFb\}$ and hence $a \lhd \{c \in S \mid cFb\}$ follows by *reflexivity.* Now, suppose $s\varepsilon F^-(W)$; then there exists $w\varepsilon W$ such that sFw and so $sF^{\lhd}w$ follows, that is, $s\varepsilon(F^{\lhd})^-(W)$.

Now, let us check that all of the conditions for $F^{\lhd}$ being a continuous relation hold.

- (function totality) Let $x \in S$. Then, by *function totality* for F, there is $y \in T$ such that xFy. Hence the result follows immediately since xFy yields $xF^{\lhd}y$.

- (function convergence) Let us suppose $a\ F^{\lhd}\ b$ and $a\ F^{\lhd}\ d$. This means that $a \lhd \{c \in S \mid cFb\}$ and $a \lhd \{e \in S \mid eFd\}$. Then $a \lhd \{c \in S \mid cFb\} \downarrow \{e \in S \mid eFd\}$, that is,

  $$a \lhd \{x \in S \mid ((\exists c \in S)\ x \lhd c\ \&\ cFb)\ \&\ ((\exists e \in S)\ x \lhd e\ \&\ eFd)\}$$

  follows by $\downarrow$*-right*. So $a \lhd \{x \in S \mid xFb\ \&\ xFd\}$ follows by *weak-saturation* and hence we get $a \lhd \{x \in S \mid x \lhd F^-(b \downarrow d)\}$ by *convergence.* Thus $a \lhd F^-(b \downarrow d)$ follows by *transitivity* and hence $a \lhd (F^{\lhd})^-(b \downarrow d)$.

- (function saturation) Assume $a \lhd W$ and $(\forall w \varepsilon W)\ wF^{\lhd}b$. Then, for all $w \varepsilon W$, $w \lhd \{x \in S \mid xFb\}$ and hence $a \lhd \{x \in S \mid xFb\}$, that is $aF^{\lhd}b$, follows by *transitivity*.
- (function continuity) Suppose $aF^{\lhd}b$ and $b \lhd V$. Then $a \lhd \{x \in S \mid xFb\}$. Now, for any $x \in S$ such that xFb, $x \lhd F^{-}(V)$ follows by *continuity* for F. Hence $a \lhd F^{-}(V)$ follows by *transitivity* and so $a \lhd (F^{\lhd})^{-}(V)$.

Assume now that G is any continuous relation which contains F and suppose that $a\ F^{\lhd}\ b$ holds. Then $a \lhd \{x \in S \mid x\ F\ b\}$ and hence $a \lhd \{x \in S \mid x\ G\ b\}$ follows since G contains F. Hence $a\ G\ b$ follows by *saturation* for G. Thus $F^{\lhd}$ is the minimal continuous relation containing F.

Corollary 2.12 *Let $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{U}$ be formal topologies and F be a continuous relation between $\mathcal{S}$ and $\mathcal{T}$ and G be a continuous relation between $\mathcal{T}$ and $\mathcal{U}$. Then the relation $F * G$ defined by setting, for any $a \in S$ and $c \in U$,*

$$a\ G * F\ c \text{ if and only if } a\ (G \circ F)^{\lhd}\ c$$

is a continuous relation between $\mathcal{S}$ and $\mathcal{U}$.

Proof. After lemma 2.10 and proposition 2.11, we have only to prove that relation composition satisfies *weak-saturation*. So, let us suppose $a \lhd e$ and $e\ G \circ F\ c$. Then there exists $y \in T$ such that $e\ F\ y$ and $y\ G\ c$; hence $a\ F\ y$ follows by *weak-saturation* for F and so $a\ G \circ F\ c$.

The following lemmas will be useful in the following.

Lemma 2.13 *Let $\mathcal{S}$, $\mathcal{T}$ be formal topologies and F be a continuous relation between $\mathcal{S}$ and $\mathcal{T}$. Then*

$$F^{\lhd} = F$$

Proof. Immediate. In fact, $F^{\lhd}$ is the minimal continuous relation containing F and hence it coincides with F when F is already a continuous relation.

Lemma 2.14 *Let $\mathcal{S}$, $\mathcal{T}$ be formal topologies and H and K be relations between $\mathcal{S}$ and $\mathcal{T}$ such that $H \subseteq K$. Then $H^{\lhd} \subseteq K^{\lhd}$.*

Proof. We only need to unfold the definitions. Indeed, for any $a \in S$ and $b \in T$, $a\ H^{\lhd}\ b$ if and only if $a \lhd \{w \in S \mid wHb\}$; then $a \lhd \{w \in S \mid wKb\}$, that is, $a\ K^{\lhd}\ b$, follows by *reflexivity* and *transitivity* since $H \subseteq K$ yields $\{w \in S \mid wHb\} \subseteq \{w \in S \mid wKb\}$.

Lemma 2.15 *Let $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{U}$ be formal topologies, F be a relation between $\mathcal{S}$ and $\mathcal{T}$ which satisfies* function continuity *and G be a relation between $\mathcal{T}$ and $\mathcal{U}$. Then*

$$(G \circ F^{\lhd})^{\lhd} = (G \circ F)^{\lhd} \text{ and } (G^{\lhd} \circ F)^{\lhd} = (G \circ F)^{\lhd}$$

Proof. We already proved that, for any relation F, $F \subseteq F^{\lhd}$. But $F \subseteq F^{\lhd}$ yields immediately $G \circ F \subseteq G \circ F^{\lhd}$. And so $(G \circ F)^{\lhd} \subseteq (G \circ F^{\lhd})^{\lhd}$ follows by lemma 2.14. The proof that $(G \circ F)^{\lhd} \subseteq (G^{\lhd} \circ F)^{\lhd}$ is completely similar.

Let us prove now the other inclusions. Suppose $a \in S$ and $c \in U$. Then

$$\begin{array}{rcl} a\ (G \circ F^{\lhd})^{\lhd}\ c & \leftrightarrow & a \lhd \{w \in S \mid w\ G \circ F^{\lhd}\ c\} \\ & \leftrightarrow & a \lhd \{w \in S \mid (\exists y \in T)\ w\ F^{\lhd}\ y\ \&\ y\ G\ c\} \\ & \leftrightarrow & a \lhd \{w \in S \mid (\exists y \in T)\ w \lhd \{z \in S \mid z\ F\ y\}\ \&\ y\ G\ c\} \\ & \rightarrow & a \lhd \{w \in S \mid w \lhd \{z \in S \mid z\ G \circ F\ c\}\} \\ \text{(by transitivity)} & \rightarrow & a \lhd \{z \in S \mid z\ G \circ F\ c\} \\ & \leftrightarrow & a\ (G \circ F)^{\lhd}\ c \end{array}$$

and

$$\begin{array}{rcl} a\ (G^{\lhd} \circ F)^{\lhd}\ c & \leftrightarrow & a \lhd \{w \in S \mid w\ G^{\lhd} \circ F\ c\} \\ & \leftrightarrow & a \lhd \{w \in S \mid (\exists y \in T)\ w\ F\ y\ \&\ y\ G^{\lhd}\ c\} \\ & \leftrightarrow & a \lhd \{w \in S \mid (\exists y \in T)\ w\ F\ y\ \&\ y \lhd \{z \in T \mid z\ G\ c\}\} \\ \text{(by funct. cont.)} & \rightarrow & a \lhd \{w \in S \mid w \lhd F^{-}(\{z \in T \mid z\ G\ c\})\} \\ \text{(by transitivity)} & \rightarrow & a \lhd F^{-}(\{z \in T \mid z\ G\ c\}) \\ & \leftrightarrow & a \lhd \{w \in S \mid (\exists y \in T)\ w\ F\ y\ \&\ y\ G\ c\} \\ & \leftrightarrow & a \lhd \{w \in S \mid w\ G \circ F\ c\} \\ & \leftrightarrow & a\ (G \circ F)^{\lhd}\ c \end{array}$$

We can now prove the main theorem of this section.

Theorem 2.16 *Formal topologies and continuous relations form a category* FTop *where the operation of composition between continuous relations is* $- * -$ *and the cover relation is its unit.*

Proof. We only need to show that the operation $- * -$ between continuous relations is associative. Recalling that relation composition is associative, we obtain

$$\begin{array}{rcll} (G * F) * H & \equiv & ((G * F) \circ H)^{\lhd} & \text{by definition of } * \\ & \equiv & ((G \circ F)^{\lhd} \circ H)^{\lhd} & \text{by definition of } * \\ & \equiv & ((G \circ F) \circ H)^{\lhd} & \text{by lemma 2.15} \\ & \equiv & (G \circ (F \circ H))^{\lhd} & \\ & \equiv & (G \circ (F \circ H)^{\lhd})^{\lhd} & \text{by lemma 2.15} \\ & \equiv & (G \circ (F * H))^{\lhd} & \text{by definition of } * \\ & \equiv & G * (F * H) & \text{by definition of } * \end{array}$$

Note that in the proof above we could apply lemma 2.15 because by lemma 2.10 we know that the composition of two continuous relations satisfies *function continuity.*

It is now trivial to realize that the cover relation is a continuous relation and we can use the previous lemma 2.12 to shorten the proof that the cover relation is the identity with respect to the operation $- * -$. Indeed, it is sufficient to prove that $F \circ \lhd = F$ and $\lhd \circ F = F$ hold since such equalities yield

$$F * \lhd = (F \circ \lhd)^{\lhd} = F^{\lhd} = F$$

and

$$\lhd * F = (\lhd \circ F)^{\lhd} = F^{\lhd} = F$$

So, suppose $a \lhd \circ F\ c$; then there exists y such that $a\ F\ y$ and $y \lhd c$; hence $a \lhd \{x \mid x\ F\ c\}$ follows by *continuity* and it yields $a\ F\ c$ by *saturation*. On the other hand, if $a\ F\ c$ then $a \lhd \circ F\ c$ is immediate since $c \lhd c$ holds by *reflexivity*. Suppose now $a\ F \circ \lhd\ c$; then, there exists y such that $a \lhd y$ and $y\ F\ c$ and hence $a\ F\ c$ follows by *weak-saturation*; on the other hand if $a\ F\ c$ then $a\ F \circ \lhd\ c$ follows immediately since $a \lhd a$ holds by *reflexivity*.

Since the conditions on continuous relations do not concern the positivity predicate the previous proofs show also the following result.

Theorem 2.17 *Formal topologies without the positivity predicate and continuous relations form a category* FTop^- *where the operation of composition between continuous relations is* $- * -$ *and the cover relation is its unit.*

The category FTop is impredicatively equivalent to the category OpLoc of open locales [JT84] (for a recent proof see [Neg02] but note that Pos is not required to be a frame morphism) while FTop^- is impredicatively equivalent to the category Loc of locales (see [BS01]). To summarize in a diagram from an impredicative point of view we have

$$\begin{array}{ccc} \mathsf{FTop} & \hookrightarrow & \mathsf{FTop}^- \\ \big| \simeq & & \big| \simeq \\ \mathsf{OpLoc} & \hookrightarrow & \mathsf{Loc} \end{array}$$

2.4.1 Properties on top element and positivity predicate

In the definition of continuous relation the top element and the positivity predicate are not involved, that is, the same definition works both in FTop and FTop^-. However, there are specific properties that depend on their presence. We start by showing that by using explicitly the top element an equivalent but simpler formulation of *function totality* is possible.

Lemma 2.18 *Let* $\mathcal{S}$ *and* $\mathcal{T}$ *be formal topologies, and* F *be a continuous relation between* S *and* T*. Then the following conditions are equivalent*

1. (function totality) $\quad (\forall a \in S)(\exists b \in T)\ aFb$
2. (anti-image totality) $\quad S \lhd_S F^-(T)$
3. (top-element totality) $\quad (\forall a \in S)\ a\ F\ \top_T$

Proof. We will show the various implication one after the other.

- (1) $\Rightarrow$ (2). Let $a \in S$. Then, by *function totality*, there is an element $b \in T$ such that aFb. Hence $a \varepsilon F^-(T)$ and so *anti-image totality* follows by *reflexivity*.

- (2) $\Rightarrow$ (3). Let us assume that *anti-image totality* holds. Then, for any $a \in S$, $a \lhd F^-(T)$. Now, by *top-element*, $T \lhd \top_T$ holds and hence lemma 2.8 shows that $F^-(T) \lhd F^-(\top_T)$. So $a \lhd F^-(\top_T)$ follows by *transitivity*. But, for any $c \varepsilon F^-(\top_T)$, $cF\top_T$ holds and hence $aF\top_T$ follows by *saturation*.

- (3) $\Rightarrow$ (1). Let $a \in S$. Then, by *top-element totality*, $aF\top_T$ holds and hence $\top_T$ is the element that we are looking for.

A clear advantage of *top-element totality* with respect to the other two formulations of the totality of a function is its simplicity, and this is the main reason why we will often use it in the following. However, both *function totality* and *anti-image totality* can be expressed with no reference to the element $\top_T$ and hence they can be used also in a context where the top element is not included in the definition of formal topology (see for instance [CSSV03]). Moreover, *anti-image totality* can also be justified with no reference to the top element. Indeed, let us assume that (X, S, ext_1) and (Y, T, ext_2) are two concrete topological spaces, ϕ is a continuous map from X to Y and F is a relation between S and T such that aFb if and only in $(\forall x \in X)\ x\varepsilon\mathsf{ext}_1(a) \to \phi(x)\varepsilon\mathsf{ext}_2(b)$. Then, we have to show that, for all $a \in S$ and all $x \in X$, if $x\varepsilon\mathsf{ext}_1(a)$ then there exists $u \in S$ such that both $x\varepsilon\mathsf{ext}_1(u)$ and $u\varepsilon F^-(T)$, that is, there exists $t \in T$ such that uFt. Now, by the condition (B_1), $x\varepsilon\mathsf{ext}_1(a)$ yields that there exists some element $t \in T$ such that $\phi(x)\varepsilon t$ and hence, by continuity of ϕ, there exists $u \in S$ such that both $x\varepsilon\mathsf{ext}_1(u)$ and uFt hold.

Now, let us show some consequence of the conditions defining continuous relations which concern explicitly the positivity predicate.

Lemma 2.19 *Let F be a continuous relation between $\mathcal{S}$ and $\mathcal{T}$. Then, for any $a \in S$ and $b \in T$, F satisfies the following condition*

$$\text{(function monotonicity)} \quad \frac{\mathsf{Pos}_S(a) \qquad aFb}{\mathsf{Pos}_T(b)}$$

Proof. Let us suppose aFb. Then, the *positivity axiom* $b \lhd b^+$ yields, by *function continuity*, that $a \lhd_S F^-(b^+)$. Hence, by *monotonicity* of the cover relation, $\mathsf{Pos}(a)$ yields that there exists some element $c\varepsilon F^-(b^+)$ such that $\mathsf{Pos}_S(c)$ holds. Therefore, there exists $y\varepsilon b^+$ such that cFy. But $y\varepsilon b^+$ yields that $y = b$ and $\mathsf{Pos}_T(y)$ hold and thus $\mathsf{Pos}_T(b)$ follows.

The condition of *function monotonicity* above was firstly part of the original definition of continuous relation in [Vir90] as a consequence of its presence in the definition of frame morphisms between formal topology in [Sam87], but it was later recognized to be derivable in [Neg02].

Another important consequence of the conditions on a continuous relation is a condition stating that two relations associated with the same function between formal points do not differ on non-positive elements of S. Before stating it let us prove the following lemma.

Lemma 2.20 *Let F be a continuous relation between $\mathcal{S}$ and $\mathcal{T}$. Then, for any $a \in S$ and $b \in T$,*

$$\mathsf{Pos}(a) \to aFb \textit{ if and only if } (\forall x \varepsilon a^+)\ xFb$$

Proof. To prove the left to right implication, suppose $x \varepsilon a^+$. Then $x = a$ and $\mathsf{Pos}_S(x)$ hold and hence $\mathsf{Pos}_S(a)$ follows. So, $\mathsf{Pos}_S(a) \to aFb$ yields aFb and hence $x = a$ yields xFb. On the other hand, assuming $\mathsf{Pos}_S(a)$, $a \varepsilon a^+$ follows and hence $(\forall x \varepsilon a^+)\ xFb$ yields aFb by logic.

Now, the following lemma is immediate.

Lemma 2.21 *Let F be a continuous relation between $\mathcal{S}$ and $\mathcal{T}$. Then, for any $a \in S$ and $b \in T$, F satisfies the following condition*

$$\text{(function positivity)} \quad \frac{\mathsf{Pos}_S(a) \to aFb}{aFb}$$

Proof. After the previous lemma we know that $\mathsf{Pos}_S(a) \to aFb$ yields that, for all $x \varepsilon a^+$, xFb. Hence $a \lhd a^+$ yields aFb by *function saturation*.

The condition of *function positivity* was first used in [SVV96] to force the faithfulness of the functor $\mathsf{Pt}(-)$ on Scott formal topologies. In that context *function saturation* cannot be used and hence *function positivity* is part of the definition of continuous relation together with *function weak-saturation* (see proposition 2.49).

An the end of this section, let us recall that in the literature there are also alternative presentations of the category of formal topologies (see for instance [GS02]) where a continuous relation is defined by requiring all of the conditions in definition 2.7 except for *function saturation*; of course, in this case one is forced to state that two continuous relations F and G are equal if $F^{\lhd}$ and $G^{\lhd}$ are equal. We prefer the approach presented here because we think that being able to use an equality between continuous relations which does not depend on the cover relation is more natural and allows a simpler technical treatment which becomes crucial in dealing with exponentiation.

2.4.2 Formal points and continuous relations

In this section we show that there is a bijective correspondence between the collection of the global elements of $\mathcal{A}$ and the collection $\mathsf{Pt}(\mathcal{A})$ of the formal points of $\mathcal{A}$. First of all, let us recall how to define a terminal object $\mathcal{T}$ in FTop.

Lemma 2.22 *Let $\mathcal{T} \equiv (T, \top, \lhd_T, \mathsf{Pos}_T)$ be the formal topology such that $T \equiv \{\top\}$ is a one element set, the top element is $\top$, the cover relation is defined by setting, for any $a \in \{\top\}$ and any subset U of $\{\top\}$,*

$$a \lhd_T U \equiv a \varepsilon U$$

and the positivity predicate is defined by setting, for any $a \in \{\top\}$,

$$\mathsf{Pos}(a) \equiv \mathsf{True}$$

Then, $\mathcal{T}$ is a terminal object in FTop*, that is, for any formal topology $\mathcal{A}$, the total relation defined by setting, for any $a \in A$, $a\ !_A \top$, is the only continuous relation between $\mathcal{A}$ and $\mathcal{T}$.*

Proof. It is immediate to check that $\mathcal{T}$ is a formal topology. Indeed,

- (top-element) Immediate.
- (reflexivity) Let us suppose that $a \in \{\top\}$, $U \subseteq \{\top\}$ and $a \varepsilon U$. Thus, $a \lhd U$ holds by definition.
- (transitivity) Let us suppose that $a \lhd U$ and $U \lhd V$ hold for some element $a \in \{\top\}$ and subsets $U, V \subseteq \{\top\}$. Then, $a \lhd U$ yields $a \varepsilon U$ and hence $U \lhd V$ yields $a \lhd V$.
- ($\downarrow$-right) Let us suppose $a \lhd U$ and $a \lhd V$. Then, by definition, $a \varepsilon U$ and $a \varepsilon V$ and hence $a \varepsilon U \downarrow V$ follows and it yields $a \lhd U \downarrow V$.
- (monotonicity) Immediate.
- (positivity) Any element $a \in \{\top\}$ is positive and hence positivity, namely $a \lhd a^+$, trivially holds.

Moreover, it is easy to see that $!_A$ is a continuous relation. Indeed, *function saturation* holds by definition while *function totality*, *function convergence* and *function continuity* hold because the anti-image of any non empty subset of T is equal to the whole set A and hence it covers any element in A by *reflexivity*. Moreover, if R is any continuous relation between $\mathcal{A}$ and $\mathcal{T}$ and $a \in A$, then $aR\top$ holds by *function totality* and hence $!_A \subseteq R$. Thus $R =!_A$ since $R \subseteq !_A$ obviously holds.

We can now state the following theorem.

Theorem 2.23 *Let $\mathcal{A}$ be a formal topology. Then there is a bijective correspondence between the collection* Pt$(\mathcal{A})$ *of the formal points of $\mathcal{A}$ and the continuous relations between $\mathcal{T}$ and $\mathcal{A}$.*

Proof. Let us suppose that α is a formal point of the formal topology $\mathcal{A}$. Then the continuous relation between $\mathcal{T}$ and $\mathcal{A}$ associated with α is defined by setting, for any $u \in \{\top\}$ and any $a \in A$,

$$uR_\alpha a \equiv a \varepsilon \alpha$$

Indeed, it is easy to prove that R_α is a continuous relation:

- (function totality) Let $u \in \{\top\}$. Recall now that any formal point in Pt$(\mathcal{A})$ is inhabited and hence there is an element $a \in A$ such that $a \varepsilon \alpha$, namely, there is an element $a \in A$ such that $uR_\alpha a$ holds.

- (function convergence) Let us suppose that $uR_\alpha a$ and $uR_\alpha c$ hold. Then, $a\varepsilon\alpha$ and $c\varepsilon\alpha$ and hence, by *point convergence*, there exists $e\varepsilon\alpha$ such that $e \lhd_A a$ and $e \lhd_A c$. Thus, $uR_\alpha e$ holds, that is, $u\varepsilon R_\alpha^-(a \downarrow c)$, and hence the result follows by *reflexivity*.

- (function saturation) Let us suppose that $u \lhd_T W$ and $(\forall w\varepsilon W)\, wR_\alpha a$. Then $u\varepsilon W$ holds by definition and hence $uR_\alpha a$ follows by logic.

- (function continuity) Let us suppose that $uR_\alpha a$ and $a \lhd_A U$. Then $a\varepsilon\alpha$ follows and hence, by *point splitness*, there exists and element $e\varepsilon\alpha$ such that $e\varepsilon U$. Thus $uR_\alpha e$ holds; hence $u\varepsilon R_\alpha^-(U)$ and so the result follows by *reflexivity*.

On the other hand, given any continuous relation R between $\mathcal{T}$ and $\mathcal{A}$, we can define a formal point α_R of $\mathcal{A}$ by setting, for any $a \in A$,

$$a\varepsilon\alpha_R \equiv \top R a$$

It is straightforward to check that α_R is a formal point. Indeed,

- (point inhabitance) By *function totality* $\top Ra$ holds for some $a \in A$ and hence $a\varepsilon\alpha_R$ holds, that is, α_R is inhabited.

- (point convergence) Let us suppose that $a\varepsilon\alpha_R$ and $c\varepsilon\alpha_R$. Then $\top Ra$ and $\top Rc$ hold and hence, by *function convergence*, $\top \lhd R^-(a \downarrow c)$. Thus $\top\varepsilon R^-(a \downarrow c)$, that is, there exists $y\varepsilon a \downarrow c$ such that $\top Ry$, and hence $(\exists y\varepsilon a \downarrow c)\ y\varepsilon\alpha_R$ follows.

- (point splitness) Let us suppose that $a\varepsilon\alpha_R$ and $a \lhd_A U$. Then $\top Ra$ holds and hence, by *function continuity*, $\top \lhd R^-(U)$. Thus $\top\varepsilon R^-(U)$, that is, there exists $y\varepsilon U$ such that $\top Ry$, and hence $(\exists y\varepsilon U)\ y\varepsilon\alpha_R$ follows.

It is now completely trivial to see that the two constructions are one the inverse of the other.

2.5 Inductively generated formal topologies

One of the main tools in formal topology is the inductive generation of the cover since this allows to develop proofs by induction. The problem of generating inductively formal topologies has been dealt with and solved in [CSSV03]. We recall here, without any proofs, only those results of [CSSV03] that we will use in the next sections.

An inductive definition of a cover starts from some axioms, which at the moment we assume to be given by means of any relation $R(a, U)$ for $a \in S$ and $U \subseteq S$. We thus want to generate the least cover $\lhd_R$ which satisfies the following condition:

$$(\text{axioms}) \qquad \frac{R(a,U)}{a \lhd_R U}$$

The first naive idea for an inductive generation of a cover relation is to use the conditions which appear in the definition of a formal topology like rules. But such conditions, though written in the shape of rules, must be understood as requirements of validity, that is, if the premises hold then also the conclusion must hold. As they stand, they are by no means acceptable rules to generate inductively a cover relation. For instance, the operation $- \downarrow -$ among subsets, which occurs in the conclusion of $\downarrow$*-right*, is not even well defined unless we already have a complete knowledge of the cover. Another problem is that admitting *transitivity* as acceptable rule for an inductive definition is equivalent to a fix-point principle, which does not have a predicative justification (see [CSSV03] for a detailed discussion of this topic).

These are the reasons why we cannot accept all the possible infinitary propositions $R(a, U)$ in the formation of an axiom and we have to impose some constraints. The solution proposed in [CSSV03] for the impredicativity problem due to the *transitivity* condition is to generate a cover relation only when we have an *axiom-set*, that is, a family $I(a)$ of sets for $a \in S$ and a family $C(a, i)$ of subsets of S for $a \in S$ and $i \in I(a)$, whose intended meaning is to state that, for all $i \in I(a)$, $a \lhd C(a, i)$. Indeed, in this case we can generate the cover relation by using the following inductive rules:

$$\text{(reflexivity)}\ \frac{a\ \varepsilon\ U}{a \lhd U} \qquad \text{(infinity)}\ \frac{i \in I(a) \qquad C(a,i) \lhd U}{a \lhd U}$$

We can now strengthen the previous rules to new ones which allow to generate a cover relation which satisfies also *top-element* and $\downarrow$*-right*. In fact, to satisfy $\downarrow$*-right*, a possibility is to add an extra primitive expressing what, in the concrete case, is the inclusion relation between two basic open subsets, that is, $\mathsf{ext}(a) \subseteq \mathsf{ext}(b)$. We can obtain this result by adding directly a pre-order relation $a \leq b$. We will show that in this way it will be possible to satisfy also the *top-element* condition.

Thus, we arrive at the following definition.

Definition 2.24 ($\leq$-formal topology) *A* $\leq$-formal topology *is a structure* $(S, \leq, \top_S, \lhd, \mathsf{Pos})$ *where S is a set, $\leq$ is a pre-order relation between elements of S, that is, $\leq$ is reflexive and transitive, $\top_S$ is a distinguished element of S and $\lhd$ is a relation between elements and subsets of S which satisfies* top-element*,* reflexivity*,* transitivity *and the two following conditions*

$$(\leq\text{-left})\quad \frac{a \leq b \quad b \lhd U}{a \lhd U} \qquad (\leq\text{-right})\quad \frac{a \lhd U \quad a \lhd V}{a \lhd U \downarrow_{\leq} V}$$

where $U \downarrow_{\leq} V \equiv \{c \in S \mid (\exists u \varepsilon U)\ c \leq u\ \&\ (\exists v \varepsilon V)\ c \leq v\}$*. Finally,* Pos *is a predicate over elements of S which satisfies* monotonicity *and* positivity*.*

It is straightforward to verify that the new conditions are valid in any concrete topological space under the intended interpretation. And only a bit more work is required to prove that any $\leq$-formal topology is a formal topology.

The proof that any formal topology is equivalent to a suitable $\leq$-formal topology is even more trivial. Indeed, it is sufficient to define an order relation between elements of S by setting $a \leq b$ if and only if $a \lhd \{b\}$ and it is obvious that all of the required conditions are satisfied.

Thus, in order to be able to generate inductively a formal topology we need only to be able to generate inductively a $\leq$-formal topology. So, let us suppose that we have a set S, an order relation $\leq$ between elements of S, a distinguished element $\top \in S$ and a given axiom-set $I(-)$ and $C(-,-)$ and that we want to generate a $\leq$-formal topology over S. To this aim we can adapt the method proposed in [CSSV03] and generate by induction a cover relation which respects the given axiom-set, *top-element*, *reflexivity*, *transitivity*, $\leq$*-left* and $\leq$*-right* and by co-induction a positivity predicate which satisfies *monotonicity* and *positivity* with respect to such a cover relation.

The first modification is to extend the axiom-set $I(-)$ and $C(-,-)$ to a new axiom-set such that it will be possible to obtain, for any $a \in S$,

$$\text{(top-element axiom)} \qquad a \lhd \{a\} \downarrow_{\leq} \{\top\}$$

Thus, the new axiom-set is defined by setting, for any $a \in S$,

$$I'(a) \equiv I(a) \cup \{*\} \qquad C'(a,i) \equiv \begin{cases} \{a\} \downarrow_{\leq} \{\top\} & \text{if } i = * \\ C(a,i) & \text{otherwise} \end{cases}$$

After this axioms are added, we can obtain $a \lhd \top$ by first showing by $\leq$*-left* that $\{a\} \downarrow_{\leq} \{\top\} \lhd \top$ and then concluding by *infinity*. Moreover, it is clear that the added axiom is valid in any $\leq$-formal topology since it is an immediate consequence of the *top element* condition by $\leq$*-right*.

The second step consists in defining a suitable positivity predicate. To this aim, let us say that a predicate $\mathsf{Pos}(-)$ satisfies $\leq$*-monotonicity* if, for any $a, b \in S$,

$$(\leq\text{-monotonicity}) \quad \frac{\mathsf{Pos}(a) \qquad a \leq b}{\mathsf{Pos}(b)}$$

holds and that it satisfies *axiom monotonicity* if, for all the axioms in the axiom-set $I'(-)$ and $C'(-,-)$ and for any $a \in S$,

$$\text{(axiom monotonicity)} \quad \frac{\mathsf{Pos}(a) \quad i \in I'(a)}{(\exists c \varepsilon C'(a,j))\ \mathsf{Pos}(c)}$$

holds.

Now, given any axiom-set $I'(-)$, $C'(-,-)$, we can use Tarski fixed-point theorem to show that it is possible to define a predicate $\mathsf{Pos}(-)$ which satisfies both $\leq$*-monotonicity* and *axiom monotonicity* by simply considering these two conditions like co-inductive rules (see appendix A). Hence, given any axiom-set, we will use such a predicate $\mathsf{Pos}(-)$ like a positivity predicate.

Finally, given any axiom-set $I'(-)$ and $C'(-,-)$ and any predicate enjoying $\leq$*-monotonicity* and *axiom monotonicity*, we can always force the validity of the *positivity* condition by adding a single axiom schema stating that, for any

$a \in S$, a is covered by the set a^+, namely, we can define a new axiom-set by setting, for any $a \in S$,

$$I''(a) \equiv I'(a) \cup \{**\} \qquad C''(a,i) \equiv \begin{cases} a^+ & \text{if } i = ** \\ C'(a,i) & \text{otherwise} \end{cases}$$

Note that if *axiom monotonicity* holds for $I'(-)$ and $C'(-,-)$ then it continues to hold also for $I''(-)$ and $C''(-,-)$ since $\mathsf{Pos}(a)$ clearly yields that there exists an element $x \varepsilon a^+$ such that $\mathsf{Pos}(x)$ holds.

We are finally ready to use the method in [CSSV03]. The only constrain for its applicability is that the following condition, which guarantees the validity of $\leq$-*right*, is satisfied.

Definition 2.25 (Localization condition) *Let S be any set and $I''(-)$ and $C''(-,-)$ be an axiom-set for a cover relation on S. Then such an axiom-set satisfies the* localization condition *if, for any $a \leq c$ and for any $i \in I''(c)$, there exists $j \in I''(a)$ such that $C''(a,j) \subseteq \{a\} \downarrow_\leq C''(c,i)$.*

It is useful to note that if an axiom-set $I(-)$ and $C(-,-)$ enjoys the *localization condition* then, for any positivity predicate Pos, also the axiom-set $I''(-)$ and $C''(-,-)$ obtained by adding to the given one both the *top-element axiom* and the *positivity axiom* satisfies such a condition. Indeed, suppose that $a \leq c$ and that we are considering the *top-element axiom* for c. Then we have to show that there exists an index $j \in I''(a)$ such that $C(a,j) \subseteq \{a\} \downarrow_\leq \{c\} \downarrow_\leq \{\top\}$. The correct choice is of course the index for $\{a\} \downarrow_\leq \{\top\}$ since it is trivial to see that $\{a\} \downarrow_\leq \{\top\} \subseteq \{a\} \downarrow_\leq \{c\} \downarrow_\leq \{\top\}$. We can provide a similar proof when the considered axiom is the *positivity axiom* for c. In this case we have to show that there exists an index $j \in I''(a)$ such that $C(a,j) \subseteq \{a\} \downarrow_\leq c^+$. The correct choice for j is now the index for a^+ since we can prove that $a^+ \subseteq \{a\} \downarrow_\leq c^+$. Indeed, let us assume that $x \varepsilon a^+$. Then, both $x = a$ and $\mathsf{Pos}(x)$ follow and hence we first obtain $\mathsf{Pos}(a)$ by logic and then $\mathsf{Pos}(c)$ by $\leq$-*monotonicity*. So $c^+ = \{c\}$ and hence both $x \leq a$ and $x \leq c$ hold since $x = a$, that is, $x \varepsilon \{a\} \downarrow_\leq c^+$.

Now, the main result in [CSSV03] is that, provided we have an axiom-set which satisfies the *localization condition*, we can define a relation between elements and subsets of S which satisfies all of the conditions for a cover by using *reflexivity*, $\leq$-*left* and *infinity* like inductive rules. Moreover, given any predicate which satisfies $\leq$-*monotonicity* and *axiom monotonicity* it is immediate to prove by induction on the length of the proof of $a \lhd U$ that if $\mathsf{Pos}(a)$ and $a \lhd U$ hold then there exists an element $u \varepsilon U$ such that $\mathsf{Pos}(u)$ holds, namely, that *monotonicity* holds. Finally, the *positivity* condition clearly holds for such cover relation and such a predicate since it is built in the axioms from which the cover relation is generated.

2.5.1 Formal points of inductively generated formal topologies

If we restrict our attention to inductively generated $\leq$-formal topologies we can simplify many of the definitions given in the previous sections. To begin with, the definition of formal point can be simplified as follows.

Definition 2.26 (Formal point) *Let* $(S, \leq, \top, \lhd, \mathsf{Pos})$ *be an inductively generated* $\leq$*-formal topology. Then, an inhabited subset* α *of* S *is a* formal point *if, for any* $a, b \in S$ *and any* $U \subseteq S$*, it satisfies the following conditions:*

$$\text{(point } \leq\text{-convergence)} \quad \frac{a\varepsilon\alpha \quad b\varepsilon\alpha}{(\exists c\varepsilon a \downarrow_{\leq} b)\ c\varepsilon\alpha}$$

$$\text{(point left-closure)} \quad \frac{a\varepsilon\alpha \quad a \leq b}{b\varepsilon\alpha}$$

$$\text{(point inductive splitness)} \quad \frac{a\varepsilon\alpha \quad i \in I(a)}{(\exists y\varepsilon C(a,i))\ y\varepsilon\alpha}$$

After observing that in the case of an inductively generated $\leq$-formal topology, $a \downarrow_{\leq} b \lhd a \downarrow b$ and $a \downarrow b \lhd a \downarrow_{\leq} b$, it is clear that the conditions in the definition above are consequences of the ones in section 2.3. On the other hand, it is possible to prove that a subset α which satisfies the conditions stated here satisfies also *point splitness*, namely, $a\varepsilon\alpha$ and $a \lhd U$ yield $(\exists y\varepsilon U)\ y\varepsilon\alpha$, by developing a proof by induction on the length of the derivation of $a \lhd U$.

2.5.2 Morphisms between inductively generated topologies

The general conditions on a continuous relation that we presented in section 2.4 can be simplified when we are dealing with morphisms between inductively generated formal topologies. The first condition that can be adapted to the new framework is *function convergence*.

Lemma 2.27 *Let* $\mathcal{A}$ *be a formal topology,* $\mathcal{B}$ *be an inductively generated formal topology and* F *be a continuous relation between* $\mathcal{A}$ *and* $\mathcal{B}$*. Then,* function convergence *is equivalent to*

$$\text{(function } \leq\text{-convergence)} \quad \frac{aFb \qquad aFd}{a \lhd F^{-}(b \downarrow_{\leq} d)}$$

Proof. We already proved that, for any inductively generated formal topology $\mathcal{B}$, if $V_1, V_2 \subseteq B$ then $V_1 \downarrow V_2 \lhd_B V_1 \downarrow_{\leq} V_2$ and $V_1 \downarrow_{\leq} V_2 \lhd_B V_1 \downarrow V_2$. Thus both $F^{-}(V_1 \downarrow V_2) \lhd_A F^{-}(V_1 \downarrow_{\leq} V_2)$ and $F^{-}(V_1 \downarrow_{\leq} V_2) \lhd_A F^{-}(V_1 \downarrow V_2)$ follows by lemma 2.8 which uses only *function continuity*. Hence the equivalence between *function convergence* and *function* $\leq$*-convergence* follows immediately by *transitivity*.

We will see now that also *function continuity* can be simplified.

Lemma 2.28 *Let* $\mathcal{A}$ *be a formal topology,* $\mathcal{B}$ *be an inductively generated formal topology and* F *be a continuous relation between* $\mathcal{A}$ *and* $\mathcal{B}$*. Then,* continuity *is equivalent to*

$$\text{(axiom continuity)} \quad \frac{aFb \quad j \in J(b)}{a \lhd F^{-}(C(b,j))} \qquad \text{(}\leq\text{-continuity)} \quad \frac{aFb \quad b \leq d}{aFd}$$

where $J(-)$ *and* $C(-,-)$ *is the axiom-set for the inductively generated formal topology* $\mathcal{B}$*.*

Proof. It is obvious that *axiom continuity* is an instance of *function continuity* and $\leq$-*continuity* is an immediate consequences of $\leq$-*left*, *function continuity* and *saturation*.

On the other hand, *continuity* can be derived from these conditions by reasoning by induction. Indeed, let us suppose aFb and $b \lhd V$. Then, we can argue according to the length of the derivation of $b \lhd V$. Let us begin by supposing that $b \lhd V$ has been derived by *reflexivity* from $b\varepsilon V$. Then $a\varepsilon F^{-}(V)$ and hence the result, that is, $a \lhd F^{-}(V)$, immediately follows by *reflexivity*. Moreover, if $b \lhd V$ has been derived by *infinity* from $C(b,j) \lhd V$ we can use *axiom continuity* to obtain $a \lhd F^{-}(C(b,j))$. Then we can obtain $a \lhd F^{-}(V)$ by proving that $F^{-}(C(b,j)) \lhd F^{-}(V)$. To this aim let us assume that $w\varepsilon F^{-}(C(b,j))$, that is, $(\exists v \varepsilon C(b,j))\ wFv$, then $C(b,j) \lhd V$ yields, by inductive hypothesis, that $w \lhd F^{-}(V)$. Finally, if $\leq$-*left* has been used, that is, we proved $b \lhd V$ from $b \leq d$ and $d \lhd V$, then we immediately obtain aFd by using $\leq$-*continuity* and hence the result follows by inductive hypothesis.

In the following we will often use the following result.

Lemma 2.29 *Let $\mathcal{A}$ be a formal topology, $\mathcal{B}$ be an inductively generated formal topology and F be a continuous relation between $\mathcal{A}$ and $\mathcal{B}$. Then the following condition*

$$\text{(weak-continuity)} \quad \frac{aFb \quad b \lhd_{\mathcal{B}} d}{aFd}$$

holds.

Proof. Immediate by *function continuity* and *saturation*.

In a similar way also *saturation* can be simplified.

Lemma 2.30 *Let $\mathcal{A}$ be an inductively generated formal topology, $\mathcal{B}$ be a formal topology and F be continuous relation between $\mathcal{A}$ and $\mathcal{B}$. Then* saturation *is equivalent to*

$$(\leq\text{-saturation})\ \frac{a \leq c \quad cFb}{aFb} \qquad (\text{axiom-saturation})\ \frac{i \in I(a) \quad (\forall x \varepsilon C(a,i))\ xFb}{aFb}$$

where $I(-)$ and $C(-,-)$ is the axiom-set used to inductively generate $\mathcal{A}$.

Proof. $\leq$-*saturation* and *axiom saturation* are obvious consequences of $\leq$-*left* and *saturation*. Thus, let us show that also the other implication holds, namely, that $a \lhd W$ and $(\forall x \varepsilon W)\ xFb$ yield aFb. The proof goes on by induction on the length of the derivation of $a \lhd W$.

First, let us assume that $a \lhd W$ has been obtained from $a\varepsilon W$ by *reflexivity*. Then $(\forall x \varepsilon W)\ xFb$ yields aFb by logic.

Suppose now that $a \lhd W$ has been obtained from $C(a,i) \lhd W$ because $i \in I(a)$. Then, by inductive hypothesis, we obtain that, for any $x\varepsilon C(a,i)$, xFb and hence aFb follows by *axiom-saturation*.

Finally, if $a \lhd W$ has been derived from $a \leq c$ and $c \lhd W$ by $\leq$-*left* then by inductive hypothesis we obtain cFb and hence aFb follows by $\leq$-*saturation*.

Let us now name the sub-categories of FTop and FTop^- whose objects are inductively generated formal topologies.

Definition 2.31 *We call* $\mathsf{FTop_i}$ *(*$\mathsf{FTop_i}^-$*) the full subcategory of* FTop *(respectively* FTop^-*) whose objects are inductively generated formal topologies.*

Note that from the impredicative point of view $\mathsf{FTop_i}$ is equivalent to FTop and $\mathsf{FTop_i}^-$ is equivalent to FTop^-; indeed, in this case, every formal topology $\mathcal{S}$ is inductively generated by the axiom-set obtained by considering all the cover relation like axioms indexed on $\mathcal{P}(S)$. On the contrary, from a predicative point of view, $\mathsf{FTop_i}$ and FTop are not equivalent because there are formal topologies which can not be generated by induction (see the last section of [CSSV03]).

2.6 Categorical product of formal topologies

In this section we recall some basic definitions about the categorical product of two inductively generated formal topologies. First of all, it is immediate to see that the terminal formal topology $\mathcal{T}$ that we introduced in section 2.4.2 is inductively generated.

Lemma 2.32 *Let $\mathcal{T}$ be the terminal formal topology defined in theorem 2.22. Then $\mathcal{T}$ can be generated inductively by using the empty set of axioms and the total order relation.*

Proof. First of all, it is worth noting that we do not need to add the *top-element* and the *positivity* axioms since they hold for $\mathcal{T}$ as a consequence of *reflexivity*. Now, we have to show that in the formal topology inductively generated by the empty set of axioms on the one element set $\{\top\}$, for any $a \in \{\top\}$ and $U \subseteq \{\top\}$, $a \varepsilon U$ if and only if $a \lhd U$. The implication from left to right is trivial by *reflexivity*; on the other hand, since there is no axiom, the only rules that one can use in the inductive generation are *reflexivity* and *$\leq$-left* and so the result is straightforward.

Now, let us recall that at present it is still open the question whether FTop is cartesian. Indeed, we are able to define the binary product of formal topologies only by means of an inductive definition and thus only $\mathsf{FTop_i}$ and $\mathsf{FTop_i}^-$ are known to be cartesian (see [CSSV03]). Since no proof appeared there we present a full proof of this result here.

Definition 2.33 *Let $\mathcal{A}$ and $\mathcal{B}$ be two inductively generated formal topologies whose axiom-sets are respectively $I_A(-)$, $C_A(-,-)$ and $I_B(-)$, $C_B(-,-)$. Then we call* binary product *of $\mathcal{A}$ and $\mathcal{B}$ the formal topology over the set $A \times B$, with order relation*

$$(a_1, b_1) \leq (a_2, b_2) \equiv (a_1 \leq_A a_2) \ \& \ (b_1 \leq_B b_2),$$

top-element $(\top_A, \top_B)$ and positivity predicate

$$\mathsf{Pos}_{A\times B}((a, b)) \equiv \mathsf{Pos}_A(a) \ \& \ \mathsf{Pos}_B(b),$$

inductively generated by the axiom-set

$$
\begin{array}{lll}
I((a,b)) & \equiv & I_A(a) + I_B(b) \\
C((a,b),i) & \equiv & \left\{ \begin{array}{ll} C_A(a,i_a) \times \{b\} & \textit{if } i \equiv \mathsf{inl}(i_a) \\ \{a\} \times C_B(b,i_b) & \textit{if } i \equiv \mathsf{inr}(i_b) \end{array} \right.
\end{array}
$$

One should note that in the previous definition we did not add the *top-element axiom* and the *positivity axiom*. In fact, we will prove that they are not necessary. Let us show first the following useful lemma.

Lemma 2.34 *Let $\mathcal{A}$ and $\mathcal{B}$ be inductively generated formal topologies, a be an element of A, b be an element of B, U be a subset of A and V be a subset of B. Then the following conditions are valid:*

$$
(1)\ \frac{a \lhd_A U}{(a,b) \lhd_{A\times B} U \times \{b\}} \qquad (2)\ \frac{b \lhd_B V}{(a,b) \lhd_{A\times B} \{a\} \times V} \qquad (3)\ \frac{a \lhd_A U \qquad b \lhd_B V}{(a,b) \lhd_{A\times B} U \times V}
$$

Proof. The proof of validity of the first two conditions can be obtained by arguing by induction on the derivation of $a \lhd_A U$ and $b \lhd_B V$ respectively. Then, the last condition can be proved as follows. Assume $a \lhd_A U$. Then, by the first condition we obtain that $(a,b) \lhd_{A\times B} U \times \{b\}$. But the second condition shows that, for every $u \varepsilon U$, $b \lhd_B V$ yields $(u,b) \lhd_{A\times B} \{u\} \times V$. Now, for every $v \varepsilon V$, from $(u,v) \varepsilon U \times V$ we get $(u,v) \lhd_{A\times B} U \times V$ by *reflexivity* and hence $(u,b) \lhd_{A\times B} U \times V$ follows by *transitivity*. Thus, again by *transitivity*, we conclude $(a,b) \lhd_{A\times B} U \times V$.

As an immediate corollary of this lemma, the product of two inductively generated formal topologies defined above is a formal topology.

Corollary 2.35 *Let $\mathcal{A}$ and $\mathcal{B}$ be inductively generated formal topologies. Then $\mathcal{A} \times \mathcal{B}$ is a formal topology.*

Proof. First of all note that it is immediate to check that the positivity predicate enjoys both $\leq$- *monotonicity* and *axiom monotonicity* and hence it is monotone with respect to the inductively generated cover.

Moreover, by using the previous lemma it is not difficult to show that *top-element* and *positivity* are satisfied for the product topology as a consequence of its validity in the component topologies. Indeed,

- (top-element) Let $(a,b) \in A \times B$. Then $a \lhd_A \top_A$ and $b \lhd_B \top_B$ yield immediately $(a,b) \lhd_{A\times B} (\top_A, \top_B)$ by lemma 2.34.
- (positivity condition) for every $a \in A$, $a \lhd_A a^+$ and, for every $b \in B$, $b \lhd_B b^+$. Then $(a,b) \lhd_{A\times B} a^+ \times b^+$ by the last condition in lemma 2.34 and hence $(a,b) \lhd_{A\times B} (a,b)^+$ follows since $a^+ \times b^+ = (a,b)^+$.

Thanks to this corollary in the following proofs by induction over the generation of the product cover we do not need to consider the case of the *top-element axiom* or the *positivity axiom* in the inductive generation.

The pairing and the projection maps are now defined.

Lemma 2.36 *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be inductively generated formal topologies and suppose that F is a continuous relation between $\mathcal{C}$ and $\mathcal{A}$ and G is a continuous relation between $\mathcal{C}$ and $\mathcal{B}$. Then the following relations*

$$\begin{array}{llcl} \textit{(pairing)} & c\ \langle F,G\rangle\ (a,b) & \equiv & c\ F\ a\ \&\ c\ G\ b \\ \textit{(first projection)} & (a,b)\ \Pi_1\ c & \equiv & (a,b) \lhd \{(x,y) \in A \times B \mid x \lhd_A c\} \\ \textit{(second projection)} & (a,b)\ \Pi_2\ d & \equiv & (a,b) \lhd \{(x,y) \in A \times B \mid y \lhd_B d\} \end{array}$$

are continuous and the following equations hold

$$\begin{array}{lcl} \Pi_1 * \langle F,G\rangle & = & F \\ \Pi_2 * \langle F,G\rangle & = & G \\ \langle \Pi_1 * H, \Pi_2 * H\rangle & = & H \end{array}$$

for any continuous relation H between $\mathcal{C}$ and $\mathcal{A} \times \mathcal{B}$.

Proof. Let us first prove that the relations we defined above are continuous.

- (pairing) We have to check that the pairing relation satisfies the required conditions.
  - (function totality) Let $c \in C$. Then, by *function totality* for F, there exists $a \in A$ such that cFa and, by *function totality* for G, there exists $b \in B$ such that cGb. Thus $c\langle F,G\rangle(a,b)$ follows, that is, we proved that *function totality* holds for $\langle F,G\rangle$.
  - (function convergence) Let $c \in C$ and suppose $c\langle F,G\rangle(a_1,b_1)$ and $c\langle F,G\rangle(a_2,b_2)$. Then cFa_1, cFa_2, cGb_1 and cGb_2 follow and hence $c \lhd F^-(a_1 \downarrow a_2)$ and $c \lhd G^-(b_1 \downarrow b_2)$ follow by *function convergence*. Thus,

    $$c \lhd F^-(a_1 \downarrow a_2) \downarrow G^-(b_1 \downarrow b_2)$$

    follows by $\downarrow$-*right* and hence we can conclude

    $$c \lhd \langle F,G\rangle^-((a_1,b_1) \downarrow (a_2,b_2))$$

    by *reflexivity* and *transitivity* since

    $$F^-(a_1 \downarrow a_2) \downarrow G^-(b_1 \downarrow b_2) \subseteq \langle F,G\rangle^-((a_1,b_1) \downarrow (a_2,b_2))$$

    Indeed, suppose $w\varepsilon F^-(a_1 \downarrow a_2) \downarrow G^-(b_1 \downarrow b_2)$. Then there exists $u\varepsilon F^-(a_1 \downarrow a_2)$ such that $w \lhd u$ and $v\varepsilon G^-(b_1 \downarrow b_2)$ such that $w \lhd v$. So, there is $u \in C$ such that $(\exists h\varepsilon a_1 \downarrow a_2)\ w \lhd u\ \&\ uFh$ and there is $v \in C$ such that $(\exists k\varepsilon b_1 \downarrow b_2)\ w \lhd v\ \&\ vGk$. Thus, by *weak-saturation* for F and G respectively, both $(\exists h\varepsilon a_1 \downarrow a_2)\ wFh$ and $(\exists k\varepsilon b_1 \downarrow b_2)\ wGk$ follow and hence $w\langle F,G\rangle(h,k)$ holds. But $h \lhd a_1$ and $k \lhd b_1$ yield $(h,k) \lhd (a_1,b_1)$ as well as $h \lhd a_2$ and $k \lhd b_2$ yield $(h,k) \lhd (a_2,b_2)$ by lemma 2.34. Hence $(h,k)\varepsilon(a_1,b_1) \downarrow (a_2,b_2)$ and thus we proved that $w\varepsilon\langle F,G\rangle^-((a_1,b_1) \downarrow (a_2,b_2))$.

– (function saturation) Suppose that c is an element of C such that $c \lhd W$ and $(\forall w \varepsilon W)\ w\langle F, G\rangle(a, b)$. Then, for all $w \varepsilon W$, both wFa and wGb hold and hence we obtain cFa and cGb by *function saturation* for F and G respectively and so $c\langle F, G\rangle(a, b)$ holds.

– (function axiom continuity) Suppose that $c \in C$, $c\langle F, G\rangle(a, b)$ and $j \in J((a, b))$. Then we have to prove that $c \lhd \langle F, G\rangle^-(C((a, b), j))$. Two cases must be considered according to the possible shape of an axiom for the product topology.

  * Suppose $C((a, b), j) \equiv \{a\} \times C_B(b, j')$. Then, from $c\ F\ a$ and $c\ G\ b$ we get $c \lhd F^-(\{a\})$ and $c \lhd G^-(C_B(b, j'))$ by *function continuity*. Thus $c \lhd F^-(\{a\}) \downarrow G^-(C_B(b, j'))$ follows by $\downarrow$*-right* and hence we obtain that $c \lhd \langle F, G\rangle^-(\{a\} \times C_B(b, j'))$ since $F^-(\{a\}) \downarrow G^-(C_B(b, j')) \subseteq \langle F, G\rangle^-(\{a\} \times C_B(b, j'))$. Indeed, let $w \varepsilon F^-(\{a\}) \downarrow G^-(C_B(b, j'))$. Then, there exist $u \varepsilon F^-(\{a\})$ and $v \varepsilon G^-(C_B(b, j'))$ such that $w \lhd u$ and $w \lhd v$. Thus, we get both that wFa by *weak-saturation* on F, and that there exists $h \varepsilon C_B(b, j')$ such that wGh by *weak-saturation* on G. Finally, we get $w\langle F, G\rangle(a, h)$ which yields $w \varepsilon \langle F, G\rangle^-(\{a\} \times C_B(b, j'))$.
  * The case $C((a, b), j) \equiv C_A(a, j') \times \{b\}$ is completely analogous to the previous one.

– ($\leq$-continuity) Suppose both that c is an element of C such that $c\langle F, G\rangle(a_1, b_1)$ and that $(a_1, b_1) \leq (a_2, b_2)$. Then, by definition, $c\ F\ a_1$ and $c\ G\ b_1$. Moreover, $(a_1, b_1) \leq (a_2, b_2)$ yields both $a_1 \leq a_2$ and $b_1 \leq b_2$. Thus $c\ F\ a_2$ and $c\ G\ b_2$ follow by $\leq$*-continuity* for F and G and hence $c\langle F, G\rangle(a_2, b_2)$ holds.

- (first projection) We have to check that the required conditions are satisfied.

  – (function totality) Suppose that $(a, b) \in A \times B$. Then $(a, b)\Pi_1 a$ trivially holds and hence Π_1 enjoys *function totality*.

  – (function convergence) Suppose that $(a, b)\ \Pi_1\ c_1$ and $(a, b)\ \Pi_1\ c_2$ hold. Then $(a, b) \lhd \{(x, y) \mid x \lhd c_1\}$ and $(a, b) \lhd \{(x, y) \mid x \lhd c_2\}$ holds and hence

$$(a, b) \lhd \{(x, y) \in A \times B \mid x \lhd c_1\} \downarrow_{\leq} \{(x, y) \in A \times B \mid x \lhd c_2\}$$

  follows by $\leq$*-right*. Thus, to conclude by *reflexivity* and *transitivity* it is sufficient to prove that

$$\{(x, y) \in A \times B \mid x \lhd c_1\} \downarrow_{\leq} \{(x, y) \in A \times B \mid x \lhd c_2\} \subseteq \Pi_1^-(c_1 \downarrow c_2)$$

  To this aim, let us suppose that

$$(x, y) \varepsilon \{(x, y) \in A \times B \mid x \lhd c_1\} \downarrow_{\leq} \{(x, y) \in A \times B \mid x \lhd c_2\}$$

Then there exist both $(x_1, y_1) \in A \times B$, such that $x_1 \lhd c_1$ and $(x, y) \leq (x_1, y_1)$, and $(x_2, y_2) \in A \times B$, such that $x_2 \lhd c_2$ and $(x, y) \leq (x_2, y_2)$. Thus, both $x \leq x_1$ and $x \leq x_2$ follow and hence we obtain $x \lhd c_1$ and $x \lhd c_2$ by $\leq$-*left* and *transitivity*. Then $x \varepsilon c_1 \downarrow c_2$ follows by $\downarrow$-*right* and hence there exists an element $z \varepsilon c_1 \downarrow c_2$, that is, x itself, such that $(x, y)\varepsilon\{(s, t) \mid s \lhd z\}$ which yields $(x, y)\ \Pi_1\ z$ by *reflexivity*. So, $(x, y)\varepsilon\Pi_1^-(c_1 \downarrow c_2)$.

  – (function saturation) Suppose that $(a, b) \lhd W$ and that, for any $(x, y)\varepsilon W$, $(x, y)\ \Pi_1\ c$. Then, for any $(x, y)\varepsilon W$, $(x, y) \lhd \{(s, t) \in A \times B \mid s \lhd_A c\}$ and hence $(a, b) \lhd \{(s, t) \in A \times B \mid s \lhd_A c\}$ follows by *transitivity*. Thus, $(a, b)\ \Pi_1\ c$ holds.
  – (function continuity) Suppose that $(a, b)\ \Pi_1\ c$ and $c \lhd U$ hold. Then $(a, b) \lhd \{(x, y) \in A \times B \mid x \lhd_A c\}$ and hence $(a, b) \lhd U \times B$ follows by *transitivity* since $\{(x, y) \in A \times B \mid x \lhd_A c\} \lhd U \times B$. Indeed, suppose $(x, y)\varepsilon\{(x, y) \in A \times B \mid x \lhd_A c\}$. Then $x \lhd_A c$ and hence $x \lhd_A U$ follows by *transitivity*. Moreover, $y \lhd_B B$ holds by *reflexivity* and hence we obtain $(x, y) \lhd U \times B$ by lemma 2.34. Now, we can conclude $(a, b) \lhd \Pi_1^-(U)$ by *reflexivity* and *transitivity* since $U \times B \subseteq \Pi_1^-(U)$. Indeed, suppose $(x, y)\varepsilon U \times B$. Then $(x, y)\varepsilon\{(s, t) \in A \times B \mid s \lhd_A x\}$ and hence $(x, y)\ \Pi_1\ x$ follows by *reflexivity*. Thus, there exists $u\varepsilon U$ such that $(x, y)\ \Pi_1\ u$, that is, $(x, y)\varepsilon\Pi_1^-(U)$.

- (second projection) Completely analogous to the previous one.

The second part of the proof consists in showing that the three required equations hold.

Let us first notice that, for any $c \in C$, $a \in A$ and $b \in B$ and for any continuous relation H between $\mathcal{C}$ and $\mathcal{A} \times \mathcal{B}$,

$$\begin{aligned} c\,\Pi_1 * H\ a \quad &\Rightarrow \quad c \lhd H^-(\{(x, y) \in A \times B \mid x \lhd a\}) \\ c\,\Pi_2 * H\ b \quad &\Rightarrow \quad c \lhd H^-(\{(x, y) \in A \times B \mid y \lhd b\}) \end{aligned}$$

Indeed, $c\,\Pi_1 * H\ a$ if and only if $c \lhd \{w \in C \mid w\,\Pi_1 \circ H\ a\}$, that is,

$$c \lhd \{w \in C \mid (\exists (x, y) \in A \times B)\ wH(x, y)\ \&\ (x, y) \lhd \{(x, y) \mid x \lhd a\}\}$$

Thus, by *function continuity* we obtain $c \lhd \{w \in C \mid w \lhd H^-(\{(x, y) \mid x \lhd a\})\}$ and hence $c \lhd H^-(\{(x, y) \mid x \lhd a\})$ follows by *transitivity*. The proof of the other implication is completely similar.

We can now proceed with the proof of validity of the equations.

- We have to prove that $\Pi_1 * \langle F, G\rangle = F$, namely, for any $c \in C$ and $a \in A$, $c\ \Pi_1 * \langle F, G\rangle\ a$ if and only if $c\ F\ a$. The right to left implication can be proved as follows. Suppose that cFa holds. Then, by *function totality* for G, there exists an element $b \in B$ such that cGb. Thus $c\langle F, G\rangle(a, b)$ follows. Moreover, $(a, b)\Pi_1 a$ trivially holds and hence we obtain that $c\ \Pi_1 \circ \langle F, G\rangle\ a$. Thus the result is immediate since $\Pi_1 \circ \langle F, G\rangle \subseteq \Pi_1 * \langle F, G\rangle$.

Now, let us prove the other implication. Suppose that $c\ \Pi_1 * \langle F, G\rangle\ a$. Then, the observation above shows that $c \lhd \langle F, G\rangle^-(\{(x,y) \mid x \lhd a\})$ and hence we obtain cFa by *saturation* since $c \lhd F^-(\{a\})$ follows by *reflexivity* and *transitivity* because we can prove that

$$\langle F, G\rangle^-(\{(x,y) \mid x \lhd a\}) \subseteq F^-(\{a\})$$

Indeed, suppose that $w\varepsilon\langle F, G\rangle^-(\{(x,y) \mid x \lhd a\})$. Then, there exists $(x,y) \in A \times B$ such that $w\langle F, G\rangle(x,y)$ and $x \lhd a$. Thus wFx follows and hence we obtain wFa by *weak-continuity* for F. So, we conclude $w\varepsilon F^-(\{a\})$.

- We have to prove that $\Pi_2 * \langle F, G\rangle = G$. Completely analogous to the previous point.

- We have to prove that for any continuous relation H between $\mathcal{C}$ and $\mathcal{A}\times\mathcal{B}$, $\langle \Pi_1 * H, \Pi_2 * H\rangle = H$.

  It is immediate to check that if $cH(a,b)$ then $c\ \langle \Pi_1 * H, \Pi_2 * H\rangle\ (a,b)$. Indeed, $(a,b)\Pi_1 a$ and $(a,b)\Pi_2 b$ clearly hold and hence $cH(a,b)$ yields both $c\ \Pi_1 \circ H\ a$ and $c\ \Pi_2 \circ H\ b$. So, $c\ \langle \Pi_1 \circ H, \Pi_2 \circ H\rangle\ (a,b)$ follows since $\Pi_1 \circ H \subseteq \Pi_1 * H$ and $\Pi_2 \circ H \subseteq \Pi_2 * H$.

  To prove the other implication let us assume that $c\ \langle \Pi_1 * H, \Pi_2 * H\rangle\ (a,b)$ holds. Then both $c\ \Pi_1 * H\ a$ and $c\ \Pi_2 * H\ b$ follows and hence by the previous observation we obtain both $c \lhd H^-(\{(x,y) \mid x \lhd a\})$ and $c \lhd H^-(\{(x,y) \mid y \lhd b\})$. Thus

$$c \lhd H^-(\{(x,y) \mid x \lhd a\}) \downarrow_\leq H^-(\{(x,y) \mid y \lhd b\})$$

  follows by $\leq$-*right*. Now we can conclude $cH(a,b)$ by *saturation* since $c \lhd H^-(\{(a,b)\})$ follows by *transitivity* because we can prove that

$$H^-(\{(x,y) \mid x \lhd a\}) \downarrow_\leq H^-(\{(x,y) \mid y \lhd b\}) \lhd H^-(\{(a,b)\})$$

  Indeed, let us suppose that

$$w\varepsilon H^-(\{(x,y) \mid x \lhd a\}) \downarrow_\leq H^-(\{(x,y) \mid y \lhd b\})$$

  Then, there is both an element $w_1 \in C$ such that $w \leq w_1$, $w_1 H(x_1, y_1)$ and $x_1 \lhd a$ for some element $(x_1, y_1) \in A \times B$ and an element $w_2 \in C$ such that $w \leq w_2$, $w_2 H(x_2, y_2)$ and $y_2 \lhd b$ for some element $(x_2, y_2) \in A \times B$. Hence $wH(x_1, y_1)$ and $wH(x_2, y_2)$ follow by $\leq$-*saturation* and hence, by *function* $\leq$-*convergence* we obtain $w \lhd H^-((x_1, y_1) \downarrow_\leq (x_2, y_2))$. Consider now any element $(s,t)\varepsilon(x_1, y_1) \downarrow_\leq (x_2, y_2)$. Then $(s,t) \leq (x_1, y_1)$ and $(s,t) \leq (x_2, y_2)$ and hence $s \leq x_1$ and $t \leq y_2$. So, by $\leq$-*left* we obtain $s \lhd a$ and $t \lhd b$ which yield that $(s,t) \lhd (a,b)$ by lemma 2.34, that is, we proved that $(x_1, y_1) \downarrow_\leq (x_2, y_2) \lhd (a,b)$. Hence $H^-((x_1, y_1) \downarrow_\leq (x_2, y_2)) \lhd H^-((a,b))$ follows by lemma 2.8 and thus we obtain $w \lhd H^-(\{(a,b)\})$ by *transitivity*.

Thus, we proved the main theorem of this section.

Proposition 2.37 $\mathsf{FTop_i}$ *and* $\mathsf{FTop_i}^-$ *are cartesian.*

Indeed, we gave all the definitions and developed all of the proofs above without using at all the positivity predicate of $\mathcal{A} \times \mathcal{B}$ and hence the same proofs work both for $\mathsf{FTop_i}$ and $\mathsf{FTop_i}^-$.

2.6.1 Properties on top element and positivity predicate

Note that all the proofs and definitions in the previous section do not require the presence of the top-element. Thus such proofs apply also when one uses a definition of formal topology which does not consider the top element as, for instance, in [CSSV03]. But, note that *function totality*, which is crucial to prove the validity of one of the equations, can be justified only by assuming the existence of the top-element.

Moreover, some of the definitions can be simplified when the top element or the positivity predicate are present. We will use such simplified definitions in the following sections.

Lemma 2.38 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be inductively generated formal topologies. Then, for any* $a, c \in A$ *and* $b, d \in B$*,*

$$(a,b)\ \Pi_1\ c \quad \textit{iff} \quad (a,b) \lhd_{A \times B} (c, \top_B)$$
$$(a,b)\ \Pi_2\ d \quad \textit{iff} \quad (a,b) \lhd_{A \times B} (\top_A, d)$$

Proof. We prove only the first of the two equivalences being the second completely similar. Let $(a,b) \in A \times B$ and $c \in A$ and suppose that $(a,b)\Pi_1 c$ holds. Then $(a,b) \lhd \{(x,y) \in A \times B \mid x \lhd c\}$. Now, we can conclude $(a,b) \lhd (c, \top_B)$ by *transitivity* since $\{(x,y) \in A \times B \mid x \lhd c\} \lhd (c, \top_B)$. Indeed, let $(x,y)\varepsilon\{(x,y) \in A \times B \mid x \lhd c\}$. Then $x \lhd c$ and $y \lhd \top_B$ and hence $(x,y) \lhd (c, \top_B)$ by lemma 2.34.

On the other hand, $(c, \top_B)\varepsilon\{(x,y) \in A \times B \mid x \lhd c\}$ and hence $(a,b) \lhd (c, \top_B)$ yields $(a,b) \lhd \{(x,y) \in A \times B \mid x \lhd c\}$ by using first *reflexivity* and hence *transitivity*.

We show now that the use of the positivity predicate allows to state and prove some attended properties on the binary product of inductively generated formal topologies.

Lemma 2.39 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be inductively generated formal topologies,* a *be a positive element of* A*,* b *be a positive element of* B *and* V *be a subset of* $A \times B$ *and suppose that* $(a,b) \lhd_{A \times B} V$ *holds. Then,* $a \lhd_A \pi_1(V)$ *and* $b \lhd_B \pi_2(V)$*, where* $\pi_1(V) \equiv \{a \in A \mid (\exists b \in B)\ (a,b)\varepsilon V\}$ *and* $\pi_2(V) \equiv \{b \in B \mid (\exists a \in A)\ (a,b)\varepsilon V\}$*.*

Proof. Since the product topology of two inductively generated formal topologies is inductively generated too, we can develop a proof by induction on the length of the derivation of $(a,b) \lhd_{A \times B} V$.

- (reflexivity) Suppose $(a,b) \lhd V$ because $(a,b)\varepsilon V$. Then $a\varepsilon\pi_1(V)$ and $b\varepsilon\pi_2(V)$. Hence $a \lhd \pi_1(V)$ and $b \lhd \pi_2(V)$ follow by *reflexivity.*
- (infinity) Two cases have to be considered.
  - Suppose $(a,b) \lhd V$ because $j \in J(a)$ and $C(a,j) \times \{b\} \lhd V$, that is, for every $x\varepsilon C(a,j)$, $(x,b) \lhd V$. Then, for every element $x\varepsilon C(a,j)^+$ we have $(x,b) \lhd V$. Hence, by inductive hypothesis we obtain both $x \lhd \pi_1(V)$ and $b \lhd \pi_2(V)$. Thus, $a \lhd \pi_1(V)$ follows by *transitivity*, since $a \lhd C(a,j)^+$, and $b \lhd \pi_2(V)$ follows because by *monotonicity* $\mathsf{Pos}_A(a)$ yields that $C(a,j)^+$ is inhabited.
  - Suppose $(a,b) \lhd V$ because $k \in K(b)$ and $\{a\} \times C(b,k) \lhd V$. The proof of this case is completely similar to the previous one.
- ($\leq$-left) Suppose $(a,b) \lhd V$ because $(a',b') \lhd V$ and $(a,b) \leq (a',b')$. Then, by inductive hypothesis, $a' \lhd \pi_1(V)$ and $b' \lhd \pi_2(V)$. But $(a,b) \leq (a',b')$ yields $a \leq a'$ and $b \leq b'$ and hence $a \lhd \pi_1(V)$ and $b \lhd \pi_2(V)$ follow by *$\leq$-left.*

The previous lemma allows to prove that there is still another equivalent formulation of the projection maps. We wanted to recall it here since the following ones are the definitions of projection maps used in [SVV96].

Lemma 2.40 *Let $\mathcal{A}$ and $\mathcal{B}$ be inductively generated formal topologies. Then, for any $a,c \in A$ and $b,d \in B$,*

$$(a,b)\ \Pi_1\ c \quad \textit{iff} \quad \mathsf{Pos}_B(b) \to a \lhd_A c$$
$$(a,b)\ \Pi_2\ d \quad \textit{iff} \quad \mathsf{Pos}_A(a) \to b \lhd_B d$$

Proof. Let us prove only the first of the two equivalences, being the second completely analogous. So, let us suppose that $(a,b)\ \Pi_1\ c$ and assume both $\mathsf{Pos}_A(a)$ and $\mathsf{Pos}_B(b)$. Then, by lemma 2.39, $a \lhd_A \{x \in A \mid x \lhd_A c\}$ and hence $a \lhd_A c$ follows by *transitivity.* Hence $\mathsf{Pos}_B(b) \to a \lhd_A c$ follows by discharging first $\mathsf{Pos}_A(a)$ by *positivity* and then $\mathsf{Pos}_B(b)$.

To prove the other implication let us assume that $\mathsf{Pos}((a,b))$ holds. Then $\mathsf{Pos}_B(b)$ follows and hence $\mathsf{Pos}_B(b) \to a \lhd_A c$ yields $a \lhd_A c$. So, we obtain that $(a,b)\varepsilon\{(x,y) \in A \times B \mid x \lhd_A c\}$ holds and hence $(a,b)\ \Pi_1\ c$ follows by *reflexivity* and *positivity.*

2.7 Unary topologies

The main result of this paper is a proof that unary topologies are exponentiable over inductively generated formal topologies. So, let us recall here the definition of unary topology.

Definition 2.41 *A formal topology $(S, \top, \lhd, \mathsf{Pos})$ is called* unary *if, for any $a \in S$ and $U \subseteq S$, $a \lhd U$ if and only if $\mathsf{Pos}(a) \to (\exists b\varepsilon U)\ a \lhd \{b\}$.*

It is trivial to see that unary formal topologies form a full sub-category of FTop; we will call unFTop such a sub-category. The definition of unary formal topology needs to be slightly modified when we work in $\mathsf{FTop_i}^-$. Indeed, in this case, there is no positivity predicate; so, we say that a formal topology $(S, \top, \lhd)$ is *unary* if, for any $a \in S$ and $U \subseteq S$, $a \lhd U$ if and only if $(\exists b \varepsilon U)\ a \lhd \{b\}$.

Unary topologies are distinguishable among formal topologies because the collection of their formal points, when it is not empty, forms an algebraic dcpo with a bottom element (see for instance [Sig90], [SVV96] or [Sam00]). In fact, supposing that $\mathsf{Alg}^0_\bot$ is the category of algebraic dcpos with a bottom element and Scott-continuous functions [AJ94] enriched with an initial object, we can prove the following theorem.

Theorem 2.42 *The category* unFTop *is impredicatively equivalent to* $\mathsf{Alg}^0_\bot$*.*

Proof. The equivalence functor from unFTop to $\mathsf{Alg}^0_\bot$ is the functor $\mathsf{Pt}(-)$, which associates to a unary formal topology $\mathcal{A}$ the collection $\mathsf{Pt}(\mathcal{A})$ of its formal points and to a continuous relation F the induced continuous function ϕ_F as defined in section 2.4 by setting $\phi_F(\alpha) \equiv \bigcup_{a\varepsilon\alpha} \overrightarrow{F}(a)$. Let us prove that this functor is well defined.

In order to check it on the objects, we have to prove that $\mathsf{Pt}(\mathcal{A})$ is an algebraic dcpo with a bottom element. So, let $\mathcal{A}$ be a unary formal topology and suppose that $(\alpha_i)_{i\in I}$ is a directed family of formal points of $\mathcal{A}$. Then $\alpha \equiv \bigcup_{i\in I} \alpha_i$ is a formal point. Indeed, α is clearly not empty and if $a\varepsilon\alpha$ and $b\varepsilon\alpha$ then there are α_i and α_j such that $a\varepsilon\alpha_i$ and $b\varepsilon\alpha_j$. But we assumed that the family I of formal points is directed and thus there is a formal point α_k such that $\alpha_i \subseteq \alpha_k$ and $\alpha_j \subseteq \alpha_k$. Hence, $a, b\varepsilon\alpha_k$ and so there exists $c\varepsilon\alpha_k$, which yields $c\varepsilon\alpha$, such that $c \lhd a$ and $c \lhd b$. Finally, if $a\varepsilon\alpha$ and $a \lhd U$ then there exists $i \in I$ such that $a\varepsilon\alpha_i$. Thus, there exists $u\varepsilon U$ such that $u\varepsilon\alpha_i$ which yields $u\varepsilon\alpha$.

The second step in the proof is to notice that, in the case of unary formal topologies, the subset $\uparrow a \equiv \{c \in A \mid a \lhd \{c\}\}$ is a formal point for any positive element $a \in A$. Indeed, $\uparrow a$ is clearly not empty and if $b_1, b_2\varepsilon \uparrow a$, that is, $a \lhd \{b_1\}$ and $a \lhd \{b_2\}$, then $a\varepsilon b_1 \downarrow b_2$ and hence there exists $c\varepsilon b_1 \downarrow b_2$ such that $a \lhd \{c\}$, that is, $c\varepsilon \uparrow a$. Finally, if $b\varepsilon \uparrow a$ and $b \lhd U$ then $a \lhd \{b\}$ and hence $a \lhd U$. But we are dealing within a unary topology and hence we obtain that $\mathsf{Pos}(a) \to (\exists u\varepsilon U)\ a \lhd \{u\}$ and hence $\mathsf{Pos}(a)$ yields $(\exists u\varepsilon U)\ a \lhd \{u\}$, that is, $(\exists u\varepsilon U)\ u\varepsilon \uparrow a$.

Now, observe that, for any point β, $\beta = \bigcup_{b\varepsilon\beta} \uparrow b$ and the union on the right is directed since $b\varepsilon\beta$ if and only if $\uparrow b \subseteq \beta$ and hence if $\uparrow b_1 \subseteq \beta$ and $\uparrow b_2 \subseteq \beta$ then there exists c such that $c \lhd \{b_1\}$, that is, $\uparrow b_1 \subseteq \uparrow c$, $c \lhd \{b_2\}$, that is, $\uparrow b_2 \subseteq \uparrow c$ and $\uparrow c \subseteq \beta$.

Thus, the formal points whose shape is $\uparrow b$, for $b\varepsilon\mathsf{Pos}_A$, are the compact elements in the ordered structure $(\mathsf{Pt}(\mathcal{A}), \subseteq)$. Indeed, if $\uparrow a \subseteq \bigcup_{i\in I} \beta_i$ then $a\varepsilon \bigcup_{i\in I} \beta_i$; hence there exists $i \in I$ such that $a\varepsilon\beta_i$ and thus $\uparrow a \subseteq \beta_i$. On the other hand, suppose that β is a compact formal point, then $\beta \subseteq \bigcup_{b\varepsilon\beta} \uparrow b$ yields that there exists $b\varepsilon\beta$ such that $\beta \subseteq \uparrow b$ and hence $\beta = \uparrow b$.

So, any formal point $\beta = \bigcup_{b\varepsilon\beta} \uparrow b$ is the directed supremum of all the compact element smaller than it, that is, $(\mathsf{Pt}(\mathcal{A}), \subseteq)$ is an algebraic dcpo.

Finally, it is clear that, provided $\mathsf{Pos}_A(\top_A)$ holds, the dcpo $(\mathsf{Pt}(\mathcal{A}), \subseteq)$ has a bottom element which is the formal point $\uparrow \top_A$ since such a formal point is clearly contained in any other formal point because of *point inhabitance* and *point splitness*.

Let us show now that $\mathsf{Pt}(-)$ is well defined also on morphisms. We have to show that any continuous function ϕ_F induced by a continuous relation F is Scott-continuous. But this result is immediate because the topology on $\mathsf{Pt}(\mathcal{A})$ is a Scott topology. Indeed, the base for such a topology is the family

$$\mathsf{ext}^{\mathsf{Pt}}(a) \equiv \{\alpha \in \mathsf{Pt}(\mathcal{A}) \mid a\varepsilon\alpha\}$$

for $a \in A$. Thus, any open subset $\mathcal{O}$ is obtained as union of elements in the base, namely,

$$\mathcal{O} \equiv \bigcup_{a\varepsilon U} \mathsf{ext}^{\mathsf{Pt}}(a)$$

Then we have to show that if $\alpha \subseteq \beta$ and $\alpha \in \mathcal{O}$ then $\beta \in \mathcal{O}$ and that if $\bigcup_{i\in I} \beta_i \in \mathcal{O}$ then there exists $i \in I$ such that $\beta_i \in \mathcal{O}$. So, let us suppose that $\alpha \subseteq \beta$ and $\alpha \in \mathcal{O}$. Then there exists $a\varepsilon U$ such that $\alpha \in \mathsf{ext}^{\mathsf{Pt}}(a)$, that is, $a\varepsilon\alpha$. Hence, $a\varepsilon\beta$ follows and thus we trivially obtain $\beta \in \mathsf{ext}^{\mathsf{Pt}}(a)$ which yields $\beta \in \mathcal{O}$. Let us suppose now that $\bigcup_{i\in I} \beta_i \in \mathcal{O}$. Then there exists $a\varepsilon U$ such that $\bigcup_{i\in I} \beta_i \in \mathsf{ext}^{\mathsf{Pt}}(a)$, that is, $a\varepsilon \bigcup_{i\in I} \beta_i$. Thus, there exists $i \in I$ such that $a\varepsilon\beta_i$, which yields $\beta_i \in \mathsf{ext}^{\mathsf{Pt}}(a)$ and so $\beta_i \in \mathcal{O}$.

Viceversa, unary topologies allows to present all the algebraic dcpos with a bottom element (provided that they have a *set* of compact elements). Indeed, we can define a functor $\downarrow (-)$ from $\mathsf{Alg}^0_\bot$ to unFTop which maps an algebraic dcpo with bottom element $(D, \preceq, \bot)$ to a formal topology $\mathcal{K}_D$ on the set K_D of its compact elements, by setting

$$\begin{array}{rcl} \mathsf{Pos}(a) & \equiv & \mathsf{True} \\ a \lhd U & \equiv & (\exists b\varepsilon U)\ a \sqsupseteq b \end{array}$$

Then, $\mathcal{K}_D$ is a unary formal topology whose top element is $\bot$ and whose formal points form an algebraic dcpo which is isomorphic to $(D, \preceq, \bot)$.

The functor $\downarrow (-)$ is extended to morphisms by mapping any continuous function f from the algebraic dcpo D_1 to the algebraic dcpo D_2 to the continuous relation defined by setting, for any $a \in K_{D_1}$ and $b \in K_{D_2}$,

$$a\ R_f\ b \equiv f(a) \sqsupseteq b$$

It is clear that the functors $\mathsf{Pt}(-)$ and $\downarrow (-)$ establish an equivalence between the categories unFTop and $\mathsf{Alg}^0_\bot$.

It is worth noting that we can obtain a category equivalent to the category $\mathsf{Alg}_\bot$ of the algebraic dcpos with a bottom element if we restrict to unary formal topologies whose top element is positive.

2.7.1 Inductive generation of a unary topology

In this section we recall the fact that all the unary formal topologies are inductively generated (see [CSSV03]). While it is obvious that this result trivially holds from an impredicative point of view, it is interesting to note that a predicative proof requires the use of the axiom of choice which is an immediate consequence of the definition of Σ-type in Martin-Löf's Type theory [Mar84].

Theorem 2.43 *Let $\mathcal{A}$ be a unary formal topology in* $\mathsf{FTop_i}$*. Then $\mathcal{A}$ can be inductively generated.*

Proof. We have to furnish an axiom-set for $\mathcal{A}$. Now, by definition, $a \lhd U$ holds if and only if $\mathsf{Pos}(a) \to (\exists u \varepsilon U)\ a \lhd \{u\}$. Thus, by the axiom of choice, there exists $f \in \mathsf{Pos}(a) \to A$ such that, for any $x \in \mathsf{Pos}(a)$, $f(x)\varepsilon U$ and $a \lhd \{f(x)\}$. It is now easy to check that the axiom-set that we are looking for is

$$\begin{array}{lcl} I(a) & \equiv & \{f : \mathsf{Pos}(a) \to A \mid (\forall x \in \mathsf{Pos}(a))\ a \lhd \{f(x)\}\} \\ C(a,f) & \equiv & \mathsf{Im}(f) \end{array}$$

where $\mathsf{Im}(f) \equiv \{c \in A \mid (\exists x \in \mathsf{Pos}(a))\ c = f(x)\}$

The statement above concerns $\mathsf{FTop_i}$, but a completely similar result can be proved within $\mathsf{FTop_i}^-$ by just substituting the positivity predicate with an always true proposition. In the rest of this section we will continue to present our results within $\mathsf{FTop_i}$ since it is usually more complex to obtain them in the presence of the positivity predicate but it is easy to check that all we do can be re-done within $\mathsf{FTop_i}^-$.

Given any formal topology $(A, \lhd, \mathsf{Pos})$ it is not difficult to obtain a unary topology out of it.

Lemma 2.44 *Let $\mathcal{A} \equiv (A, \top, \lhd, \mathsf{Pos})$ be a formal topology and set*

$$a \lhd_{\mathsf{Un}} U \equiv \mathsf{Pos}(a) \to (\exists u \varepsilon U)\ a \lhd \{u\}$$

Then $\mathsf{Un}(A) \equiv (A, \top, \lhd_{\mathsf{Un}}, \mathsf{Pos})$ is a unary topology called the unary image *of $\mathcal{A}$.*

Proof. First of all, we have to check that all the conditions for $\mathsf{Un}(A)$ being a formal topology are satisfied.

- (top-element) We have to show that, for any $a \in A$, $a \lhd_{\mathsf{Un}} \top$ but this is a trivial consequence of the fact that $a \lhd \top$.
- (Reflexivity) Suppose $a\varepsilon U$. Then $a \lhd a$ yields $(\exists u \varepsilon U)\ a \lhd u$, which yields immediately $a \lhd_{\mathsf{Un}} U$.
- (Transitivity) Suppose $a \lhd_{\mathsf{Un}} U$ and $U \lhd_{\mathsf{Un}} V$ and assume that $\mathsf{Pos}(a)$ holds. Then there exists $u\varepsilon U$ such that $a \lhd \{u\}$ and hence both $\mathsf{Pos}(u)$ and $u \lhd_{\mathsf{Un}} V$ follow. Hence there exists $v\varepsilon V$ such that $u \lhd \{v\}$ and hence $a \lhd \{v\}$ follows by *transitivity*. Thus, $a \lhd_{\mathsf{Un}} V$ follows by discharging the assumption $\mathsf{Pos}(a)$.

- ($\downarrow$-right) Suppose $a \lhd_{\mathsf{Un}} U$ and $a \lhd_{\mathsf{Un}} V$ and assume that $\mathsf{Pos}(a)$ holds. Then there exists $u \varepsilon U$ and $v \varepsilon V$ such that $a \lhd \{u\}$ and $a \lhd \{v\}$. Then $a \varepsilon U \downarrow V$ and hence there exists $z \varepsilon U \downarrow V$ such that $a \lhd \{z\}$; thus we obtain $a \lhd_{\mathsf{Un}} U \downarrow V$ by discharging the assumption $\mathsf{Pos}(a)$.

- (monotonicity) Suppose that $\mathsf{Pos}(a)$ and $a \lhd_{\mathsf{Un}} U$ holds. Then there exists $u \varepsilon U$ such that $a \lhd \{u\}$ and hence $\mathsf{Pos}(u)$ follows by *monotonicity*.

- (Positivity) Immediate, by logic.

It is now obvious that $\mathsf{Un}(A)$ is a unary formal topology.

2.7.2 Embedding in a unary topology

One can prove that any formal topology embeds continuously into its unary image. This result follows easily after the next lemma.

Lemma 2.45 *Let $\mathcal{A}_1 \equiv (A, \top_1, \lhd_1, \mathsf{Pos}_1)$ and $\mathcal{A}_2 \equiv (A, \top_2, \lhd_2, \mathsf{Pos}_2)$ be two formal topologies on the same base A and suppose that, for any $a, b \in A$ and $U \subseteq A$,*

$$\textit{(cover embedding)} \quad \frac{a \lhd_2 U}{a \lhd_1 U} \qquad \textit{(convergence embedding)} \quad a \downarrow_1 b \lhd_1 a \downarrow_2 b$$

Then $\top_1 \lhd_1 \top_2$ and, for any $a \in A$, $\mathsf{Pos}_1(a) \rightarrow \mathsf{Pos}_2(a)$. Moreover, $\lhd_1$ is a continuos relation between $\mathcal{A}_1$ and $\mathcal{A}_2$ and, for any point α in $\mathsf{Pt}(\mathcal{A}_1)$, α is also a point in $\mathsf{Pt}(\mathcal{A}_2)$.

Proof. By *top element* for $\mathcal{A}_2$, $\top_1 \lhd_2 \top_2$ and hence $\top_1 \lhd_1 \top_2$ follows immediately by *cover embedding*. Moreover, for any $a \in A$, $a \lhd_2 a^{+_2}$. Then, by *cover embedding*, $a \lhd_1 a^{+_2}$ and hence $\mathsf{Pos}_1(a)$ yields that there exists $x \varepsilon a^{+_2}$ such that $\mathsf{Pos}_1(x)$. But $x \varepsilon a^{+_2}$ yields $\mathsf{Pos}_2(a)$.

It is now straightforward to check that $\lhd_1$ is a continuous relation. Indeed, *function totality* and *function saturation* hold trivially. Moreover, *function continuity* can be proved as follows. Suppose that $a \lhd_1 b$ and $b \lhd_2 V$; then $b \lhd_1 V$ follows by *cover embedding* and hence we obtain $a \lhd_1 V$ by *transitivity*; thus $a \lhd_1 \lhd_1^{-}(V)$ follows by *reflexivity* and *transitivity* since $V \subseteq \lhd_1^{-}(V)$. Finally, also *function convergence* holds. Indeed, if $a \lhd_1 b$ and $a \lhd_1 d$ then $a \varepsilon \lhd_1^{-}(b \downarrow_1 d)$; now, $b \downarrow_1 d \lhd_1 b \downarrow_2 d$ holds by *convergence embedding* and hence $\lhd_1^{-}(b \downarrow_1 d) \lhd_1 \lhd_1^{-}(b \downarrow_2 d)$ follows by lemma 2.8 which only requires *function continuity* of $\lhd_1$ that we already proved; so $a \lhd_1 \lhd_1^{-}(b \downarrow_2 d)$ follows by *reflexivity* and *transitivity*.

Let us suppose now that $\alpha \in \mathsf{Pt}(\mathcal{A}_1)$. Then α is obviously inhabited. Moreover if $a, b \varepsilon \alpha$ then by *point convergence* there exists $c \varepsilon a \downarrow_1 b$ such that $c \varepsilon \alpha$; hence $c \lhd_1 a \downarrow_1 b$ and so $c \lhd_1 a \downarrow_2 b$ follows by *convergence embedding* and thus, by *point splitness*, there exists $d \varepsilon a \downarrow_2 b$ such that $d \varepsilon \alpha$. Finally, if $a \varepsilon \alpha$ and $a \lhd_2 V$ then $a \lhd_1 V$ follows by *cover embedding* and hence, by *point splitness*, there exists an element $c \varepsilon V$ such that $c \varepsilon \alpha$.

In the case of inductively generated formal topologies this lemma can be simplified as follows.

Corollary 2.46 *Let* $\mathcal{A}_1 \equiv (A, \top_1, \leq, \lhd_1, \mathsf{Pos}_1)$ *and* $\mathcal{A}_2 \equiv (A, \top_2, \leq, \lhd_2, \mathsf{Pos}_2)$ *be two inductively generated formal topologies, on the same base* A *and with the same pre-order relation* $\leq$*, whose axiom-sets are respectively* $I_1(-)$*,* $C_1(-,-)$ *and* $I_2(-)$*,* $C_2(-,-)$*. Then, if, for any* $a \in A$ *and* $i \in I_2(a)$*,* $a \lhd_1 C_2(a,i)$ *then* $\lhd_1$ *is a continuous relation between* $\mathcal{A}_1$ *and* $\mathcal{A}_2$*.*

Proof. After lemma 2.45, to prove the claim it is sufficient to prove that, for any $a, b \in A$ and $U \subseteq A$, $a \lhd_2 U$ yields $a \lhd_1 U$ and $a \downarrow_1 b \lhd_1 a \downarrow_2 b$.

So, let us suppose that $a \lhd_2 U$. Then we can prove $a \lhd_1 U$ by induction on the length of the derivation of $a \lhd_2 U$. Indeed, if $a \lhd_2 U$ has been obtained by *reflexivity* from $a \varepsilon U$ then $a \lhd_1 U$ follows by *reflexivity*. If $a \lhd_2 U$ has been obtained by $\leq$*-left* from $a \leq c$ and $c \lhd_2 U$ then, by inductive hypothesis, we obtain $c \lhd_1 U$ and hence $a \lhd_1 U$ by $\leq$*-left*. Finally, if $a \lhd_2 U$ has been obtained by *infinity* from $C_2(a,i) \lhd_2 U$ then by inductive hypothesis we obtain $C_2(a,i) \lhd_1 U$ and hence $a \lhd_1 U$ follows by *transitivity* since $a \lhd_1 C_2(a,i)$ holds by assumption.

As regard to the second condition, let us note that $a \downarrow_\leq b \lhd_2 a \downarrow_2 b$ holds and hence $a \downarrow_\leq b \lhd_1 a \downarrow_2 b$ follows since we proved above that *cover embedding* holds. Hence $a \downarrow_1 b \lhd_1 a \downarrow_2 b$ follows by *transitivity* since $a \downarrow_1 b \lhd_1 a \downarrow_\leq b$ holds.

The following lemma follows immediately.

Lemma 2.47 (embedding) *Let* $\mathcal{A}$ *be any formal topology. Then the cover relation* $\lhd_A$ *is a continuous relation between* $\mathcal{A}$ *and* $\mathsf{Un}(\mathcal{A})$*.*

Proof. We have only to show that *cover embedding* and *convergence embedding* hold. So, let us suppose that $a \lhd_{\mathsf{Un}} U$ and assume that $\mathsf{Pos}(a)$ holds; then there exists an element $u \varepsilon U$ such that $a \lhd u$ holds. Then $a \lhd U$ follows by *reflexivity* and *transitivity* and hence we can discharge the assumption $\mathsf{Pos}(a)$ by *positivity*, that is, we proved that $a \lhd_{\mathsf{Un}} U$ yields $a \lhd U$. Finally, for any $w \in A$, $w \lhd a$ and $w \lhd b$ immediately yield $w \lhd_{\mathsf{Un}} a$ and $w \lhd_{\mathsf{Un}} b$; hence $a \downarrow b \subseteq a \downarrow_{\mathsf{Un}} b$ and so *convergence embedding* follows by *reflexivity*.

2.7.3 Binary product with a unary formal topology

We present now a lemma that characterizes the topological product of formal topologies in the case one of them is unary.

Lemma 2.48 *Consider the topological product of an inductively generated formal topology* $\mathcal{C} \equiv (C, \top_C, \leq_C, \lhd_C, \mathsf{Pos}_C)$ *and a unary formal topology* $\mathcal{A} \equiv (A, \top_A, \leq_A, \lhd_A, \mathsf{Pos}_A)$ *in* $\mathsf{FTop_i}$ *and suppose that* $c \in C$*,* $a \in A$ *and* $W \subseteq C \times A$*. Hence, if* $\mathsf{Pos}_A(a)$ *and* $(c,a) \lhd_{C\times A} W$ *hold then there exists a subset* W_1 *of* C *such that* $c \lhd_C W_1$ *and, for every* $w_1 \varepsilon W_1$ *there exists* $w_2 \in A$ *such that* $a \lhd_A w_2$ *and* $(w_1, w_2) \varepsilon W$*.*

Proof. The statement is proved by induction on the length of the derivation of $(c, a) \lhd_{C\times A} W$.

If $(c, a) \lhd_{C\times A} W$ has been obtained from $(c, a)\varepsilon W$ by *reflexivity* then we can simply set $W_1 \equiv \{c\}$ and $w_2 \equiv a$ and the result is obvious.

On the other hand, if $(c, a) \lhd_{C\times A} W$ follows by *infinity* then we have to consider two cases.

- $(c, a) \lhd_{C\times A} W$ follows from $\{c\} \times C_A(a, j) \lhd_{C\times A} W$. Then, since $\mathsf{Pos}_A(a)$ holds and $\mathcal{A}$ is a unary formal topology, there exists a positive element $v\varepsilon C_A(a, j)$ such that $a \lhd_A \{v\}$ and $(c, v) \lhd_{C\times A} W$. Hence, by inductive hypothesis, there exists a subset W_v of C such that $c \lhd_C W_v$ and for every $w_1\varepsilon W_v$ there exists $w_2 \in A$ such that $v \lhd_A w_2$ and $(w_1, w_2)\varepsilon W$. Then, to satisfy the statement we put $W_1 = W_v$ and for every $w_1\varepsilon W_1$ we keep the same element w_2 since from $v \lhd_A \{w_2\}$ and $a \lhd_A \{v\}$ we obtain $a \lhd_A \{w_2\}$ by *transitivity*.

- $(c, a) \lhd_{C\times A} W$ follows from $C_C(c, j) \times \{a\} \lhd_{C\times A} W$. Then $c \lhd_C C_C(c, j)$ and, for all $u\varepsilon C_C(c, j)$, $(u, a) \lhd_{C\times A} W$. Then, for every $u\varepsilon C_C(c, j)$, by inductive hypothesis there exists a subset W_u of C such that $u \lhd_C W_u$ and, for every $w_1\varepsilon W_u$, there exists $w_{2u} \in A$ such that $a \lhd_A w_{2u}$ and $(w_1, w_{2u})\varepsilon W$. Then, we set $W_1 \equiv \bigcup_{u\varepsilon C_C(c,j)} W_u$ and we obtain that, for every $w_1\varepsilon W_1$, there exists $u\varepsilon C_C(c, j)$ such that $w_1\varepsilon W_u$ and then the result follows.

Finally, suppose that $(c, a) \lhd_{C\times A} W$ follows by $\leq$-*left* from $(c_1, a_1) \lhd_{C\times A} W$ and $(c, a) \leq (c_1, a_1)$. Then $(c, a) \leq (c_1, a_1)$ yields $a \leq a_1$ and hence $\mathsf{Pos}_A(a)$ yields $\mathsf{Pos}_A(a_1)$ by $\leq$-*monotonicity*. Hence, by inductive hypothesis, there exists a subset W_1 of C such that $c_1 \lhd_C W_1$ and for every $w_1\varepsilon W_1$, there exists $w_2 \in A$ such that $a_1 \lhd_A w_2$ and $(w_1, w_2)\varepsilon W$. Thus, to obtain the result, we can simply use the same subset W_1. Indeed, $(c, a) \leq (c_1, a_1)$ yields $c \leq c_1$ and $a \leq a_1$ and then, by using $\leq$-*left* on the formal topologies $\mathcal{C}$ and $\mathcal{A}$ respectively, we obtain both that $c \lhd_C W_1$ holds and that for every $w_1\varepsilon W_1$ there exists $w_2 \in A$ such that $a \lhd_A w_2$ with $(w_1, w_2)\varepsilon W$.

It is straightforward to check that a statement analogous to the one above above holds also in $\mathsf{FTop_i}^-$ by simply substituting the positivity predicate with an always true predicate. The same holds for all of the statements in the following sections when it is a matter of working in $\mathsf{FTop_i}^-$ instead that in $\mathsf{FTop_i}$.

2.7.4 Continuous relation of a unary formal topology

The definition of continuous relation between formal topologies can be substantially simplified if we restrict our attention to the case of continuous relations between a unary formal topology and a generic one. This simplification is the key for the possibility to define the exponential of a unary topology over an inductively generated one (see section 3.1).

Proposition 2.49 *Suppose* $\mathcal{A} = (A, \top_A, \lhd_A, \mathsf{Pos}_A)$ *is a unary formal topology and* $\mathcal{B} = (B, \top_B, \lhd_B, \mathsf{Pos}_B)$ *is any formal topology. Then a* continuous relation *between* $\mathcal{A}$ *and* $\mathcal{B}$ *is a binary proposition* aFb *between* A *and* B *which satisfies* function totality*,* function weak-saturation *and* function positivity*, the following special case of* function convergence

$$\text{(unary function convergence)} \quad \frac{\mathsf{Pos}(a) \quad aFb \quad aFd}{(\exists y \varepsilon b \downarrow d)\ aFy}$$

and the following special case of function continuity

$$\text{(unary function continuity)} \quad \frac{\mathsf{Pos}(a) \quad aFb \quad b \lhd_B V}{(\exists v \varepsilon V)\ aFv}$$

Proof. We show that the conditions here are equivalent to the standard ones when working with unary formal topologies. Indeed, we already observed that *function weak saturation* and *function positivity* are consequences of *function saturation*. As regard to the validity of *unary function convergence*, let us suppose that $\mathsf{Pos}(a)$, aFb and aFd hold. Then, *function convergence* yield $a \lhd F^-(b \downarrow d)$ and hence $\mathsf{Pos}(a)$ yields that there exists $x \in A$ such that $x \varepsilon F^-(b \downarrow d)$ and $a \lhd \{x\}$ since A is a unary formal topology. Thus, there exists $y \varepsilon b \downarrow d$ such that xFy and hence aFy follows by *function weak-saturation*. Finally, the validity of *unary function continuity* can be proved as follows. Let us suppose that $\mathsf{Pos}(a)$, aFb and $b \lhd V$ hold. Then we obtain $a \lhd \{w \mid (\exists v \varepsilon V)\ wFv\}$ by *function continuity* and hence $\mathsf{Pos}(a) \rightarrow (\exists w \in A)(\exists v \varepsilon V)\ wFv \ \&\ a \lhd w$ follows since $\mathcal{A}$ is a unary formal topology. And it yields $(\exists w \in A)(\exists v \varepsilon V)\ wFv \ \&\ a \lhd w$, since we supposed that $\mathsf{Pos}(a)$ holds. Hence we can conclude that $(\exists v \varepsilon V)\ aFv$ by *function weak-saturation*.

On the other hand, *function saturation* can be proved as follows: suppose that $a \lhd_A W$; then, assuming $\mathsf{Pos}(a)$, we obtain that there exists $w \varepsilon W$ such that $a \lhd w$, since $\mathcal{A}$ is a unary formal topology, and so $(\forall w \varepsilon W)\ wFb$ yields wFb and hence aFb follows by *weak saturation*; thus, we can conclude aFb by *function positivity*. Moreover, *function convergence* can be proved as follows. Suppose that aFb and aFd and assume that $\mathsf{Pos}(a)$ holds. Then, by *unary function convergence* there exists $y \varepsilon b \downarrow d$ such that aFy holds. Hence $a \varepsilon F^-(b \downarrow d)$ holds and it yields $a \lhd F^-(b \downarrow d)$ by first using *reflexivity* and then *positivity* which allows to discharge the assumption $\mathsf{Pos}(a)$. Finally, also *function continuity* is valid. Indeed, if aFb and $b \lhd_B V$ then, under the assumption that $\mathsf{Pos}(a)$ holds, we obtain, by *unary continuity*, that $(\exists v \varepsilon V)\ aFv$ and thus $a \varepsilon F^-(V)$; hence $a \lhd F^-(V)$ follows by *reflexivity* and *positivity* which allows to discharge the assumption $\mathsf{Pos}(a)$.

Note that, even if the domain of a continuous relation is a unary formal topology we cannot just simplify *saturation* by requiring only *weak-saturation*. In fact, the latter is not sufficient, because it works only for positive elements. Indeed, recalling that $\mathsf{Pos}(a) \rightarrow aFb$ is equivalent to $(\forall x \varepsilon a^+)\ xFb$, the premise of the *function positivity* condition yields aFb by *saturation*. Hence *saturation* yields both *function positivity* and *weak-saturation* and so it is clear that need to require both of them to obtain back *saturation* as we saw in the proof of the previous lemma.

As we already did in lemma 2.28, the *unary continuity* condition can be further simplified if the co-domain formal topology is inductively generated.

Lemma 2.50 *Let $\mathcal{A}$ be a unary topology, $\mathcal{B}$ be an inductively generated formal topology and F be a continuous relation between $\mathcal{A}$ and $\mathcal{B}$. Then* unary function continuity *is equivalent to*

$$\text{(unary axiom continuity)} \quad \frac{\mathsf{Pos}(a) \quad aFb \quad j \in J(b)}{(\exists v \varepsilon C(b,j))\ aFv}$$

and

$$\text{(unary }\leq\text{-continuity)} \quad \frac{\mathsf{Pos}(a) \quad aFb \quad b \leq d}{aFd}$$

Proof. First of all, it is obvious that *unary axiom continuity* and *unary $\leq$-continuity* are immediate consequences of *unary function continuity* and *$\leq$-left.*

Vice-versa, we can prove that *function unary continuity* is a consequence of the conditions above. Let us suppose that $\mathsf{Pos}(a)$, aFb and $b \lhd V$. Then *unary continuity* can be derived from the conditions here by reasoning by induction on the length of the derivation of $b \lhd V$. So, let us assume that $b \lhd V$ has been derived from $b\varepsilon V$ by *reflexivity.* Then it is clear that $(\exists v \varepsilon V)\ aFv$ holds by logic. Moreover, if $b \lhd V$ has been derived from $C(b,j) \lhd V$ by *infinity* then we can use *unary axiom continuity* to get that there exists $w \varepsilon C(b,j)$ such that aFw. Then, by inductive hypothesis, $w \lhd V$ yields that there exists $v \varepsilon V$ such that aFv. Finally if *$\leq$-left* was used, that is, we proved $b \lhd V$ from $b \leq d$ and $d \lhd V$ then the result follows by induction because from $\mathsf{Pos}(a)$, aFb and $b \leq d$ we obtain aFd by using *unary $\leq$-continuity.*

3 The construction of the exponential object

We are now ready to prove the main result of the paper, namely, the exponentiation of unary topologies over inductively generated ones.

3.1 The exponential topology

In this section, given a unary formal topology $\mathcal{A}$ and an inductively generated one $\mathcal{B}$, we show how to build an inductively generated formal topology, that we indicate by $\mathcal{A} \to \mathcal{B}$, whose formal points are (in bijective correspondence with) the continuous relations between $\mathcal{A}$ and $\mathcal{B}$.

The basic neighbourhoods of $\mathcal{A} \to \mathcal{B}$ are lists whose elements are couples in the set $\mathsf{Pos}_A \times B \equiv \{(a,b) \in A \times B \mid \mathsf{Pos}_A(a)\}$. The intended meaning of a list $l \in A \to B$ is to give a partial information on a continuous relation R between A and B. To indicate that the list l approximates the continuous relation R we introduce the following definition

$$R \Vdash l \equiv (\forall (a,b) \epsilon l)\ aRb$$

where the proposition $x\epsilon l$ is defined by induction on the construction of l by setting $x\epsilon \mathsf{nil} \equiv \mathsf{False}$ and $x\epsilon(a,b)\cdot l \equiv (x=(a,b)) \vee x\epsilon l$.

The reason to consider only couples in $\mathsf{Pos}_A \times B$ is that *function positivity* guarantees that every non positive element of A is in relation with every element of B and hence it is useless to keep the information on the non-positive elements of A.

Since we want to obtain an inductively generated formal topology, in order to apply the method in section 2.5, we have to introduce also a pre-order relation among lists. The obvious choice is to set

$$l \preceq m \equiv (\forall (a,b) \in \mathsf{Pos}_A \times B)\ (a,b)\epsilon m \to (a,b)\epsilon l$$

stating that the list l is more precise, that is, it is contained into fewer continuous relations, than the list m. This order relation is a refinement of the reverse sub-list relation, which states that m is a sub-list of l, because $- \preceq -$ does not consider the order among the elements in a list and their repetitions.

According to the explanation in section 2.5 we have now to find the axiom-set that specifies the main properties of the cover relation for the exponential, then we will add the *top-element axiom*, successively we will define the suitable positivity predicate by co-induction, then we will add to the axiom-set so far obtained the *positivity axiom* and finally we will generate the cover relation by induction.

Now, the inspiring idea for the axiom-set is to look for those axioms which will force a point of the exponential formal topology to be a continuous relation. Thus, each axiom has to explain how an information l on a continuous relation can be made more precise and still be part of a continuous relation. So, we add a new axiom schema in correspondence with each of the conditions defining a continuous relation. Meanwhile, we need to justify such axioms. According to the intended meaning of the cover relation in section 2.2, given an axiom $l \lhd U$, such a justification amounts to show that $\mathsf{ext}(l) \subseteq \mathsf{Ext}(U)$, that is, every formal point containing l also contains a basic neighbourhood of U. Recalling that formal points are expected to be continuous relations, this means that we have to prove that, for any continuous relation F, if $F \Vdash l$ then there exists $m \varepsilon U$ such that $F \Vdash m$.

So we have now a clear plan for finding our axiom-set. In order to keep the exposition clear, we are not going to formalize the axiom-set completely by specifying a set $I(-)$ of indexes for every list l and a family $C(-,-)$ of subsets defining all of the subsets which cover l by axiom. In fact, we are just going to write down which subsets have to appear in the family $C(-,-)$. We hope that it will be clear how such a formalization can be actually performed.

The first axiom schema that we require is the formalization of *function totality*, namely, $a\ F\ \top_B$ for any $a \in A$. It is expressed by stating that, for any $l \in A \to B$ and any positive element $a \in A$, there is an index $k \in I(l)$ such that

$$\text{(totality axiom)} \quad C(l,k) \equiv \{(a, \top_B)\cdot l\}$$

Now, if F is any continuous relation which contains l, that is, such that $(a,b)\epsilon l$ yields aFb, then it also contains $(a,\top_B)\cdot l$, because of *function totality*.

The second axiom schema is a formalization of *unary function convergence.* This condition states that if $\mathsf{Pos}(a)$, aFb and aFd hold then there exists $y \varepsilon b \downarrow d$ such that aFy. The corresponding axiom states that, provided $(a,b)\epsilon l$ and $(a,d)\epsilon l$, there is an index $k \in I(l)$ such that

$$\text{(unary convergence axiom)} \quad C(l,k) \equiv \{(a,y) \cdot l \mid y \varepsilon b \downarrow d\}$$

Now, if F is a continuous relation which contains l and $(a,b)\epsilon l$ and $(a,d)\epsilon l$ then we get $\mathsf{Pos}(a)$, aFb and aFd; hence by *unary convergence* there exists $y \varepsilon b \downarrow d$ such that aFy; so F contains $(a,y) \cdot l$.

The third required condition is *weak-saturation*, that is, if $a \lhd c$ and cFb then aFb. The corresponding axiom states that, provided $(c,b)\epsilon l$, $\mathsf{Pos}(a)$ and $a \lhd c$, there is an index $k \in I(l)$ such that

$$\text{(weak-saturation axiom)} \quad C(l,k) \equiv \{(a,b) \cdot l\}$$

Now, suppose that $(c,b)\epsilon l$, $\mathsf{Pos}(a)$ and $a \lhd c$ and that F is any continuous relation containing l. Then cFb holds and hence $a \lhd c$ yields aFb by *weak-saturation*; so F contains $(a,b) \cdot l$.

Since we are considering the exponentiation of a unary formal topology $\mathcal{A}$ over an inductively generated one $\mathcal{B}$, we have to consider now *axiom unary continuity* and *unary $\leq$-continuity.* The first condition states that if $\mathsf{Pos}(a)$, aFb and $j \in J(b)$, where $J(b)$ is the axiom-indexing set for $\mathcal{B}$, then there exists $y \varepsilon C(b,j)$ such that aFy. The corresponding axiom states that, provided $(a,b)\epsilon l$ and $j \in J(b)$, there is an index $k \in I(l)$ such that

$$\text{(unary continuity axiom)} \quad C(l,k) \equiv \{(a,y) \cdot l \mid y \varepsilon C(b,j)\}$$

Now, if F is any continuous relation which contains l then $(a,b)\epsilon l$ yields both $\mathsf{Pos}(a)$ and aFb and hence $j \in J(b)$ yields that there exists $y \varepsilon C(b,j)$ such that aFy; so F contains $(a,y) \cdot l$.

Finally *unary $\leq$-continuity* states that $\mathsf{Pos}(a)$, aFb and $b \leq d$ yield aFd. The corresponding axiom states that, provided $(a,b)\epsilon l$ and $b \leq d$, there is an index $k \in I(l)$ such that

$$\text{($\leq$-continuity axiom)} \quad C(l,k) \equiv \{(a,d) \cdot l\}$$

Now, if F is any continuous relation which contains l, then from $(a,b)\epsilon l$ we get $\mathsf{Pos}(a)$ and aFb and hence we conclude aFd by *unary $\leq$-continuity* since $b \leq d$. So F contains $(a,d) \cdot l$.

It is not too difficult to show that the axioms above form an axiom-set. However, it is interesting to note that to obtain this result it is necessary that the formal topology $\mathcal{B}$ is inductively generated; indeed the continuity axiom for a general topology would have required that, provided $(a,b)\epsilon l$ and $b \lhd V$, there is an index in $I(l)$ for all the subsets $\{(a,v) \cdot l \mid v \varepsilon V\}$. But, in general, this cannot be possible since it would be necessary to quantify over the collection of all the subsets of B.

Now, we should add the *top-element axiom*, namely, for every list l, we should add a new index for the subset $\{l\} \downarrow_{\preceq} \{\mathsf{nil}\}$. But we can skip this step here since, after the exponential formal topology will be generated, $l \lhd \mathsf{nil}$ will clearly holds as a consequence by $\preceq$-*left* of $l \preceq \mathsf{nil}$ which obviously holds.

Let us turn now our attention to the positivity predicate. As we already explained, for its definition we follow the method suggested in section 2.5, that is, we are going to use the following co-inductive rules:

($\preceq$-monotonicity)	$\dfrac{\mathsf{Pos}(l) \quad l \preceq m}{\mathsf{Pos}(m)}$
(totality positivity)	$\dfrac{\mathsf{Pos}(l) \quad \mathsf{Pos}(a)}{\mathsf{Pos}((a,\top)\cdot l)}$
(unary convergence positivity)	$\dfrac{\mathsf{Pos}(l) \quad (a,b)\epsilon l \quad (a,d)\epsilon l}{(\exists y \varepsilon b \downarrow d)\ \mathsf{Pos}((a,y)\cdot l)}$
(weak-saturation positivity)	$\dfrac{\mathsf{Pos}(l) \quad \mathsf{Pos}(a) \quad a \lhd c \quad (c,b)\epsilon l}{\mathsf{Pos}((a,b)\cdot l)}$
(unary continuity positivity)	$\dfrac{\mathsf{Pos}(l) \quad (a,b)\epsilon l \quad j \in J(b)}{(\exists y \varepsilon C(b,j))\ \mathsf{Pos}((a,y)\cdot l)}$
($\leq$-continuity positivity)	$\dfrac{\mathsf{Pos}(l) \quad (a,b)\epsilon l \quad b \leq d}{\mathsf{Pos}((a,d)\cdot l)}$

It is worth noting that we did not add the co-inductive rule for the *top element axiom* since we did not to add such an axiom.

After the definition of the positivity predicate, in order to force *positivity* to hold for the cover relation, for every list l, we have to add to the axiom-set so far obtained a new index $*$ and a new subset, namely

$$\text{(positivity axiom)} \quad C(l,*) \equiv \{m \in A \to B \mid m = l \ \&\ \mathsf{Pos}(m)\}$$

Thus we completed the definition of the axiom-set for the formal topology $\mathcal{A} \to \mathcal{B}$ and it is not too difficult to verify that such an axiom-set satisfies the localization condition of section 2.5. So we can finally generate by induction the formal topology $\mathcal{A} \to \mathcal{B}$ whose cover relation satisfies the following conditions:

(top-element axiom)	$l \lhd \mathsf{nil}$
(totality axiom)	$l \lhd (a,\top_B)\cdot l$
(unary convergence axiom)	$l \lhd \{(a,y)\cdot l \mid y \varepsilon b \downarrow d\}$ if $(a,b)\epsilon l$ and $(a,d)\epsilon l$
(weak saturation axiom)	$l \lhd (a,b)\cdot l$ if $(c,b)\epsilon l, \mathsf{Pos}(a)$ and $a \lhd_A c$
($\leq$-continuity axiom)	$l \lhd (a,d)\cdot l$ if $(a,b)\epsilon l$ and $b \leq d$
(unary continuity axiom)	$l \lhd \{(a,y)\cdot l \mid y \varepsilon C(b,j)\}$ if $(a,b)\epsilon l, j \in J(b)$

3.1.1 Some immediate lemmas on the exponential

In this section we prove some lemmas which are immediate consequence of the definition of exponential topology and that we will use in the following.

Lemma 3.1 *Let $l \in A \to B$, $(c,b)\epsilon l$, $\mathsf{Pos}(a)$ and $a \lhd_A c$. Then*

$$l \lhd_{A\to B} (a,b)\cdot \mathsf{nil}$$

Proof. From $(c,b)\epsilon l$, $\mathsf{Pos}(a)$ and $a \lhd_A c$, by *weak-saturation axiom*, we obtain $l \lhd_{A\to B} (a,b)\cdot l$. Then we conclude $l \lhd_{A\to B} (a,b)\cdot \mathsf{nil}$ by *transitivity* since, by $\preceq$*-left*, we get $(a,b)\cdot l \lhd_{A\to B} (a,b)\cdot \mathsf{nil}$ from $(a,b)\cdot l \preceq (a,b)\cdot \mathsf{nil}$.

Lemma 3.2 *Let $l \in A \to B$, $(a,b)\epsilon l$ and $b \le d$. Then*

$$l \lhd_{A\to B} (a,d)\cdot \mathsf{nil}$$

Proof. From $(a,b)\epsilon l$, and $b \le d$ we obtain $l \lhd_{A\to B} (a,d)\cdot l$ by $\le$*-continuity axiom*. Then we conclude $l \lhd_{A\to B} (a,d)\cdot \mathsf{nil}$ by *transitivity* since, by $\le$*-left*, we get $(a,d)\cdot l \lhd_{A\to B} (a,d)\cdot \mathsf{nil}$ from $(a,d)\cdot l \preceq (a,d)\cdot \mathsf{nil}$.

Let us recall now the operation of appending two lists since we are going to use it in the next lemmas: given two lists m_1 and m_2 in $A \to B$ we will write $m_1 \cdot m_2$ to mean the result of appending the list m_1 to the list m_2 and, given two subsets $U_1, U_2 \subseteq A \to B$, we will write $U_1 \cdot U_2$ to mean the subset $U_1 \cdot U_2 \equiv \{m_1 \cdot m_2 \mid m_1 \varepsilon U_1 \ \& \ m_2 \varepsilon U_2\}$.

Lemma 3.3 *Let $l \in A \to B$ and $U_1, U_2 \subseteq A \to B$. Then the following condition holds:*

$$(\cdot\text{-right}) \quad \frac{l \lhd U_1 \qquad l \lhd U_2}{l \lhd U_1 \cdot U_2}$$

Proof. The premises of $\cdot$*-right* yield $l \lhd U_1 \downarrow_\preceq U_2$ by $\preceq$-right. Then, we conclude $l \lhd U_1 \cdot U_2$ by *transitivity* since $U_1 \downarrow_\preceq U_2 \lhd U_1 \cdot U_2$. Indeed, for any $l \varepsilon U_1 \downarrow_\preceq U_2$, there exist $u_1 \varepsilon U_1$ and $u_2 \varepsilon U_2$ such that $l \preceq u_1$ and $l \preceq u_2$. Thus, $l \preceq u_1 \cdot u_2$ and hence $l \lhd u_1 \cdot u_2$ follows by $\preceq$*-left* and hence $l \lhd U_1 \cdot U_2$.

Lemma 3.4 *Let F be a continuous relation from $\mathcal{C}$ to $\mathcal{A} \to \mathcal{B}$. Then the following condition holds, for any $c \in C$ and any $l_1, l_2 \in A \to B$,*

$$\frac{c\ F\ l_1 \qquad c\ F\ l_2}{c\ F\ l_1 \cdot l_2}$$

Proof. Since $c\ F\ l_1$ and $c\ F\ l_2$ we get $c \lhd \{w \in C \mid (\exists l \varepsilon l_1 \downarrow l_2)\ wFl\}$ by *function convergence*. Now, for any $w \in C$ such that wFl and $l \varepsilon l_1 \downarrow l_2$, we get $l \lhd l_1 \cdot l_2$ by lemma 3.3, since $l \lhd l_1$ and $l \lhd l_2$, and hence $w\ F\ l_1 \cdot l_2$ follows by *weak-continuity*. Thus, we conclude $c\ F\ l_1 \cdot l_2$ by *saturation*.

Even if the axioms for the exponential topology $\mathcal{A} \to \mathcal{B}$ that we introduced use directly the particular axiom-set used to generate the formal topology $\mathcal{B}$, the next two lemmas show that the resulting topology does not depend on these particular axiom-set but on the cover of $\mathcal{B}$.

Lemma 3.5 *Let $\mathcal{A}$ be a unary formal topology and $\mathcal{B}$ an inductively generated one. Then, for any list $l \in A \to B$, the following condition holds*

$$\text{(generalized unary continuity positivity)}\quad \frac{\mathsf{Pos}(l) \quad (a,b)\epsilon l \quad b \lhd_B V}{(\exists y \varepsilon V)\ \mathsf{Pos}((a,y)\cdot l)}$$

Proof. The proof goes on by induction on the generation of $b \lhd_B V$.

- If $b \lhd_B V$ has been obtained by *reflexivity*, then the result follows immediately by $\leq$-*continuity positivity* since $b \leq b$.
- If $b \lhd_B V$ has been obtained by $\leq$-*left* from $b \leq d$ and $d \lhd_B V$, then $\mathsf{Pos}((a,d)\cdot l)$ follows by $\leq$-*continuity positivity.* Then by inductive hypothesis we obtain $(\exists y \varepsilon V)\ \mathsf{Pos}((a,y)\cdot(a,d)\cdot l)$ and so we can conclude $(\exists y \varepsilon V)\ \mathsf{Pos}((a,y)\cdot l)$ by $\preceq$-*monotonicity* since $(a,y)\cdot(a,d)\cdot l \preceq (a,y)\cdot l$.
- If $b \lhd_B V$ has been obtained by *infinity* from $C_B(b,j) \lhd_B V$, then $(\exists y \varepsilon C_B(b,j))\ \mathsf{Pos}((a,y)\cdot l)$ holds by *unary continuity positivity.* Now, by inductive hypothesis, from $\mathsf{Pos}((a,y)\cdot l)$ we get $(\exists z \varepsilon V)\ \mathsf{Pos}((a,z)\cdot(a,y)\cdot l)$ and hence we conclude $(\exists z \varepsilon V)\ \mathsf{Pos}((a,z)\cdot l)$ by $\preceq$-*monotonicity* since $(a,z)\cdot(a,y)\cdot l \preceq (a,z)\cdot l$.

Lemma 3.6 *Let $\mathcal{A}$ be a unary formal topology and $\mathcal{B}$ be an inductively generated one. Then, for any list $l \in A \to B$ and any $(a,b)\epsilon l$, if $b \lhd_B V$ then*

$$\text{(unary continuity cover)}\quad l \lhd_{A\to B} \{(a,y)\cdot l \mid y \varepsilon V\}$$

Proof. The proof goes on by induction on the generation of $b \lhd_B V$.

- If $b \lhd_B V$ has been obtained by *reflexivity* then $(a,b)\cdot l \varepsilon \{(a,y)\cdot l \mid y\varepsilon V\}$. Hence $(a,b)\cdot l \lhd \{(a,y)\cdot l \mid y \varepsilon V\}$ follows by *reflexivity* and hence we obtain $l \lhd_{A\to B} \{(a,y)\cdot l \mid y\varepsilon V\}$ by *transitivity* since $l \preceq (a,b)\cdot l$ yields $l \lhd (a,b)\cdot l$.
- If $b \lhd_B V$ has been obtained by $\leq$-*left* from $b \leq d$ and $d \lhd_B V$, then, by inductive hypothesis, we obtain $(a,d)\cdot l \lhd_{A\to B} \{(a,y)\cdot(a,d)\cdot l \mid y\varepsilon V\}$. Now, for any $y\varepsilon V$, $(a,y)\cdot(a,d)\cdot l \preceq (a,y)\cdot l$ yields $(a,y)\cdot(a,d)\cdot l \lhd_{A\to B} (a,y)\cdot l$ and hence $(a,d)\cdot l \lhd_{A\to B} \{(a,y)\cdot l \mid y\varepsilon V\}$ follows by *transitivity.* But $b \leq d$ yields $l \lhd_{A\to B} (a,d)\cdot l$ by $\leq$-*continuity axiom.* So, we conclude $l \lhd_{A\to B} \{(a,y)\cdot l \mid y\varepsilon V\}$ by *transitivity.*
- If $b \lhd_B V$ has been obtained by *infinity* from $C_B(b,j) \lhd V$, then, for every $z\varepsilon C_B(b,j)$, $(a,z)\cdot l \lhd_{A\to B} \{(a,y)\cdot(a,z)\cdot l \mid y\varepsilon V\}$ holds by inductive hypothesis. Observe now that, for any $y\varepsilon V$, $(a,y)\cdot(a,z)\cdot l \preceq (a,y)\cdot l$ yields $(a,y)\cdot(a,z)\cdot l \lhd_{A\to B} (a,y)\cdot l$. Hence $(a,z)\cdot l \lhd_{A\to B} \{(a,y)\cdot l \mid y\varepsilon V\}$ follows by *transitivity.* Now, $(a,b)\epsilon l$ and hence, by *unary continuity axiom*, we get $l \lhd_{A\to B} \{(a,z)\cdot l \mid z\varepsilon C_B(b,j)\}$ and so we conclude $l \lhd_{A\to B} \{(a,y)\cdot l \mid y\varepsilon V\}$ by *transitivity.*

3.1.2 Embedding of exponential topologies

Thanks to the lemmas of the previous section we can prove the following embedding lemma.

Proposition 3.7 *The exponential $\mathcal{A} \to \mathcal{B}$ of a unary formal topology $\mathcal{A}$ over an inductively generated formal topology $\mathcal{B}$ embeds into the exponential $\mathcal{A} \to \mathsf{Un}(\mathcal{B})$ via the cover relation $- \lhd_{\mathcal{A}\to\mathcal{B}} -$.*

Proof. After corollary 2.46, the claim is a consequence of proving that, for any $l \in A \to B$ and $j \in I_{A\to\mathsf{Un}(B)}(l)$, $l \lhd_{A\to B} C_{A\to\mathsf{Un}(B)}(l, j)$ holds. This is immediate for all the axioms in the axiom-set of $\mathcal{A} \to \mathsf{Un}(\mathcal{B})$, except for the *positivity axiom* and *unary continuity axiom.*

To prove that the result holds for the *positivity axiom*, namely to prove that

$$l \lhd_{A\to B} \{m \in A \to B \mid \mathsf{Pos}_{A\to\mathsf{Un}(B)}(m) \;\&\; m = l\}$$

holds, it is enough to observe that $\mathsf{Pos}_{A\to B}(l)$ yields $\mathsf{Pos}_{A\to\mathsf{Un}(B)}(l)$. Now, this fact can be proved by co-induction by observing that $Q(l) \equiv \mathsf{Pos}_{A\to B}(l)$ satisfies all the co-inductive conditions defining $\mathsf{Pos}_{A\to\mathsf{Un}(B)}(l)$. Indeed, this result is immediate for all of the conditions except for the *unary continuity positivity* condition that can be proved as follows. First, observe that if $I_{\mathsf{Un}(B)}(-)$ and $C_{\mathsf{Un}(B)}(-,-)$ is the axiom set of $\mathsf{Un}(\mathcal{B})$ then, for any $b \in B$ and $j \in I_{\mathsf{Un}(B)}(b)$, $b \lhd_B C_{\mathsf{Un}(\mathcal{B})}(b, j)$. Then *unary continuity positivity* follows by lemma 3.5.

Finally, observe that the validity in $\mathcal{A} \to \mathcal{B}$ of the *unary-continuity* axiom of $A \to \mathsf{Un}(\mathcal{B})$, that is,

$$l \lhd_{A\to B} \{(a, y) \cdot l \mid y \varepsilon C_{\mathsf{Un}(\mathcal{B})}(b, j)\}$$

follows by lemma 3.6 since, for any $b \in B$ and $j \in I_{\mathsf{Un}(B)}(b)$, $b \lhd_B C_{\mathsf{Un}(\mathcal{B})}(b, j)$.

After this proposition and lemma 2.45, it follows that the formal points of $\mathcal{A} \to \mathcal{B}$ are also formal points of $\mathcal{A} \to \mathsf{Un}(\mathcal{B})$. Since in the next section we will prove that the collection of the formal points of $\mathcal{A} \to \mathcal{B}$ is in bijection with the collection of the continuous relations between $\mathcal{A}$ and $\mathcal{B}$, this embedding means that continuous relations between a unary formal topology $\mathcal{A}$ and any inductively generated formal topology $\mathcal{B}$ form a subcollection of continuous relations between $\mathcal{A}$ and $\mathsf{Un}(\mathcal{B})$, as expected.

3.2 Bijection between points and relations

In this section we prove that there is a bijective correspondence between the continuous relations between a unary formal topology $\mathcal{A}$ and an inductively generated one $\mathcal{B}$ and the formal points of the formal topology $\mathcal{A} \to \mathcal{B}$. It is clear that this result is an immediate consequence of the bijective correspondence between the collection of the formal points of the formal topology $\mathcal{A}$ and the morphisms between the terminal formal topology $\mathcal{T}$ and $\mathcal{A}$ that we proved in section 2.4.2 and the proof that, for any unary formal topology $\mathcal{A}$ and any inductively generated formal topology $\mathcal{B}$, the formal topology $\mathcal{A} \to \mathcal{B}$ is the exponential of $\mathcal{A}$ and $\mathcal{B}$ that we will show in the next sections. However, we decided to insert here a direct proof since we think that it is more straight and perspicuous to understand how the axioms for the exponential have been found.

In order to simplify the proof of the next theorem 3.9, it is useful to observe first that the following lemma holds. It suggests how to get rid of the positivity predicate when we define the continuous relation associated with one formal point of $\mathcal{A} \to \mathcal{B}$, that is, it allows to use the same definition both in $\mathsf{FTop_i}$ and $\mathsf{FTop_i}^-$.

Lemma 3.8 *Let $\mathcal{A}$ be a unary formal topology and $\mathcal{B}$ be any inductively generated formal topology. Suppose that $a \in A$, $b \in B$ and $\Phi \in \mathsf{Pt}(\mathcal{A} \to \mathcal{B})$. Then $\mathsf{Pos}(a) \to (a,b) \cdot \mathsf{nil} \varepsilon \Phi$ if and only if $a \lhd \{c \in A \mid (c,b) \cdot \mathsf{nil} \varepsilon \Phi\}$.*

Proof. One direction is trivial: if $\mathsf{Pos}(a) \to (a,b) \cdot \mathsf{nil} \varepsilon \Phi$ then, assuming that $\mathsf{Pos}(a)$ holds, we get $a \varepsilon \{c \in A \mid (c,b) \cdot \mathsf{nil} \varepsilon \Phi\}$ and hence $a \lhd \{c \in A \mid (c,b) \cdot \mathsf{nil} \varepsilon \Phi\}$, with no assumption, follows by *reflexivity* and *positivity*. In order to prove the other implication, let us suppose that $a \lhd \{c \in A \mid (c,b) \cdot \mathsf{nil} \varepsilon \Phi\}$ holds. Then, supposing that $\mathsf{Pos}(a)$ holds, one deduces that there exists an element $c \in A$ such that both $(c,b) \cdot \mathsf{nil} \varepsilon \Phi$ and $a \lhd c$ hold since $\mathcal{A}$ is a unary formal topology. So, $(c,b) \epsilon (c,b) \cdot \mathsf{nil}$, $\mathsf{Pos}(a)$ and $a \lhd c$ yield $(c,b) \cdot \mathsf{nil} \lhd (a,b) \cdot \mathsf{nil}$ by lemma 3.1. Hence $(c,b) \cdot \mathsf{nil} \varepsilon \Phi$ yields $(a,b) \cdot \mathsf{nil} \varepsilon \Phi$ by *point splitness.*

Now we can prove the main theorem of this section.

Theorem 3.9 *Let $\mathcal{A}$ be a unary formal topology and $\mathcal{B}$ be an inductively generated formal topology. Then there exists a bijective correspondence between the collection of the formal points of $\mathcal{A} \to \mathcal{B}$ and the collection of the continuous relations between $\mathcal{A}$ and $\mathcal{B}$.*

Proof. Let us begin by defining the continuous relation R_Φ associated with the formal point $\Phi \in \mathsf{Pt}(\mathcal{A} \to \mathcal{B})$:

$$aR_\Phi b \equiv a \lhd \{c \in A \mid (c,b) \cdot \mathsf{nil} \varepsilon \Phi\}$$

Then we have to prove that R_Φ is a continuous relation between $\mathcal{A}$ and $\mathcal{B}$.

- (function totality) We have to prove that, for every $a \in A$, $aR_\Phi \top_B$ holds. So, let us assume that $\mathsf{Pos}(a)$ holds. Then the result is immediate since $(a, \top_B) \cdot \mathsf{nil} \varepsilon \Phi$ follows by *point splitness* because $\mathsf{nil} \varepsilon \Phi$ holds by *point inhabitance* and $\mathsf{nil} \lhd (a, \top_B) \cdot \mathsf{nil}$ by *totality axiom.*

- (unary function convergence) Suppose that $aR_\Phi b$, $aR_\Phi d$ and $\mathsf{Pos}(a)$ hold. Then we have to prove that there exists $y \varepsilon b \downarrow d$ such that $aR_\Phi y$. By lemma 3.8, the assumptions yield $(a,b) \cdot \mathsf{nil} \varepsilon \Phi$ and $(a,d) \cdot \mathsf{nil} \varepsilon \Phi$ and hence, by *point $\leq$-convergence*, there exists a list l such that both $l \preceq (a,b) \cdot \mathsf{nil}$, $l \preceq (a,d) \cdot \mathsf{nil}$ and $l \varepsilon \Phi$. Then $(a,b) \epsilon l$ and $(a,d) \epsilon l$ and hence $l \lhd \{(a,y) \cdot l \mid y \varepsilon b \downarrow d\}$ follows by *unary convergence axiom.* Then, by *point splitness*, it follows that there exists $y\varepsilon b \downarrow d$ such that $(a, y) \cdot l\varepsilon\Phi$ and thus also $(a, y) \cdot \mathsf{nil}\varepsilon\Phi$ since $(a, y) \cdot l \lhd (a, y) \cdot \mathsf{nil}$ is a consequence of $(a, y) \cdot l \preceq (a, y) \cdot \mathsf{nil}$ by $\preceq$-*left*. So, by lemma 3.8, $aR_\Phi y$ holds.

- (function weak saturation) Suppose that $a \lhd_A c$ and $cR_\Phi b$. Then we get $c \lhd \{x \in A \mid (x, b) \cdot \mathsf{nil}\varepsilon\phi\}$ and hence $a \lhd \{x \in A \mid (x, b) \cdot \mathsf{nil}\varepsilon\phi\}$ follows by *transitivity*, that is, $aR_\Phi b$ holds.
- (unary axiom continuity) Suppose that $aR_\Phi b$, $\mathsf{Pos}(a)$ and $j \in J(b)$ hold. Then, by lemma 3.8, $(a, b) \cdot \mathsf{nil}\varepsilon\Phi$. Moreover, by *unary continuity axiom*, $(a, b) \cdot \mathsf{nil} \lhd \{(a, y) \cdot (a, b) \cdot \mathsf{nil} \mid y\varepsilon C(b, j)\}$ holds and hence there exists $y\varepsilon C(b, j)$ such that $(a, y) \cdot (a, b) \cdot \mathsf{nil}\varepsilon\Phi$ by *point splitness*. But then also $(a, y) \cdot \mathsf{nil}\varepsilon\Phi$ follows since $(a, y) \cdot (a, b) \cdot \mathsf{nil} \lhd (a, y) \cdot \mathsf{nil}$ is a consequence of $(a, y) \cdot (a, b) \cdot \mathsf{nil} \preceq (a, y) \cdot \mathsf{nil}$ by $\preceq$-*left*. So, by lemma 3.8, $aR_\Phi y$ holds.
- (unary function $\leq$-continuity) Suppose that $aR_\Phi b$, $\mathsf{Pos}(a)$ and $b \leq d$ hold. Then $(a, b) \cdot \mathsf{nil}\varepsilon\Phi$ follows immediately by lemma 3.8. Hence, by lemma 3.2, $b \leq d$ yields $(a, b) \cdot \mathsf{nil} \lhd (a, d) \cdot \mathsf{nil}$ since $(a, b)\epsilon(a, b) \cdot \mathsf{nil}$. So, by *point splitness*, $(a, d) \cdot \mathsf{nil}\varepsilon\Phi$ follows, that is, $aR_\Phi d$ holds.
- (function positivity) Immediate by *positivity* for the cover.

The definition of the point Φ_R associated to the continuous relation R is the following:

$$l\varepsilon\Phi_R \text{ iff } R \Vdash l$$

It is not difficult to prove that Φ_R is indeed a formal point:

- (point inhabitance) $\mathsf{nil}\varepsilon\Phi_R$ because $R \Vdash \mathsf{nil}$ holds by logic.
- (point $\leq$-directness) Let us assume that $l\varepsilon\Phi_R$ and $m\varepsilon\Phi_R$; then $R \Vdash l$ and $R \Vdash m$ and hence there exists a list k, namely, $l \cdot m$, such that $k \preceq l$, $k \preceq m$ and $R \Vdash k$ which yields $k\varepsilon\Phi_R$.
- (point left-closure) Let us assume that $l\varepsilon\Phi_R$ and $l \preceq m$. Then, by hypothesis $R \Vdash l$ and hence $R \Vdash m$ follows trivially by logic. Thus $m\varepsilon\Phi_R$.
- (point inductive splitness) Let us argue according to the shape of possible axioms.
  - (axiom totality) Suppose that $l\varepsilon\Phi_R$ holds and that $l \lhd (a, \top) \cdot l$. Then $R \Vdash l$ holds by definition and $aR\top$ holds by *function totality* and hence $R \Vdash (a, \top) \cdot l$, that is, $(a, \top) \cdot l\varepsilon\Phi_R$, follows.
  - (axiom unary convergence) Suppose $l\varepsilon\Phi_R$ and $l \lhd \{(a, y) \cdot l \mid y\varepsilon b \downarrow d\}$ because $(a, b)\epsilon l$ and $(a, d)\epsilon l$. Then $R \Vdash l$ and so aRb and aRd hold. Hence, there exists $y\varepsilon b \downarrow d$ such that aRy by *convergence*. Thus $R \Vdash (a, y) \cdot l$ and hence $(a, y) \cdot l\varepsilon\Phi_R$.
  - (weak-saturation axiom) Suppose $l\varepsilon\Phi_R$ and $l \lhd (a, b) \cdot l$ because $(c, b)\epsilon l$, $\mathsf{Pos}(a)$ and $a \lhd c$. Then $R \Vdash l$ and hence cRb. So aRb follows by *weak-saturation* and hence $R \Vdash (a, b) \cdot l$, that is, $(a, b) \cdot l\varepsilon\Phi_R$.

– (unary continuity axiom) Suppose $l\varepsilon\Phi_R$ and $l \lhd \{(a,y)\cdot l \mid y\varepsilon C(b,j)\}$ because $(a,b)\epsilon l$ and $j \in J(b)$. Then $R \Vdash l$ and hence aRb holds. Moreover, $\mathsf{Pos}(a)$ holds since $(a,b)\epsilon l$, and hence $j \in J(b)$ yields that there exists $y\varepsilon C(b,j)$ such that aRy holds by *unary continuity axiom*. So $R \Vdash (a,y)\cdot l$ holds and hence $(a,y)\cdot l\varepsilon\Phi_R$.

– ($\leq$-continuity axiom) Suppose $l\varepsilon\Phi_R$ and $l \lhd (a,d)\cdot l$ because $(a,b)\epsilon l$ and $b \leq d$. Then $R \Vdash l$ and hence aRb holds. But also $\mathsf{Pos}(a)$ holds and hence $b \leq d$ yields aRd by *unary $\leq$-continuity*. So, $R \Vdash (a,d)\cdot l$ holds and hence $(a,d)\cdot l\varepsilon\Phi_R$.

– (positivity axiom) Suppose $l\varepsilon\Phi_R$. We have to prove that there exists $m\varepsilon l^+$ such that $m\varepsilon\Phi_R$. To this aim we show by co-induction that $\mathsf{Pos}(l)$ holds. Set $Q(m) \equiv R \Vdash m$. Then $Q(m)$ satisfies all the conditions for the positivity predicate. Indeed,

  * ($\leq$-monotonicity) If $R \Vdash m$ and $m \preceq k$ then $R \Vdash k$ trivially follows.
  * (totality positivity) Assume that $R \Vdash m$ and $\mathsf{Pos}(a)$ hold. Now, for any $a \in A$, $a\ R\ \top$ holds by *function totality* and hence $R \Vdash (a,\top)\cdot m$ follows.
  * (unary convergence positivity) Assume that $R \Vdash m$, $(a,b)\epsilon m$ and $(a,d)\epsilon m$. Then aRb, aRd and $\mathsf{Pos}(a)$ follows and hence, by *unary convergence*, there exists $y\varepsilon b \downarrow d$ such that aRy. Thus, by using again $R \Vdash m$, we conclude that $R \Vdash (a,y)\cdot m$.
  * (weak-saturation positivity) Assume that $R \Vdash m$, $\mathsf{Pos}(a)$, $a \lhd c$ and $(c,b)\epsilon m$ hold. Then, cRb follows and hence we can obtain aRb by *weak-saturation*. We conclude $R \Vdash (a,b)\cdot m$ since $R \Vdash m$ holds by hypothesis.
  * (unary continuity positivity) Assume that $R \Vdash m$, $(a,b)\epsilon m$ and $j \in J(b)$ hold. Then aRb and $\mathsf{Pos}(a)$ follows and hence, by *unary continuity*, there exists $y\varepsilon C(b,j)$ such that aRy. Thus $R \Vdash (a,y)\cdot m$ follows by using again $R \Vdash m$.
  * ($\leq$-continuity positivity) Assume that $R \Vdash m$, $(a,b)\epsilon m$ and $b \leq d$ hold. Then, we get aRb and $\mathsf{Pos}(a)$ and so aRd follows by *$\leq$-continuity*. Hence, we conclude $R \Vdash (a,d)\cdot m$ by using again $R \Vdash m$.

  Now $l\varepsilon\Phi_R$ yields $R \Vdash l$, that is, $Q(l)$ holds, and hence $\mathsf{Pos}(l)$ follows by the maximality of $\mathsf{Pos}(-)$.

To conclude the proof we have only to show that the two constructions are one the inverse of the other. Indeed

$$\begin{array}{rll} aR_{\Phi_F}b & \text{iff} & a \lhd \{c \in A \mid (c,b)\cdot \mathsf{nil}\varepsilon\Phi_F\} \\ \text{by lemma 3.8} & \text{iff} & \mathsf{Pos}(a) \to (a,b)\cdot\mathsf{nil}\varepsilon\Phi_F \\ & \text{iff} & \mathsf{Pos}(a) \to F \Vdash (a,b)\cdot\mathsf{nil} \\ & \text{iff} & \mathsf{Pos}(a) \to aFb \\ \text{by function positivity} & \text{iff} & aFb \end{array}$$

Moreover,

$$\begin{array}{rll} l\varepsilon\Phi_{R_\Psi} & \text{iff} & R_\Psi \Vdash l \\ & \text{iff} & (\forall(a,b)\epsilon l)\ aR_\Psi b \\ & \text{iff} & (\forall(a,b)\epsilon l)\ a \lhd \{c \in A \mid (c,b)\cdot \mathsf{nil}\varepsilon\Psi\} \\ \text{by lemma 3.8} & \text{iff} & (\forall(a,b)\epsilon l)\ \mathsf{Pos}(a) \to (a,b)\cdot \mathsf{nil}\varepsilon\Psi \\ & \text{iff} & (\forall(a,b)\epsilon l)\ (a,b)\cdot \mathsf{nil}\varepsilon\Psi \\ & \text{iff} & l\varepsilon\Psi \end{array}$$

where the last but one step holds because the first component of each element of l is positive while the last step is proved as follows. If $l\varepsilon\Psi$ then, for each $(a,b)\epsilon l$, $(a,b)\cdot\mathsf{nil}\varepsilon\Psi$ follows by *point splitness* because $l \lhd (a,b)\cdot\mathsf{nil}$ follows from $l \preceq (a,b)\cdot\mathsf{nil}$ by $\preceq$*-left*. To prove the other implication observe that if, for all $i = 1,...,\mathsf{length}(l)$, $(a_i,b_i)\epsilon l$ yields $(a_i,b_i)\cdot\mathsf{nil}\varepsilon\Psi$; then by *point convergence* there exists a list k such that $k\varepsilon\Psi$ and, for any $i = 1,...,\mathsf{length}(l)$, $k \preceq (a_i,b_i)\cdot\mathsf{nil}$. Then $k \preceq l$ and hence by $\preceq$*-left* we obtain $k \lhd l$ and thus $l\varepsilon\Psi$ follows by *point splitness*.

3.3 Application and abstraction

In this section we show that the formal topology introduced in the previous section is the exponential of a unary topology over an inductively generated formal topology. From a categorical point of view this means that, for any unary topology $\mathcal{A}$ the functor $- \times \mathcal{A}\ :\ \mathsf{FTop_i} \Rightarrow \mathsf{FTop_i}$ has got a right adjoint $\mathcal{A} \to -\ :\ \mathsf{FTop_i} \Rightarrow \mathsf{FTop_i}$. Equivalently, this amounts to define, for any unary topology $\mathcal{A}$ and any formal topology $\mathcal{B}$ in $\mathsf{FTop_i}$, an *application relation* Ap between $(\mathcal{A} \to \mathcal{B}) \times \mathcal{A}$ and $\mathcal{B}$ such that for any continuous relation F between $\mathcal{C} \times \mathcal{A}$ and $\mathcal{B}$ there exists a continuous relation $\Lambda(F)$, called *abstraction* of F, between $\mathcal{C}$ and $\mathcal{A} \to \mathcal{B}$ such that, for any continuous relation G between $\mathcal{C}$ and $\mathcal{A} \to \mathcal{B}$, the following equations are satisfied

$$\begin{array}{rcl} \mathsf{Ap} * \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle & = & F \\ \Lambda(\mathsf{Ap} * \langle G * \Pi_1, \Pi_2 \rangle) & = & G \end{array}$$

Let $l \in A \to B$, $a \in A$ and $b \in B$. Then, we propose the following definitions for the application and the abstraction:

$$\begin{array}{lcl} (l,a)\ \mathsf{Ap}\ b & \equiv & \mathsf{Pos}(a) \to (l \lhd (a,b)\cdot\mathsf{nil}) \\ c\ \Lambda(F)\ l & \equiv & (\forall(a,b)\epsilon l)\ (c,a)Fb \end{array}$$

In the next sections we will prove that they are continuous relations and that the required equations hold.

3.3.1 The application

The next lemma states that the candidate relation for *application* is a continuous relation.

Lemma 3.10 *Let $\mathcal{A}$ be a unary topology and $\mathcal{B}$ be an inductively generated one. Then* Ap *is a continuous relation.*

Proof. We have to check that all of the required conditions hold.

- (function totality) Let $a \in A$ and $l \in A \to B$. Then, we need to prove that $(l, a)\mathsf{Ap}\top_B$, that is, $\mathsf{Pos}(a) \to (l \lhd (a, \top_B) \cdot \mathsf{nil})$. Since $l \preceq \mathsf{nil}$, by $\preceq$*-left* we get $l \lhd \mathsf{nil}$. Then, supposing $\mathsf{Pos}(a)$, by *axiom totality* we obtain $\mathsf{nil} \lhd (a, \top_B) \cdot \mathsf{nil}$ and hence $l \lhd (a, \top_B) \cdot \mathsf{nil}$ follows by *transitivity*. Finally, we conclude by discharging $\mathsf{Pos}(a)$.

- (function convergence) Suppose (l, a) Ap b and (l, a) Ap d. Then, we have to show that $(l, a) \lhd \mathsf{Ap}^-(b \downarrow d)$. To this aim let us assume that $\mathsf{Pos}(a)$ holds. Then, we get both $l \lhd (a, b) \cdot \mathsf{nil}$ and $l \lhd (a, d) \cdot \mathsf{nil}$. Hence, $l \lhd \{(a, b) \cdot \mathsf{nil}\} \downarrow_\preceq \{(a, d) \cdot \mathsf{nil}\}$ follows by $\preceq$*-right* and so, by lemma 2.34, we obtain $(a, l) \lhd \{a\} \times \{(a, b) \cdot \mathsf{nil}\} \downarrow_\preceq \{(a, d) \cdot \mathsf{nil}\}$. Thus, we can conclude by *transitivity* provided we prove that $(\{(a, b) \cdot \mathsf{nil}\} \downarrow \{(a, d) \cdot \mathsf{nil}\}) \times \{a\}$ is covered by $\mathsf{Ap}^-(b \downarrow d)$. To this aim, let us consider any couple $(x, a) \in (A \to B) \times A$ such that $x \preceq (a, b) \cdot \mathsf{nil}$ and $x \preceq (a, d) \cdot \mathsf{nil}$. Then, $(a, b)\epsilon x$ and $(a, d)\epsilon x$ and thus $x \lhd \{(a, y) \cdot x \mid y\varepsilon b \downarrow d\}$ by *unary convergence axiom*. But, for any $y\varepsilon b \downarrow d$, $(a, y) \cdot x \lhd (a, y) \cdot \mathsf{nil}$ follows by $\preceq$*-left* from $(a, y) \cdot x \preceq (a, y) \cdot \mathsf{nil}$. Then, $x \lhd \{(a, y) \cdot \mathsf{nil} \mid y\varepsilon b \downarrow d\}$ follows by *transitivity* and hence we obtain $(x, a) \lhd \{(a, y) \cdot \mathsf{nil} \mid y\varepsilon b \downarrow d\} \times \{a\}$ by lemma 2.34. Observe now that any couple $((a, y) \cdot \mathsf{nil}, a)$ belongs to the subset $\mathsf{Ap}^-(b \downarrow d)$, since $(\exists y\varepsilon b \downarrow d)\ \mathsf{Pos}(a) \to (a, y) \cdot \mathsf{nil} \lhd (a, y) \cdot \mathsf{nil}$ clearly holds, and hence $(x, a) \lhd \mathsf{Ap}^-(b \downarrow d)$ follows by *reflexivity* and *transitivity*.

- (axiom continuity) Suppose (l, a) Ap b and $j \in J(b)$. Then, after assuming $\mathsf{Pos}(a)$, we obtain $l \lhd (a, b) \cdot \mathsf{nil}$. Then, by *unary continuity axiom* we get $(a, b) \cdot \mathsf{nil} \lhd \{(a, y) \cdot (a, b) \cdot \mathsf{nil} \mid y\varepsilon C(b, j)\}$. Since, for any $y\varepsilon C(b, j)$, we obtain $(a, y) \cdot (a, b) \cdot \mathsf{nil} \lhd (a, y) \cdot \mathsf{nil}$ from $(a, y) \cdot (a, b) \cdot \mathsf{nil} \preceq (a, y) \cdot \mathsf{nil}$ by $\preceq$*-left*, we conclude $(a, b) \cdot \mathsf{nil} \lhd \{(a, y) \cdot \mathsf{nil} \mid y\varepsilon C(b, j)\}$ by *transitivity*. So, we obtain $l \lhd \{(a, y) \cdot \mathsf{nil} \mid y\varepsilon C(b, j)\}$ by *transitivity* and hence we can deduce $(l, a) \lhd \{((a, y) \cdot \mathsf{nil}, a) \mid y\varepsilon C(b, j)\}$ by lemma 2.34. Thus $(l, a) \lhd \mathsf{Ap}^-(C(b, j))$ follows by *transitivity* since $\{((a, y) \cdot \mathsf{nil}, a) \mid y\varepsilon C(b, j)\}$ is a subset of $\mathsf{Ap}^-(C(b, j))$.

- (function $\leq$-continuity) Suppose (l, a) Ap b and $b \leq d$ and assume that $\mathsf{Pos}(a)$ holds. Then, $l \lhd (a, b) \cdot \mathsf{nil}$ follows. Now, by lemma 3.2, $b \leq d$ yields $(a, b) \cdot \mathsf{nil} \lhd (a, d) \cdot \mathsf{nil}$ and so we obtain $l \lhd (a, d) \cdot \mathsf{nil}$ by *transitivity*. Finally, we conclude (l, a) Ap d by discharging the assumption $\mathsf{Pos}(a)$.

- (axiom saturation) We have to show that if $k \in J((l, a))$ is an index for an axiom of the product topology and, for any $y\varepsilon C((l, a), k)$, y Ap b holds then also (l, a) Ap b holds. We will argue according to the shape of the considered axiom.

  Axioms whose shape is $(l, a) \lhd C(l, j) \times \{a\}$:

– (totality ax., weak-saturation ax., $\leq$-continuity ax.) Suppose that (m, a) Ap b holds and that $l \lhd m$ has been obtained by applying one of the considered axiom. Moreover, assume that $\mathsf{Pos}(a)$ holds. Then, $m \lhd (a, b) \cdot \mathsf{nil}$ and hence $l \lhd (a, b) \cdot \mathsf{nil}$ follows by *transitivity*. Thus, by discharging the assumption $\mathsf{Pos}(a)$, we obtain (l, a) Ap b.

– (unary convergence axiom) Suppose that $l \lhd \{(c, y) \cdot l \mid y \varepsilon d_1 \downarrow d_2)\}$ holds because $(c, d_1) \epsilon l$ and $(c, d_2) \epsilon l$. Moreover, suppose that, for every $y \varepsilon d_1 \downarrow d_2$, $((c, y) \cdot l, a)$ Ap b holds and assume that $\mathsf{Pos}(a)$ holds. Then, for any $y \varepsilon d_1 \downarrow d_2$, from $((c, y) \cdot l, a)$ Ap b we obtain $(c, y) \cdot l \lhd (a, b) \cdot \mathsf{nil}$. Hence, $l \lhd (a, b) \cdot \mathsf{nil}$ follows by *transitivity*. Thus, we obtain (l, a) Ap b by discharging the assumption $\mathsf{Pos}(a)$.

– (unary continuity axiom) Suppose $l \lhd \{(c, y) \cdot l \mid y \varepsilon C(d, j)\}$ because $(c, d) \epsilon l$ and $j \in J(d)$. Moreover, suppose that, for every $y \varepsilon C(d, j)$, $((c, y) \cdot l, a)$ Ap b and assume that $\mathsf{Pos}(a)$ holds. Then, for any $y \varepsilon C(d, j)$, from $((c, y) \cdot l, a)$ Ap b we obtain $(c, y) \cdot l \lhd (a, b) \cdot \mathsf{nil}$. Hence, $l \lhd (a, b) \cdot \mathsf{nil}$ follows by *transitivity*. Thus, we obtain (l, a) Ap b by discharging the assumption $\mathsf{Pos}(a)$.

– (positivity axiom) Suppose (x, a) Ap b for every $x \varepsilon l^+$ and let us assume that $\mathsf{Pos}(a)$ holds. Then, for any $x \varepsilon l^+$, we obtain $x \lhd (a, b) \cdot \mathsf{nil}$ and so $l \lhd (a, b) \cdot \mathsf{nil}$ follows by *transitivity* since $l \lhd l^+$. Thus (l, a) Ap b follows by discharging the assumption $\mathsf{Pos}(a)$.

Axioms whose shape is $(l, a) \lhd \{l\} \times C(a, j)$:

– Let us assume that $\mathsf{Pos}(a)$ holds. Then there exists an element $c \varepsilon C(a, j)$ such that $a \lhd c$, since $\mathcal{A}$ is a unary formal topology, and hence $\mathsf{Pos}(c)$ holds by *monotonicity*. Recalling now that (l, c) Ap b holds by hypothesis we obtain that $l \lhd (c, b) \cdot \mathsf{nil}$. But, $a \lhd c$ yields $(c, b) \cdot \mathsf{nil} \lhd (a, b) \cdot \mathsf{nil}$ by lemma 3.1 and hence $l \lhd (a, b) \cdot \mathsf{nil}$ follows by *transitivity*. So, we obtain (l, a) Ap b by discharging the assumption $\mathsf{Pos}(a)$.

- ($\leq$-saturation) We have to prove that if $(l, a) \leq (m, c)$ and (m, c) Ap b then (l, a) Ap b. First, note that $(l, a) \leq (m, c)$ yields $a \leq c$ and $l \preceq m$. Now, let us suppose that both $\mathsf{Pos}(a)$ and $\mathsf{Pos}(l)$ hold. Then, $\mathsf{Pos}(a)$ yields $\mathsf{Pos}(c)$ by $\leq$-*monotonicity* and hence (m, c) Ap b yields $m \lhd (c, b) \cdot \mathsf{nil}$. Therefore, $l \lhd (c, b) \cdot \mathsf{nil}$ follows by $\preceq$-*left*. Moreover, by lemma 3.1, $a \lhd c$, which is a consequence of $a \leq c$, yields $(c, b) \cdot \mathsf{nil} \lhd (a, b) \cdot \mathsf{nil}$ and hence we get $l \lhd (a, b) \cdot \mathsf{nil}$ by *transitivity* and we can discharge the assumption $\mathsf{Pos}(l)$ by *positivity*. Finally, we conclude (l, a) Ap b by discharging also the assumption $\mathsf{Pos}(a)$.

3.3.2 The abstraction

Now we have to prove that the *abstraction* of a continuous relation is a continuous relation.

Lemma 3.11 *Let $\mathcal{A}$ be a unary formal topology and $\mathcal{C}$ and $\mathcal{B}$ be generated formal topologies. Suppose that F is any continuous relation between $\mathcal{C} \times \mathcal{A}$ and $\mathcal{B}$. Then $\Lambda(F)$ is a continuous relation between $\mathcal{C}$ and $\mathcal{A} \to \mathcal{B}$.*

Proof. Let us check that all of the required conditions are satisfied.

- (function totality) $c\ \Lambda(F)\ \mathsf{nil}$ is immediate by intuitionistic logic.
- (function convergence) If $c\ \Lambda(F)\ l$ and $c\ \Lambda(F)\ m$ then $(\forall(a,b)\epsilon l)\ (c,a)Fb$ and $(\forall(a,b)\epsilon m)\ (c,a)Fb$. Thus, $(\forall(a,b)\epsilon l \cdot m)\ (c,a)Fb$ and so $c\ \Lambda(F)\ l \cdot m$. Hence, $c\varepsilon\{w \in C \mid (\exists k\varepsilon l \downarrow m)\ w\ \Lambda(F)\ k\}$, since $l \cdot m \varepsilon l \downarrow m$ because $l \cdot m \lhd l$ and $l \cdot m \lhd m$ follow by *$\preceq$-left* from $l \cdot m \preceq l$ and $l \cdot m \preceq m$. So, $c \lhd \{w \in C \mid (\exists k\varepsilon l \downarrow m)\ w\ \Lambda(F)\ k\}$ follows by *reflexivity*.
- (continuity axiom) We have to check that if $c\ \Lambda(F)\ l$ and $j \in J(l)$ then $c \lhd \Lambda(F)^{-}(C(l,j))$ holds for all the possible axioms indexed by $J(l)$.
  - (totality axiom) We have to show that, for any positive element a of A, c is covered by the subset $\{w \in C \mid w\ \Lambda(F)\ (a, \top_B)\cdot l\}$. This result follows immediately by *reflexivity* since $c\varepsilon\{w \in C \mid e\ \Lambda(F)\ (a, \top_B)\cdot l\}$ because, for all $(x,y)\epsilon(a,\top_B)\cdot l$, $(c,x)Fy$ holds; indeed, for all $(x,y)\epsilon l$, $(c,x)Fy$ holds by assumption and $(c,a)F\top_B$ holds by *function totality*.
  - (unary convergence axiom) We have to show that, provided $(a,b)\epsilon l$ and $(a,d)\epsilon l$ hold, then $c \lhd \Lambda(F)^{-}(\{(a,y)\cdot l \mid y\varepsilon b \downarrow d\})$ follows, that is, $c \lhd \{w \in C \mid (\exists y\varepsilon b \downarrow d)\ w\ \Lambda(F)\ (a,y)\cdot l\}$. Now, $c\,\Lambda(F)\,l$ yields $(c,a)\,F\,b$ and $(c,a)\,F\,d$ and hence $(c,a) \lhd F^{-}(b \downarrow d)$ follows by *unary convergence*. Note now that $\mathsf{Pos}(a)$ holds since $(a,b)\epsilon l$ and $\mathcal{A}$ is a unary formal topology. Thus, by lemma 2.48, we can find a subset W_1 of C such that $c \lhd_C W_1$ and for any $w_1 \varepsilon W_1$ there exists an element $w_2 \in A$ such that $a \lhd_A w_2$ and $(w_1,w_2)\varepsilon F^{-}(b \downarrow d)$, that is, $(\exists y\varepsilon b \downarrow d)\ (w_1,w_2)Fy$. Then it is easy to see that W_1 is a subset of $\{w \in C \mid (\exists y\varepsilon b \downarrow d)\ (w,a)Fy\}$; indeed, $(w_1,w_2)Fy$ yields $(w_1,a)Fy$ by *weak-saturation*, since $a \lhd w_2$ yields $(w_1,a) \lhd (w_1,w_2)$.

    Therefore, we know both that $c \lhd W_1$ and that $W_1 \subseteq \{w \in C \mid (\exists y\varepsilon b \downarrow d)\ (w,a)Fy\}$. Hence, by *reflexivity* and *transitivity*, we get $c \lhd \{w \in C \mid (\exists y\varepsilon b \downarrow d)\ (w,a)Fy\}$. Then, by *$\downarrow$-right*, we obtain $c \lhd \{c\} \downarrow \{w \in C \mid (\exists y\varepsilon b \downarrow d)\ (w,a)Fy\}$.

    We will prove now that $\{c\} \downarrow \{w \in C \mid (\exists y\varepsilon b \downarrow d)\ (w,a)Fy\}$ is a subset of $\{w \in C \mid (\exists y\varepsilon b \downarrow d)\ w\ \Lambda(F)\ (a,y)\cdot l\}$. Indeed, suppose $x \lhd c$ and $x \lhd w$ for some $w \in C$ such that $(w,a)Fy$ for some $y\varepsilon b \downarrow d$. Then $(x,a)Fy$ follows by *weak-saturation* since $x \lhd w$ yields $(x,a) \lhd (w,a)$ by lemma 2.34. Moreover, for any $(s,t)\epsilon l$, $(c,s)Ft$ holds, since by hypothesis $c\ \Lambda(F)\ l$. Hence, $(x,s)Ft$ follows by *weak-saturation* since $x \lhd c$ yields $(x,s) \lhd (c,s)$ by lemma 2.34. Thus, we proved that $x\ \Lambda(F)\ (a,y)\cdot l$, that is, we proved that $x\varepsilon\{w \in C \mid (\exists y\varepsilon b \downarrow d)\ w\ \Lambda(F)\ (a,y)\cdot l\}$.

Now, we can finally conclude. Indeed, by *transitivity*, we get $c \lhd \{w \in C \mid (\exists y \varepsilon b \downarrow d)\ w\ \Lambda(F)\ (a,y) \cdot l\}$, that is, $c \lhd \Lambda(F)^-(\{(a,y) \cdot l \mid y \varepsilon b \downarrow d\})$.

- (weak-saturation axiom) We have to show that, if $(a_2, b)\epsilon l$, $\mathsf{Pos}(a_1)$ and $a_1 \lhd a_2$ then $c \lhd \Lambda(F)^-((a_1, b) \cdot l)$ holds. Now, $(a_2, b)\epsilon l$ and $c\Lambda(F)l$ yield $(c, a_2)Fb$ and hence $(c, a_1)Fb$ follows by *weak-saturation* since $a_1 \lhd a_2$ yields $(c, a_1) \lhd (c, a_2)$ by lemma 2.34. Therefore, for all $(x, y)\epsilon(a_1, b) \cdot l$ we have $(c, x)Fy$, that is, $c\varepsilon\Lambda(F)^-((a_1, b) \cdot l)$ and hence the result follows by *reflexivity*.
- (unary continuity axiom) We have to show that, supposing $(a, b)\epsilon l$ and $j \in J(b)$, $c \lhd \Lambda(F)^-(\{(a, y) \cdot l \mid y\varepsilon C(b, j)\})$ follows, that is, $c \lhd \{w \in C \mid (\exists y\varepsilon C(b, j))\ w\ \Lambda(F)\ (a, y) \cdot l\}$ holds. Now, $c\,\Lambda(F)\,l$ yields that $(c, a)\,F\,b$ and hence $(c, a) \lhd F^-(C(b, j))$ follows by *unary continuity* for F. Note now that $\mathsf{Pos}(a)$ holds since $(a, b)\epsilon l$ and $\mathcal{A}$ is a unary formal topology. Thus, we can apply lemma 2.48 to find a subset W_1 of C such that $c \lhd_C W_1$ and for any $w_1\varepsilon W_1$ there exists $w_2 \in A$ such that $a \lhd_A w_2$ and $(w_1, w_2)\varepsilon F^-(C(b, j))$, that is, $(\exists y\varepsilon C(b, j))\ (w_1, w_2)Fy$.

  Then it is easy to see that $W_1 \subseteq \{w \in C \mid (\exists y\varepsilon C(b, j))\ (w, a)Fy\}$; indeed, $(w_1, w_2)Fy$ yields $(w_1, a)Fy$ by *weak-saturation*, since $a \lhd w_2$ yields $(w_1, a) \lhd (w_1, w_2)$ by lemma 2.34. So, we know that $c \lhd W_1$ and $W_1 \subseteq \{w \in C \mid (\exists y\varepsilon C(b, j))(w, a)Fy\}$ holds and hence, by *reflexivity* and *transitivity* $c \lhd \{w \in C \mid (\exists y\varepsilon C(b, j))\ (w, a)Fy\}$ follows. Then, by $\downarrow$*-right*, we get $c \lhd \{c\} \downarrow \{w \in C \mid (\exists y\varepsilon C(b, j))(w, a)Fy\}$. We will prove now that $\{c\} \downarrow \{w \in C \mid (\exists y\varepsilon C(b, j))\ (w, a)Fy\}$ is a subset of $\{w \in C \mid (\exists y\varepsilon C(b, j))\ w\ \Lambda(F)\ (a, y) \cdot l\}$. Indeed, let $x \lhd c$ and $x \lhd w$ for some $w \in C$ such that $(w, a)Fy$ for some $y\varepsilon C(b, j)$. Then $(x, a)Fy$ follows by *weak-saturation* since $x \lhd w$ yields $(x, a) \lhd (w, a)$ by lemma 2.34. Moreover, for any $(s, t)\epsilon l$, $(c, s)Ft$ holds, since by hypothesis $c\,\Lambda(F)\,l$, and hence $(x, s)Ft$ follows by *weak-saturation* since $x \lhd c$ yields $(x, s) \lhd (c, s)$ by lemma 2.34. Thus, we proved that $x\ \Lambda(F)\ (a, y) \cdot l$, that is, $x\varepsilon\{w \in C \mid (\exists y\varepsilon C(b, j))\ w\ \Lambda(F)\ (a, y) \cdot l\}$. Now, we can finally conclude. Indeed, by *transitivity*, we obtain $c \lhd \{w \in C \mid (\exists y\varepsilon C(b, j))\ w\,\Lambda(F)\ (a, y)\cdot l\}$, that is, $c \lhd \Lambda(F)^-(\{(a, y)\cdot l \mid y\varepsilon C(b, j)\})$.
- ($\leq$-continuity axiom) We have to show that, if both $c\ \Lambda(F)\ l$ and $l \lhd (a, d) \cdot l$ because $(a, b)\epsilon l$ and $b \leq d$, then $c \lhd \Lambda(F)^-((a, d) \cdot l)$ holds. Actually, we show that for all $(x, y)\epsilon(a, d) \cdot l$, $(c, x)Fy$ holds, and from this fact the result follows by *reflexivity*. To this purpose, it is enough to observe that $c\Lambda(F)l$ yields $(c, a)Fb$ since $(a, b)\epsilon l$ and hence $(c, a)Fd$ follows by *function* $\leq$*-continuity*.
- (positivity axiom) We have to prove that $c\,\Lambda(F)\,l$ yields $c \lhd \Lambda(F)^-(l^+)$. To obtain this result let us prove first that *function monotonicity* holds for $\Lambda(F)$. To this aim let us suppose that $\mathsf{Pos}(c)$ and $c\ \Lambda(F)\ l$

hold. Then we have to prove that $\mathsf{Pos}(l)$ holds. We obtain this result by co-inductive reasoning. Indeed, let us consider the following predicate over elements of $\mathcal{A} \to \mathcal{B}$:

$$Q(k) \equiv (\exists x \in C)\ \mathsf{Pos}(x)\ \&\ x\ \Lambda(F)\ k$$

Then, it is trivial to see that $Q(l)$ holds since $\mathsf{Pos}(c)$ and $c\ \Lambda(F)\ l$ hold. We show now that $Q(-)$ satisfies all of the conditions to be a positivity predicate and hence $\mathsf{Pos}(l)$ will follow by the maximality of Pos.

* (order positivity) The proof that $Q(k)$ and $k \preceq m$ yield $Q(m)$ is immediate. Indeed, $Q(k)$ means that there exists a positive element $x \in C$ such that, for all $(s,t)\epsilon k$, $(x,s)Ft$, which immediately yields that, for all $(s,t)\epsilon m$, $(x,s)Ft$ since $k \preceq m$. Hence $x\Lambda(F)m$ holds.
* (totality positivity) Suppose that $Q(k)$ holds. Then there exists a positive element $x \in C$ such that, for all $(s,t)\epsilon k$, $(x,s)Ft$ holds. But, for all $a \in A$, $(x,a)\ F\ \top_B$ holds by *function totality* and hence, for all $(y,z)\epsilon(a,\top_B)\cdot k$, $(x,y)Fz$ follows, that is, $x\ \Lambda(F)\ (a,\top_B)\cdot k$ holds and hence $Q((a,\top_B)\cdot k)$ follows.
* (unary convergence positivity) Suppose that $Q(k)$, $(a,b)\epsilon k$ and $(a,d)\epsilon k$ hold. Then, we have to show that there exists an element $y\varepsilon b \downarrow d$ such that $Q((a,y)\cdot k)$. Now, $Q(k)$ means that there exists a positive element $x \in C$ such that, for all $(s,t)\epsilon k$, $(x,s)\ F\ t$. Hence, we get both that $(x,a)\ F\ b$ and $(x,a)\ F\ d$ hold and thus $(x,a) \lhd \{(w_1,w_2) \mid (\exists y\varepsilon b \downarrow d)\ (w_1,w_2)\ F\ y\}$ follows by *function convergence*. Note now that $\mathsf{Pos}(a)$ holds since $(a,b)\epsilon k$ and hence, by lemma 2.48, there exists a subset W_1 of C such that $x \lhd W_1$ and, for any $w_1\varepsilon W_1$, there exists $w_2 \in A$ such that both $a \lhd w_2$ and $(\exists y\varepsilon b \downarrow d)\ (w_1,w_2)\ F\ y$ hold. Then $(\exists y\varepsilon b \downarrow d)\ (w_1,a)\ F\ y$ follows by *weak-saturation* since $a \lhd w_2$ yields $(w_1,a) \lhd (w_1,w_2)$. Moreover, $x \lhd W_1$ yields $x \lhd \{x\} \downarrow W_1$ by $\downarrow$*-right* and hence, by *monotonicity* of Pos, there exists $z \in C$ such that $z \lhd x$, $(\exists w_1\varepsilon W_1)\ z \lhd w_1$ and $\mathsf{Pos}(z)$ hold. Thus, we immediately obtain that $(\exists y\varepsilon b \downarrow d)\ (z,a)\ F\ y$ holds by *weak-saturation* since, for any w_1 such that $z \lhd w_1$, $(z,a) \lhd (w_1,a)$ follows by lemma 2.34. Moreover, for any $(s,t)\epsilon k$, $(x,s)\ F\ t$ holds and hence also $(z,s)\ F\ t$ follows by *weak-saturation* since $(z,s) \lhd (x,s)$ is a consequence of $z \lhd x$ by lemma 2.34. Therefore, we get $(\exists y\varepsilon b \downarrow d)\ z\ \Lambda(F)\ (a,y)\cdot k$ and so we conclude $(\exists y\varepsilon b \downarrow d)\ Q((a,y)\cdot k)$ since $\mathsf{Pos}(z)$ holds.
* (weak-saturation positivity) Suppose that $Q(k)$, $\mathsf{Pos}(a_1)$, $a_1 \lhd a_2$ and $(a_2,b)\epsilon k$ hold. Then, there exists a positive element $x \in C$ such that, for all $(s,t)\epsilon k$, $(x,s)\ F\ t$ holds. Hence $(x,a_2)\ F\ b$ follows. Observe now that $a_1 \lhd a_2$ yields $(x,a_1) \lhd (x,a_2)$ and hence $(x,a_1)\ F\ b$ follows by *weak-saturation*. Thus, for any

$(s,t)\epsilon(a_1,b)\cdot k$, $(x,s)\ F\ k$ follows. Therefore, $x\Lambda(F)(a_1,b)\cdot k$ and hence we conclude $Q((a_1,b)\cdot k)$.

* (unary continuity positivity) Suppose that $Q(k)$, $(a,b)\epsilon k$ and $j \in J(b)$ hold. Then we have to show that there exists an element $y\varepsilon C(b,j)$ such that $Q((a,y)\cdot k)$. Now, $Q(k)$ means that there exists a positive element $x \in C$ such that, for all $(s,t)\epsilon k$, $(x,s)\ F\ t$ holds. So we deduce that $(x,a)\ F\ b$ holds and hence $(x,a) \lhd \{(w_1,w_2) \mid (\exists y\varepsilon C(b,j))\ (w_1,w_2)\ F\ y\}$ follows by *continuity*. Note now that $\mathsf{Pos}(a)$ holds since $(a,b)\epsilon k$ and hence, by lemma 2.48, there exists a subset W_1 of C such that both $x \lhd W_1$ and, for any $w_1\varepsilon W_1$, there exists $w_2 \in A$ such that $a \lhd w_2$ and $(\exists y\varepsilon C(b,j))\ (w_1,w_2)\ F\ y$. Now, by lemma 2.34, from $a \lhd w_2$ we get $(w_1,a) \lhd (w_1,w_2)$ and hence $(\exists y\varepsilon b\downarrow d)\ (w_1,a)\ F\ y$ follows by *weak-saturation*. Moreover, $x \lhd W_1$ yields $x \lhd \{x\} \downarrow W_1$ by *$\downarrow$-right* and hence, by *monotonicity* of Pos, there exists $z \in C$ such $z \lhd x$, $(\exists w_1\varepsilon W_1)\ z \lhd w_1$ and $\mathsf{Pos}(z)$ hold. Thus, we immediately obtain that $(\exists y\varepsilon C(b,j))\ (z,a)\ F\ y$ by *weak-saturation* since $(z,a) \lhd (w_1,a)$. Moreover, for any $(s,t)\epsilon k$, $(x,s)\ F\ t$ holds and hence also $(z,s)\ F\ t$ follows by *weak-saturation* since $(z,s) \lhd (x,s)$ is a consequence of $z \lhd x$ by lemma 2.34. Therefore, we get $(\exists y\varepsilon C(b,j))\ z\ \Lambda(F)\ (a,y)\cdot k$ and hence we conclude $(\exists y\varepsilon C(b,j))\ Q((a,y)\cdot k)$ since $\mathsf{Pos}(z)$ holds.
* ($\leq$-continuity positivity) Suppose that both $Q(k)$, $(a,b)\epsilon k$ and $b \leq d$ hold. Then, there exists a positive element $x \in C$ such that, for all $(s,t)\epsilon k$, $(x,s)Ft$. Hence, we deduce $(x,a)\ F\ b$ and hence $(x,a)\ F\ d$ follows by *$\leq$-continuity* since $b \leq d$. Thus, we conclude that, for all $(s,t)\epsilon(a,d)\cdot k$, $(x,s)Ft$ holds and hence also $Q((a,d)\cdot k)$ follows since $\mathsf{Pos}(x)$ holds.

This finishes the proof that $Q(k)$ satisfies all of the conditions defining Pos on $\mathcal{A} \to \mathcal{B}$. Now, we can conclude the proof that $c\ \Lambda(F)\ l$ yields $c \lhd \Lambda(F)^{-}(l^{+})$. Assuming $\mathsf{Pos}(c)$, by *function monotonicity* we obtain that $\mathsf{Pos}(l)$ holds and hence $l^{+} = \{l\}$. Then, $c\ \Lambda(F)\ l$ yields $c\varepsilon\Lambda(F)^{-}(l^{+})$ and hence, $c \lhd \Lambda(F)^{-}(l^{+})$ follows by applying first *reflexivity* and then *positivity* to discharge the assumption $\mathsf{Pos}(c)$.

- (function $\leq$-continuity) We have to show that $c\ \Lambda(F)\ l$ and $l \preceq m$ yields $c\ \Lambda(F)\ m$. The result is immediate. Indeed, let $(x,y)\epsilon m$. Then $(x,y)\epsilon l$, since $l \preceq m$, and hence $(c,x)\,F\,y$ follows, since $c\ \Lambda(F)\ l$. So, $c\ \Lambda(F)\ m$ holds.

- ($\leq$-saturation) We have to show that if $c_1 \leq c_2$ and $c_2\ \Lambda(F)\ l$ then $c_1\ \Lambda(F)\ l$. Now, $c_2\ \Lambda(F)\ l$ yields that, for all $(a,b)\epsilon l$, $(c_2,a)Fb$ holds. But, for all $(a,b)\epsilon l$, $c_1 \leq c_2$ yields $(c_1,a) \leq (c_2,a)$, and hence, for all $(a,b)\epsilon l$, $(c_1,a)Fb$, that is, $c_1\ \Lambda(F)\ l$, follows by *$\leq$-saturation* of F.

- (axiom saturation) Suppose $c \in C$ and $j \in J(c)$, that is, $c \lhd_C C(c,j)$ is an axiom of the inductively generated formal topology $\mathcal{C}$, and suppose that

$y\ \Lambda(F)\ l$ holds for every $y \varepsilon C(c,j)$. Then, we want to show that $c\ \Lambda(F)\ l$ holds. Now, for every $(a,b) \epsilon l$ we have $(c,a) \lhd_{C\times A} C(c,j) \times \{a\}$ and for every $(y,a) \varepsilon C(c,j) \times \{a\}$, $(y,a)Fb$ is a consequence of the fact that for every $y \varepsilon C(c,j)$, $y\ \Lambda(F)\ l$ holds. Then, $(c,a)Fb$ follows by *saturation* of F and hence, we obtain $c\ \Lambda(F)\ l$ by universal quantification.

Thus, we have finished with the proof that, for any continuos relation F, $\Lambda(F)$ is also a continuous relation. In the next section we will prove that the required equations hold.

3.3.3 The equations

To finish the proof that the formal topology $\mathcal{A} \to \mathcal{B}$ is the exponential of $\mathcal{A}$ over $\mathcal{B}$ we have to show that the adjunction equations hold with respect to *application* and *abstraction*.

Proposition 3.12 *Let $\mathcal{A}$ be a unary formal topology and $\mathcal{C}$ and $\mathcal{B}$ be inductively generated formal topologies. Then*

1. *for every continuous relation F between $\mathcal{C} \times \mathcal{A}$ and $\mathcal{B}$,*

$$\mathsf{Ap} * \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle = F$$

2. *for every continuous relation G between $\mathcal{C}$ and $\mathcal{A} \to \mathcal{B}$,*

$$\Lambda(\mathsf{Ap} * \langle G * \Pi_1, \Pi_2 \rangle) = G$$

Proof. We prove the two implications of the considered equations one after the other.

- *(1. Right to left)* We have to prove that, for any $c \in C$, $a \in A$ and $b \in B$, if $(c,a)Fb$ then $(c,a)\ \mathsf{Ap} * \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle\ b$. Now, $(c,a)Fb$ yields $c\ \Lambda(F)\ (a,b) \cdot \mathsf{nil}$ and hence $(c,a) \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle ((a,b) \cdot \mathsf{nil}, a)$ follows since $(c,a)\ \Pi_2\ a$ holds. Then, we conclude $(c,a)\ \mathsf{Ap} \circ \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle\ b$ since $((a,b) \cdot \mathsf{nil}, a)\ \mathsf{Ap}\ b$ holds.
- *(1. Left to right)* We have to prove that $(c,a)\ \mathsf{Ap} * \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle\ b$ yields $(c,a)Fb$. The proof will have the following structure.

    1. First, for any positive element $(x,y) \in C \times A$ such that

    $$(x,y)\ \mathsf{Ap} \circ \langle \Lambda(F) * \Pi_1, \Pi_2 \rangle\ b$$

    that is, such that there exist $l_{(x,y)} \in A \to B$ and $a_{(x,y)} \in A$ such that $(x,y)\ \Lambda(F) * \Pi_1\ l_{(x,y)}$ and $(x,y)\ \Pi_2\ a_{(x,y)}$ and $(l_{(x,y)}, a_{(x,y)})\ \mathsf{Ap}\ b$, we will prove that

    $$(x,y) \lhd \{(x',y') \varepsilon \mathsf{Pos}_{C\times A} \mid (x',y')\ \Lambda(F) \circ \Pi_1\ l_{(x,y)}\} \downarrow_{\leq} \{(\top_C, a_{(x,y)})\}$$

    holds.

2. Then, we will prove that

$$\{(x',y')\varepsilon\mathsf{Pos}_{C\times A} \mid (x',y')\ \Lambda(F)\circ\Pi_1\ l_{(x,y)}\} \downarrow_{\leq} \{(\top_C, a_{(x,y)})\} \lhd \{(x'',y'')\varepsilon\mathsf{Pos}_{C\times A} \mid (x'',y'')Fb\}$$

3. Finally, from $(c,a)\ \mathsf{Ap} * \langle\Lambda(F)*\Pi_1,\Pi_2\rangle\ b$ we get

$$(c,a) \lhd \{(x,y)\in C\times A \mid (x,y)\ \mathsf{Ap}\circ\langle\Lambda(F)*\Pi_1,\Pi_2\rangle\ b\}$$

and hence by *positivity* we obtain

$$(c,a) \lhd \{(x,y)\varepsilon\mathsf{Pos}_{C\times A} \mid (x,y)\ \mathsf{Ap}\circ\langle\Lambda(F)*\Pi_1,\Pi_2\rangle\ b\}$$

Thus, by *transitivity*, we will get

$$(c,a) \lhd \{(x'',y'')\varepsilon\mathsf{Pos}_{C\times A} \mid (x'',y'')Fb\}$$

and hence we will conclude $(c,a)\ F\ b$ by *saturation*.

So, let us prove now points (1) and (2):

1. $(x,y)\ \Lambda(F)*\Pi_1\ l_{(x,y)}$ means that

$$(x,y) \lhd \{(x',y')\in C\times A \mid (x',y')\ \Lambda(F)\circ\Pi_1\ l_{(x,y)}\}$$

and hence by *positivity* we obtain

$$(x,y) \lhd \{(x',y')\varepsilon\mathsf{Pos}_{C\times A} \mid (x',y')\ \Lambda(F)\circ\Pi_1\ l_{(x,y)}\}$$

Moreover, $(x,y)\ \Pi_2\ a_{(x,y)}$ yields that $(x,y)\lhd(\top_C, a_{(x,y)})$ and hence

$$(x,y) \lhd \{(x',y')\varepsilon\mathsf{Pos}_{C\times A} \mid (x',y')\ \Lambda(F)\circ\Pi_1\ l_{(x,y)}\} \downarrow_{\leq} \{(\top_C, a_{(x,y)})\}$$

follows by $\leq$-*right*.

2. Let (x'',y'') be an element in $C\times A$ such that

$$(x'',y'')\varepsilon\{(x',y')\varepsilon\mathsf{Pos}_{C\times A} \mid (x',y')\ \Lambda(F)\circ\Pi_1\ l_{(x,y)}\} \downarrow_{\leq} \{(\top_C, a_{(x,y)})\}$$

Therefore there exists a positive element (x',y') in $C\times A$ such that both $(x'',y'')\leq(x',y')$ and $(x',y')\ \Lambda(F)\circ\Pi_1\ l_{(x,y)}$ hold and also $(x'',y'')\leq(\top_C, a_{(x,y)})$. Now, $(x',y')\ \Lambda(F)\circ\Pi_1\ l_{(x,y)}$ yields that there exists an element $c_{(x,y)}\in C$ such that both $(x',y')\lhd(c_{(x,y)},\top_A)$ and $c_{(x,y)}\ \Lambda(F)\ l_{(x,y)}$ hold. Moreover, $(x'',y'')\leq(x',y')$ and $(x',y')\lhd(c_{(x,y)},\top_A)$ yields $(x'',y'')\lhd(c_{(x,y)},\top_A)$ by $\leq$-*left* and *transitivity*. Finally, $(x'',y'')\lhd(\top_C, a_{(x,y)})$ follows from $(x'',y'')\leq(\top_C, a_{(x,y)})$ by $\leq$-*left* and hence we get $(x'',y'')\lhd(c_{(x,y)},\top_A)\downarrow_{\leq}(\top_C, a_{(x,y)})$ by $\leq$-*right*. Now, we can conclude

$$(x'',y'')\lhd(c_{(x,y)}, a_{(x,y)})$$

by *transitivity* since $(c_{(x,y)}, \top_A) \downarrow_\leq (\top_C, a_{(x,y)}) \lhd (c_{(x,y)}, a_{(x,y)})$. Indeed, for any $(s,t) \in C \times A$ such that $(s,t) \leq (c_{(x,y)}, \top_A)$ and $(s,t) \leq (\top_C, a_{(x,y)})$ we get immediately that $(s,t) \leq (c_{(x,y)}, a_{(x,y)})$ and hence $(s,t) \lhd (c_{(x,y)}, a_{(x,y)})$ follows by $\leq$-*left*.

Recall now that $(l_{(x,y)}, a_{(x,y)})$ Ap b. Then, by definition, we obtain $\mathsf{Pos}(a_{(x,y)}) \to l_{(x,y)} \lhd \{(a_{(x,y)}, b) \cdot \mathsf{nil}\}$. But $(x,y)\ \Pi_2\ a_{(x,y)}$ holds and (x,y) is a positive element of $C \times A$ and hence $\mathsf{Pos}(a_{(x,y)})$ follows by *function positivity* for Π_2. Thus we get $l_{(x,y)} \lhd \{(a_{(x,y)}, b) \cdot \mathsf{nil}\}$.

Recall also that $c_{(x,y)}\ \Lambda(F)\ l_{(x,y)}$. Then $l_{(x,y)} \lhd \{(a_{(x,y)}, b) \cdot \mathsf{nil}\}$ yields $c_{(x,y)}\ \Lambda(F)\ (a_{(x,y)}, b) \cdot \mathsf{nil}$ by *weak-continuity* of $\Lambda(F)$, and hence we obtain that $(c_{(x,y)}, a_{(x,y)})Fb$. Then, by *weak-continuity* of F, from $(x'', y'') \lhd (c_{(x,y)}, a_{(x,y)})$ we obtain $(x'', y'')Fb$ and so $(x'', y'')\varepsilon\{(x'', y'')\varepsilon\mathsf{Pos}_{C\times A} \mid (x'', y'')Fb\}$. Thus

$$(x'', y'') \lhd \{(x'', y'')\varepsilon\mathsf{Pos}_{C\times A} \mid (x'', y'')Fb\}$$

follows by *reflexivity*.

- *(2. Right to left)* We have to prove that, for any $c \in C$ and for any $l \in A \to B$, if $c\ G\ l$ holds then also $c\ \Lambda(\mathsf{Ap} * \langle G * \Pi_1, \Pi_2\rangle)\ l$ holds, that is, for any $(a,b)\epsilon l$, $(c,a)\ \mathsf{Ap} * \langle G * \Pi_1, \Pi_2\rangle\ b$. So, suppose $(a,b)\epsilon l$. Then, we get $c\ G\ \{(a,b) \cdot \mathsf{nil}\}$ by *weak-continuity* of G since $l \lhd \{(a,b) \cdot \mathsf{nil}\}$ follows by $\leq$-*left* from $l \preceq \{(a,b) \cdot \mathsf{nil}\}$. Now, $(c,a)\ \Pi_1\ c$ and $(c,a)\ \Pi_2\ a$ clearly hold and hence we get $(c,a)\ \langle G * \Pi_1, \Pi_2\rangle\ ((a,b) \cdot \mathsf{nil}, a)$. Finally, we obtain $(c,a)\ \mathsf{Ap} * \langle G * \Pi_1, \Pi_2\rangle\ b$ since $((a,b) \cdot \mathsf{nil}, a)\ \mathsf{Ap}\ b$ obviously holds.

- *(2. Left to right)* We have to prove that, for every $c \in C$ and $l \in A \to B$, $c\ \Lambda(\mathsf{Ap}*\langle G*\Pi_1, \Pi_2\rangle)\ l$ yields $c\ G\ l$. So, suppose that $c\ \Lambda(\mathsf{Ap}*\langle G*\Pi_1, \Pi_2\rangle)\ l$ holds. Then, for every $(a,b)\epsilon l$, $(c,a)\ \mathsf{Ap} * \langle G * \Pi_1, \Pi_2\rangle\ b$ and hence

$$(c,a) \lhd \{(x,y)\varepsilon\mathsf{Pos}_{C\times A} \mid (x,y)\ \mathsf{Ap} \circ \langle G * \Pi_1, \Pi_2\rangle\ b\}$$

follows by *positivity*. Now, the proof of this point will go on as follows:

1. First, for every positive element $(x,y) \in C \times A$ such that

$$(x,y)\ \mathsf{Ap} \circ \langle G * \Pi_1, \Pi_2\rangle\ b$$

that is, such that there exist $l_{(x,y)} \in A \to B$ and $a_{(x,y)} \in A$ such that $(l_{(x,y)}, a_{(x,y)})\ \mathsf{Ap}\ b$ and $(x,y)\ \langle G * \Pi_1, \Pi_2\rangle\ (l_{(x,y)}, a_{(x,y)})$, we will prove that

$$(x,y) \lhd \{(x',y')\varepsilon\mathsf{Pos}_{C\times A} \mid (x',y')\ G \circ \Pi_1\ l_{(x,y)}\} \downarrow_\leq (\top_C, a_{(x,y)})$$

holds.

2. Then, we will prove that

$$\{(x',y')\varepsilon\mathsf{Pos}_{C\times A} \mid (x',y')\ G \circ \Pi_1\ l_{(x,y)}\} \downarrow_\leq (\top_C, a_{(x,y)}) \lhd \{(x'',y'')\varepsilon\mathsf{Pos}_{C\times A} \mid x''\ G\ (a,b) \cdot \mathsf{nil}\}$$

3. So, by *transitivity*, we obtain

$$(c, a) \lhd \{(x'', y'')\varepsilon\mathsf{Pos}_{C\times A} \mid x''\ G\ (a, b)\cdot\mathsf{nil}\}$$

and hence by lemma 2.48, that we can apply since a is positive because $(a, b)\epsilon l$, there exists a subset W_1 of C such that $c \lhd_C W_1$ and for every $w_1\varepsilon W_1$ there exists $w_2 \in A$ such that $a \lhd_A w_2$ and $w_1\ G\ (a, b)\cdot\mathsf{nil}$. Hence, by *saturation* of G, we conclude $c\ G\ (a, b)\cdot\mathsf{nil}$ from $c \lhd_C W_1$.

Since $c\ G\ (a, b)\cdot\mathsf{nil}$ holds for every $(a, b)\epsilon l$, by successive applications of lemma 3.4 we conclude $c\ G\ l$. So, let us prove the points (1) and (2) above.

1. $(x, y)\ \langle G * \Pi_1, \Pi_2\rangle\ (l_{(x,y)}, a_{(x,y)})$, yields both $(x, y)\, G * \Pi_1\, l_{(x,y)}$ and $(x, y)\, \Pi_2\, a_{(x,y)}$. Now, (x, y) is positive and hence $(x, y)\, \Pi_2\, a_{(x,y)}$ yields that $a_{(x,y)}$ is positive by *function monotonicity* for Π_2.
   Hence $(l_{(x,y)}, a_{(x,y)})\ \mathsf{Ap}\ b$ yields $l_{(x,y)} \lhd \{(a_{(x,y)}, b)\cdot\mathsf{nil}\}$. Moreover, $(x, y)\ G * \Pi_1\ l_{(x,y)}$ yields

   $$(x, y) \lhd \{(x', y')\varepsilon\mathsf{Pos}_{C\times A} \mid (x', y')\ G\circ\Pi_1\ l_{(x,y)}\}$$

   by *positivity* and so, by $\leq$*-right*, we obtain

   $$(x, y) \lhd \{(x', y')\varepsilon\mathsf{Pos}_{C\times A} \mid (x', y')\ G\circ\Pi_1\ l_{(x,y)}\} \downarrow_{\leq} (\top_C, a_{(x,y)})$$

2. Then, we prove that

   $$\{(x', y')\varepsilon\mathsf{Pos}_{C\times A} \mid (x', y')\ G\circ\Pi_1\ l_{(x,y)}\} \downarrow_{\leq} (\top_C, a_{(x,y)}) \lhd \{(x'', y'')\varepsilon\mathsf{Pos}_{C\times A} \mid x''\ G\ (a, b)\cdot\mathsf{nil}\}$$

   Indeed, suppose

   $$(x'', y'')\varepsilon\{(x', y')\varepsilon\mathsf{Pos}_{C\times A} \mid (x', y')\, G\circ\Pi_1\, l_{(x,y)}\} \downarrow_{\leq} (\top_C, a_{(x,y)})$$

   Then, there exists a positive element $(x', y') \in C \times A$ such that $(x'', y'') \leq (x', y')$ and $(x', y')\, G\circ\Pi_1\, l_{(x,y)}$ and $(x'', y'') \leq (\top_C, a_{(x,y)})$. Now, $(x', y')\, G\circ\Pi_1\, l_{(x,y)}$ yields that there exists $c_{(x,y)} \in C$ such that $(x', y') \lhd (c_{(x,y)}, \top_A)$ and $c_{(x,y)}\, G\, l_{(x,y)}$.
   Recall now that $(l_{(x,y)}, a_{(x,y)})\ \mathsf{Ap}\ b$. Then, by definition, we obtain $\mathsf{Pos}(a_{(x,y)}) \to l_{(x,y)} \lhd \{(a_{(x,y)}, b)\cdot\mathsf{nil}\}$. But $(x, y)\ \Pi_2\ a_{(x,y)}$ holds and (x, y) is a positive element of $C \times A$ and hence $\mathsf{Pos}(a_{(x,y)})$ follows by *function positivity* for Π_2. Thus we get $l_{(x,y)} \lhd \{(a_{(x,y)}, b)\cdot\mathsf{nil}\}$. So, from $c_{(x,y)}\, G\, l_{(x,y)}$ by *weak-continuity* of G we obtain

   $$c_{(x,y)}\, G\, (a, b)\cdot\mathsf{nil}$$

   Now, $(x'', y'') \leq (x', y')$ and $(x', y') \lhd (c_{(x,y)}, \top_A)$ yield $(x'', y'') \lhd (c_{(x,y)}, \top_A)$ by $\leq$*-left*.

Moreover, again by $\leq$-*left*, $(x'', y'') \leq (\top_C, a_{(x,y)})$ yields $(x'', y'') \lhd (\top_C, a_{(x,y)})$. So, by $\leq$-*right*, $(x'', y'') \lhd (c_{(x,y)}, \top_A) \downarrow_{\leq} (\top_C, a_{(x,y)})$ follows and hence we conclude

$$(x'', y'') \lhd (c_{(x,y)}, a_{(x,y)})$$

by *transitivity* since $(c_{(x,y)}, \top_A) \downarrow_{\leq} (\top_C, a_{(x,y)}) \lhd (c_{(x,y)}, a_{(x,y)})$.
Since (x'', y'') is positive, by lemma 2.48 there exists a subset W_1 of C such that $x'' \lhd_C W_1$ and for every $w_1 \varepsilon W_1$ there exists $w_2 \in A$ such that $y'' \lhd_A w_2$ and $(w_1, w_2) \varepsilon \{(c_{(x,y)}, a_{(x,y)})\}$. So, $w_1 = c_{(x,y)}$ and hence $x'' \lhd c_{(x,y)}$. Therefore, $c_{(x,y)}\, G\, (a, b) \cdot \mathsf{nil}$ yields $x''\, G\, (a, b) \cdot \mathsf{nil}$ by *weak saturation* of G.
Thus, $(x'', y'') \varepsilon \{(x'', y'') \varepsilon \mathsf{Pos}_{C \times A} \mid x'\ G\ (a, b) \cdot \mathsf{nil}\}$ and hence by applying *reflexivity* we get

$$\{(x', y') \varepsilon \mathsf{Pos}_{C \times A} \mid (x', y')\ G \circ \Pi_1\ l\} \downarrow_{\leq} (\top, a') \lhd \\ \{(x'', y'') \varepsilon \mathsf{Pos}_{C \times A} \mid x''\ G\ (a, b) \cdot \mathsf{nil}\}$$

So, we are arrived at the main theorem of the paper.

Theorem 3.13 *Unary topologies are exponentiable in* $\mathsf{FTop_i}$.

Let us remark that the proof of this theorem is valid also intuitionistically since no use of the axiom of choice is required in an impredicative approach.

4 Concluding remarks

We add here some observations that can be useful for a more complete understanding of the topic of the paper and which are immediate consequences of our work.

4.1 Why our result is limited to unary formal topologies

We showed that all the conditions on a continuous relation F between a unary formal topology $\mathcal{A}$ and an inductively generated one $\mathcal{B}$ have in general one of the following shapes, for $a, a' \in A$, $b, b' \in B$ and $V \subseteq B$:

$$\frac{a\ R\ b \quad P(a, b, a', b')}{a'\ R\ b'} \qquad \frac{a\ R\ b \quad Q(a, b, V)}{(\exists y \varepsilon V)\ a\ R\ y}$$

Now, in section 3.1 we showed how to obtain an axiom out of each kind of condition. In fact, to any condition whose shape is

$$\frac{a\ R\ b \quad P(a, b, a', b')}{a'\ R\ b'}$$

corresponds the axiom

$$l \lhd (a', b') \cdot l$$

for any $l \in A \to B$ such that $(a, b)\epsilon l$ and $P(a, b, a', b')$ hold, and to any condition whose shape is

$$\frac{a\ R\ b \quad Q(a, b, V)}{(\exists y \varepsilon V)\ a\ R\ y}$$

corresponds the axiom

$$l \lhd \{(a, y) \cdot l \mid y \varepsilon V\}$$

for any $l \in A \to B$ such that $(a, b)\epsilon l$ and $Q(a, b, V)$ hold.

Thus, we can define the exponential formal topology of an inductively generated formal topology over another one provided that we can express the general conditions on a continuous relation by using one of the shapes above. At present, we have obtained this result only in the case of having a unary formal topology as exponent.

4.2 Exponentiation without the positivity predicate

The exponentiation of unary topologies in $\mathsf{FTop_i}$ clearly yields the exponentiation of unary topologies in $\mathsf{FTop_i}^-$.

Theorem 4.1 *Unary topologies are exponentiable in the category* $\mathsf{FTop_i}^-$.

Proof. It is sufficient to observe that all the proofs work as in the case of $\mathsf{FTop_i}$ by simply substituting Pos with an always true predicate.

Note that in this case the proof of our main theorem becomes entirely predicative since there is no need to justify the definition of the positivity predicate by co-induction. In fact, this result constitutes a partial but completely predicative version of [Hyl81, Joh84, Sig95]. Indeed, in these papers, if $\mathcal{A}$ is a locally compact locale and $\mathcal{B}$ is any locale then the cover for $\mathcal{A} \to \mathcal{B}$ is generated from axioms on a new proposition, denoted, for instance in [Hyl81], for $a \in A$ and $b \in B$ by $a \ll f^*b$, which represents the collection of locale morphisms f such that a is way-below f^*b. Now, when $\mathcal{A}$ is a locale representing an algebraic dcpo, this proposition corresponds exactly to our $\mathsf{ext}^{Pt}((a, b) \cdot \mathsf{nil})$ since the latter represents the collection of all the continuous relations R such that $a\ R\ b$.

4.3 Unary topologies are not closed under exponentiation

It is known that algebraic dcpos with bottom element lack function spaces, that is, the category of algebraic dcpos is not cartesian closed [AJ94]. Our work makes explicit where the problem is. Indeed, it is clear that all of the axioms of the exponential topology between unary topologies satisfy the unary conditions except for the axiom on unary convergence since we can not limit it to a single element. This is to be contrasted with what happens in the case of the category of unary formal topologies equipped with a monoid operation on the elements of the base which expresses intersection of open subsets. Indeed, this category turns out to be equivalent to the category of Scott Domains [SVV96] and it can be proved to be *predicatively* cartesian closed [Val03].

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A On the definition by co-induction

It is not difficult to realize that we can formalize the problem of defining the positivity predicate by expressing it as the problem of finding the maximal subset K of a set S satisfying the following conditions

$$\frac{x \varepsilon K \quad A(x,y)}{y \varepsilon K} \qquad \frac{x \varepsilon K \quad y \varepsilon B(x)}{(\exists v \varepsilon C(x,y))\ v \varepsilon K}$$

for some propositions $A(x,y)$, $B(x)$ and $C(x,y)$.

Now, it is easy to see that one can use Tarski fixed point theorem in order to solve them in an impredicative way. Indeed, the map $\tau : \mathcal{P}(S) \to \mathcal{P}(S)$ defined by setting

$$\begin{array}{rcl}\tau(X) & \equiv & \{x \in S \mid (\forall y \in S)\ A(x,y) \to y \varepsilon X\} \\ & & \cap\ \{x \in S \mid (\forall y \varepsilon B(x))(\exists v \varepsilon C(x,y))\ v \varepsilon X\}\end{array}$$

is clearly monotone and hence it admits a maximal fixed point which obviously satisfies the required conditions.