1 Holomorphic maps

1.1 The field $\mathbb{C}$ of Complex Numbers

The topology on $\mathbb{C}$ is the usual one, namely the Euclidean structure inherited from the vector space identification $\mathbb{C} \sim \mathbb{R}^2$. This identifications is nothing but the one induced by the $\mathbb{R}$-isomorphism $x + iy \to (x, y)$. Hereinafter a subset $E \subset \mathbb{C}$, will be sometimes regarded as a subset of $\mathbb{R}^2$, confusion being clearly avoided by the context. On the other hand $\mathbb{C}$ has an extra structure, namely the field structure given, together with the sum, by the product operation: $(x + iy) \cdot (a + ib) = (xa - yb) + i(xb + ya)$. (notice that this is just the *formal* extension of the usual product in $\mathbb{R}$ and the product rule $i^2 = -1$). Hence the norm of $z = x + iy \in C$ is given by

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}},$$

where $\bar{z} \doteq x - iy$ is the *conjugate of* z. As for the division, let us remind that that if $z = x + iy \in \mathbb{C}$, $z \neq 0$, $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x-iy}{x^2+y^2}$.

1.2 Complex derivative

Let $\Omega \subseteq \mathbb{C}$ be an open set and let $f : \Omega \to \mathbb{C}$ be a map. Let $z_0 = x_0 + iy_0 \in \Omega$ $(x_0, y_0 \in \mathbb{R})$. Proceeding as in the case of real functions, one can consider the limit of the difference quotient

$$\frac{f(z) - f(z_0)}{z - z_0}. \tag{1.1}$$

Definition 1.1 *If the there exists the limit of the differential quotient (1.1) for z going to z_0 then it is called the* (complex) derivative of f at z_0 *and is denoted by $f'(z_0)$ or $\frac{df}{dz}(z_0)$:*

$$\frac{df}{dz}(z_0) = f'(z_0) \doteq \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{1.2}$$

As in the case of the real variable, a complex map f has complex derivative $f'(z_0)$ at z_0 if and only if f is (complex) *differentiable at* z_0 with, which means

$$f(z) - f(z_0) = f'(z_0)(z - z_0) + o(z - z_0).$$

Definition 1.2 *If the derivative $f'(z_0)$ exists at every $z_0 \in \Omega$ and is continuous, f is called a* holomorphic *function.*

Remark 1.1 Though practically useful, the requirement that the complex derivative f' is continuous is, in fact, redundant. Indeed, it can be proved that the only existence of f' at each point of Ω is sufficient in order f' to be continuous, i.e., f to be of class C^1. As a matter of fact, much stronger regularity holds true (see [1]: f turns out to be C^∞ and *analytic*, i.e., locally equal to the sum of its (convergent) Taylor's series. For this reason, referring to complex variable functions, someone utilizes the term *analytic*, instead of *holomorphic*.).

1.3 Relation with partial derivatives: the Cauchy-Riemann equations

The fact that the limit involves z approaching z_0 from *any* direction has strong geometrical consequences. Actually one has:

Proposition 1.1 *If $f'(z_0)$ exists, then*

$$u_x = v_y \qquad u_y = -v_x \ , \tag{1.3}$$

where we mean that the partial derivatives are computed in (x_0, y_0). Identities (1.3) are called the Cauchy-Riemann conditions.

Proof. Let us set $z = x + iy$, $z_0 = x_0 + iy_0$, $f(z) = u(x, y) + iv(x, y)$. If $f'(z_0)$ exists, the limit in (1.2) must be valid, in particular, for $z = z_0 + h = (x_0 + h) + iy$ with $h \in R$ $h \to 0$. So

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0}{h} = f_x(z_0) =$$

$$\lim_{h\to 0} \left(\frac{u(x_0 + h, y) - u(x_0, y_0)}{h} + i \frac{v(x_0 + h, y) - v(x_0, y_0)}{h} \right) = \tag{1.4}$$

$$u_x(x_0, y_0) + iv_x(x_0, y_0)$$

Similarly, the limit in (1.2) must be valid for $z = z_0 + ih = x_0 + i(y + h)$ with $h \in \mathbb{R}$ $h \to 0$:

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0 + ih) - f(z_0}{ih} = f_y(z_0) =$$

$$\lim_{h\to 0} \left(\frac{u(x_0, y + h) - u(x_0, y_0)}{ih} + \lim_{h\to 0} i \frac{v(x_0, y + h) - v(x_0, y_0)}{ih} \right) = \tag{1.5}$$

$$-iu_y(x_0, y_0) + v_y(x_0, y_0)$$

By (1.4),(1.5) one gets

$$u_x = v_y \qquad u_y = -v_x,$$

so the proof is concluded.

In particular, if f is holomorphic the Cauchy-Riemann conditions (1.3) are verified at each $z_0 = x_0 + iy_0 \in \Omega$.

A natural question is now whether or not the Cauchy-Riemann conditions are also *sufficient* for the derivative $f'(z_0)$ to exist. Actually this is the case:

Proposition 1.2 *Assume that u and v (as real maps defined on $\Omega \subset \mathbb{R}^2$) are (real-) differentiable at (x_0, y_0) and that Cauchy-Riemann conditions (1.3) are verified. Then $f'(z_0)$ exists. Furthermore,* *(i) one has*

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = i^{-1}(u_y(x_0, y_0) + iv_y(x_0, y_0)) = v_y(x_0, y_0) - iu_y(x_0, y_0); \tag{1.6}$$

(ii) the map $(u, v) : \Omega \subset I\!R^2 \to I\!R^2$ *is* $I\!R$*-differentiable, i.e.,*

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{pmatrix} + \begin{pmatrix} u_x(x_0,y_0) & u_y(x_0,y_0) \\ v_x(x_0,y_0) & v_y(x_0,y_0) \end{pmatrix} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + o(|(x - x_0, y - y_0)|) =$$

$$\begin{pmatrix} u(x_0,y_0) \\ v(x_0,y_0) \end{pmatrix} + \begin{pmatrix} u_x(x_0,y_0) & u_y(x_0,y_0) \\ -u_y(x_0,y_0) & u_x(x_0,y_0) \end{pmatrix} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + o(|(x - x_0, y - y_0)|) \tag{1.7}$$

Proof. See [1] for a proof that $f'(z_0)$ does exist (or try it by exercise). To prove (ii) is equivalent to show that

$$\begin{pmatrix} u_x(x_0,y_0) & u_y(x_0,y_0) \\ -u_y(x_0,y_0) & u_x(x_0,y_0) \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = f'(z_0) \cdot (h_1 + ih_2)$$

as soon as the complex number on the right hand-side is identified with the corresponding vector in $I\!R^2$. This is trivial, as

$$f'(z_0) \cdot (h_1 + ih_2) = (u_x - iu_y)(h_1 + ih_2) = (u_x h_1 - u_y h_2) + i(u_x h_2 + u_y h_1).$$

1.4 Geometric interpretation

When $f'(z_0) \neq 0$, Cauchy-Riemann conditions, which , as we have seen, characterize the existence of the complex derivative, have a nice geometrical interpretation. Indeed, suppose that f is holomorphic and $f'(z_0) \neq 0$. By

$$f'(z_0) = -iu_y(x_0, y_0) + v_y(x_0, y_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$$

one has $\nabla u(x_0, y_0) \neq 0$ and $\nabla u(x_0, y_0) \neq 0$. So, near (x_0, y_0) the level sets (in Ω) $u = u(x_0, y_0)$ and $v = v(x_0, y_0)$ are in fact C^1 lines. In particular, the Cauchy-Riemann conditions

$$(v_x, v_y) = (-u_y, u_x)$$

say that:

(A) $|\nabla u| = |\nabla v|$, which can be interpreted as the fact that locally u and v grow at the same rate along the directions of maximal rate, which coincide with the directions of ∇u and ∇v, respectively.

(B) ∇u is orthogonal to ∇v. Since ∇u and ∇v are orthogonal to the level line of u and v, respectively, this means that these level lines intersect each other orthogonally.

(C) Interpreting $1(\in \mathbb{C})$ and i as the $\mathbb{R}^2$ vectors $(1,0)$ and $(0,1)$, respectively, and regarding f as a map from Ω to $\mathbb{R}^2$, Cauchy-Riemann conditions can also be written in the form

$$(v_x + iv_y) = i(u_x + iu_y) \tag{1.8}$$

Again, this shows that ∇v is obtained by a counter-clockwise rotation of $\pi/2$ of ∇u. Actually, multiplication by i can be interpreted as counter-clockwise rotation of $\pi/2$, (namely, if complex numbers are interpreted as column vectors, multiplication by i is equivalent to left multiplication by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Indeed,

$$iz = i(x+iy) = -y + ix = \begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

(D) Actually (A)-(C) are consequences of the following fact:

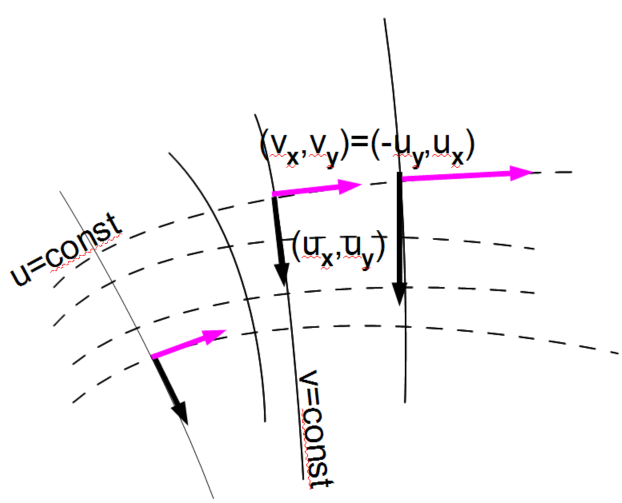
The multiplication by $f'(z_0)$ is equivalent, when $\mathbb{C}$ is identified with $\mathbb{R}^2$, with the left multiplication by the matrix

$$\begin{pmatrix} u_x(x_0,y_0) & u_y(x_0,y_0) \\ -u_y(x_0,y_0) & u_x(x_0,y_0) \end{pmatrix}$$

Moreover, this matrix is a rotation *(= special orthogonal matrix =orthogonal matrix with determinant equal to* 1*) times a positive constant. More precisely,*

$$\begin{pmatrix} u_x(x_0,y_0) & u_y(x_0,y_0) \\ -u_y(x_0,y_0) & u_x(x_0,y_0) \end{pmatrix} = \rho \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

where (ρ, ϕ) are the modulus and and the argument of $f'(z_0)$, namely $f'(z_0) = \rho e^{i\phi}$.



1.5 Operations, compositions, and inversion of holonomic functions

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Proposition 1.3 *Let $f, g : \Omega \to C$ have complex derivative at $z_0 \in \Omega$ differentiable. Then $f+g, fg$ have complex derivative at z_0, and*

$$(f+g)(z_0) = f(z_0) + g(z_0), \qquad (fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0) \tag{1.9}$$

. *If $g(z_0) \neq 0$ then $\frac{f}{g}$ has complex derivative and*

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}.$$

Proof. The proof does not differ anyhow from the proof in the real case. In particular one can use the fact that f the derivative $f'(z_0)$ if and only if

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(z - z_0) \tag{1.10}$$

For instance, the equality,

$$f(z)g(z) = [f(z_0) + f'(z_0)(z - z_0) + o(z - z_0)] \cdot [g(z_0) + g'(z_0)(z - z_0) + o(z - z_0)] =$$

$$f(z_0)g(z_0) + (f'(z_0)g(z_0) - f(z_0)g'(z_0))(z - z_0) + o(z - z_0)$$

implies the existence of $(fg)'(z_0)$ and the formula in (1.9).

Proposition 1.4 *If $f : \Omega \to A \subset \mathbb{C}$ has derivative at a point $z_0 \in \Omega$ and $g : A \to \mathbb{C}$ has derivative at $w_0 \doteq f(z_0)$, then $g \circ f$ has derivative at z_0, and*

$$(g \circ f)'(z_0) = g'(w_0)f'(z_0).$$

In particular, a composition of holomorphic maps is holomorphic.

Proof. Again the proof is a straightforward application of the differentiability relation (1.10):

$$\lim_{z \to z_0} \frac{g \circ f(z) - g \circ f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{g'(w_0)(f(z) - f(z_0)) + o(f(z) - f(z_0))}{z - z_0} =$$

$$\lim_{z \to z_0} \frac{g'(w_0)(f'(z_0)(z - z_0) + o(z - z_0)) + o(f'(z_0)(z - z_0))}{z - z_0} = g'(w_0)f'(z_0)$$

Examples

- The identity $z \mapsto$ is holomorphic, and $f'(z) = 1$ for all $z \in \mathbb{C}$.

- Every power $z \mapsto z^n$, $n \in \mathbb{Z}$ is holomorphic (on $\mathbb{C}$ if $n \geq 0$, and on $\mathbb{C}\backslash 0$ if $n < 0$.). Moreover
$$\frac{d(z^n)}{dz} = nz^{n-1}$$
For instance, $z^2 = (x^2 - y^2) + i2xy$ is holomorphic on $\mathbb{C}$ and its derivative is equal to the holomorphic function $2z = 2x + i2y$. Similarly,
$$z^3 = (x + iy)[(x^2 - y^2) + i2xy] = (x^3 - 3xy^2) + i(-y^3 + 3x^2y)$$
$$(z^3)' = 3z^2 = 3(x^2 - y^2) + i6xy(= (z^3)_x)$$
$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}.$$

- We have already claimed that every holomorphic function is in fact of class C^∞. In particular its derivative is a holomorphic function. For instance, on the subset $I\!R^2\backslash\{(x,0),\ x \leq 0\}$, the (holomorphic map) $\frac{1}{z}$ is the derivative of the holomorphic function
$$\log z \doteq \log(x^2 + y^2)^{\frac{1}{2}} + i\theta(x, y),$$
where the *argument* function $\theta(x, y)$ is defined by
$$\theta(x,y) = \begin{cases} \arctan\frac{y}{x} & \text{for all } \ x > 0 \\ -\arctan\frac{x}{y} + \pi/2 & \text{for all } \ y > 0 \\ -\arctan\frac{x}{y} - \pi/2 & \text{for all } \ y < 0. \end{cases}$$
Notice that $\theta(x, y)$ is *well* defined and continuous on the domain $A \doteq R^2\backslash\{(x,0),\ x \leq 0\}$, and $\theta(A) =]-\pi, \pi[$.
The map $\log : A \to I\!R \times]-\pi, \pi[$ is called the "principal branch" of the (multi-valued) log-function. Actually it is the unique inverse of the restriction to $I\!R + i]-\pi, \pi[$ of the *complex exponential map*
$$e^z \doteq e^x(\cos x + i \sin x) = e^x e^{iy}$$
.

- We know by (complex) *power series* theory that every convergent power series can be differentiated term to term, so its sum has a complex derivative. This is the case, e.g., of e^z, $\sin z$, $\cos z$, $\sinh z$ $\cosh z$ (defined as power series).

- As we have seen composition of holomorphic functions is holomorphic. For instance $e^{-z^2} = e^{(-x^2+y^2)}[\cos(2xy) + i\sin(2xy)]$ is holomorphic.

- While the exponential map
$$\rho + i\theta \to e^\rho(\cos\theta + i\sin\theta(= e^{\rho+i\theta})$$

is holomorphic, the *polar coordinates* map

$$\rho + i\theta \to \rho(\cos\theta + i\sin\theta(= \rho e^{i\theta})$$

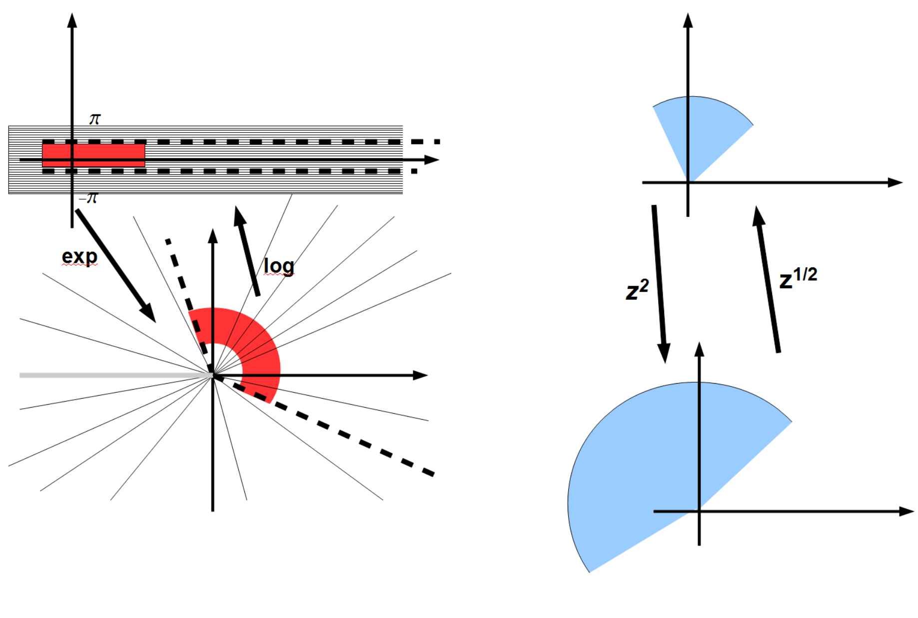
IS NOT holomorphic, for Riemann-Cauchy conditions are not verified. Notice however that the level lines of the real and the imaginary parts *are orthogonal at each point.* Yet, their gradients have different norm. As a direct consequence of this fact (see next subsection), the Laplace equation *is not invariant* by passing from Euclidean coordinates to polar coordinates. Actually, it is a matter of straightforward computation to verify that a C^2 map $(x, y) \to u(x, y)$ is harmonic, namely

$$\Delta u = u_{xx} + u_{yy} = 0$$

, if and only if the function $U(\rho, \theta) = u(\rho\cos\theta, \rho\sin\theta)$ solves the equation

$$U_{rr} + \frac{U_{r\theta}}{r} + \frac{U_{\theta\theta}}{r^2} = 0. \tag{1.11}$$

Equation (1.11) is called the *Laplace equation in polar coordinates.*



1.6 Holomorphic functions and harmonic maps

We have already claimed (without proving it) that holomorphic functions are in fact of class C^∞. In particular they have second derivatives. It turns out that both the real and the imaginary part are harmonic, that is they solve Laplace equation:

Proposition 1.5 *If $f : \Omega \in \mathbb{C}$ is holomorphic then (u and v are indefinitely differentiable) and*

$$\Delta u = 0 \qquad \Delta v = 0 \qquad on \quad \Omega$$

Proof. By the Cauchy-Riemann conditions and Schwarz Theorem one has:

$$u_{xx} = v_{yx} = v_{yx} = -u_{yy}, \qquad v_{xx} = -u_{yx} = -u_{xy} = -v_{yy}$$

A map like v is called a conjugate harmonic of u (see below).

For instance,

- $z^2 = (x^2 - y^2) + i2xy$ is holomorphic, so $u(x,y) = x^2 + y^2$ and $v(x,y) = 2xy$ are both harmonic (and $2xy$ is a conjugate harmonic of $x^2 - y^2$).

Let us try to "invert" Proposition 1.5. Clearly, *it is not true* that if $u = u(x,y)$ and $v = v(x,y)$ are harmonic functions then $f = u + iv$ is holomorphic. For instance, if $u(x,y) = v(x,y) = x$ then both u and v are harmonic but $f \doteq u + iv = x + ix$ is not holomorphic: $u_x = 1 \neq 0 = v_y = 0$. However, instead of $v = x$ we considered the (harmonic) map $v = y$, things would work: indeed in this case $u_x = 1 = v_y$, and $u_y = 0 = -v_x$. In fact, $f(x+iy) = x + iy$ is the identity map.

So there is hope that starting from a harmonic map u one can find a (necessarily harmonic) v such that $f = u + iv$ is holomorphic. Such a map u is called a *harmonic conjugate* to v. Clearly, if Ω is connected, v is a conjugate harmonic to u if and only if for every $c \in I\!R$ $v+c$ is a conjugate harmonic to u. Indeed, if $f = u + iv$ is holomorphic the first derivatives of v are determined by the Cauchy-Riemann conditions: $(v_x, v_y) = (-u_y, u_x)$. In other words, v is a primitive of the form $\omega \doteq -u_y dx + v_x dy$ —in the language of vector fields: v is a potential of of the vector field $(-u_y, v_x)$. On the other hand primitives of a form (=potentials of a vector field) on a connected domain are determined up to a real constant.

Actually, a conjugate harmonic v does exist as soon as the set Ω is *simply connected*:

Theorem 1.1 *Let $\Omega \subset \mathbb{C}$ be a simply connected open subset, and let $u : \Omega \to I\!R$ be a harmonic map. Then there exists a map v which is conjugate harmonic to u.*

Proof. Since u is harmonic, the C^1 vector field $F \doteq (-u_y, u_x)$ is irrotational:

$$F_{1y} = -u_{yy} = u_{xx} = F_{2x}.$$

Therefore F is conservative, for Ω is simply connected. Hence there exists a C^2 map v such that $\nabla v = (v_x, v_y) = (F_1, F_2) = (-u_y, u_x)$, so $f \doteq u + iv$ verifies the Cauchy-Riemann conditions, i.e., it is holomorphic.

2 Incompressible potential fluids and holomorphic functions

If $v(x) = (v_1, v_2, v_3)(x_1, x_2, x_3)$ is the velocity of a homogeneous (=with constant density) fluid in a region $D \subset I\!R^3$ we can express the property of being **incompressible** by assuming that

$$\operatorname{div} v = \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = 0 \tag{2.12}$$

(If the density $\rho = \rho(t, x)$ depends on time and space, then (2.12) must be replaced by $\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0$). This is indeed equivalent, via Gauss' Divergence Theorem, to imposing that the flow of the velocity through the boundary of any open subset $A \subset D$ be equal to zero (respectively, to the rate of change in time of the mass of A).

If the *circulation* (i.e. the line-integral) of v along γ

$$\int_\gamma v \doteq \int_a^b \langle v \cdot \gamma(t) \rangle dt \tag{2.13}$$

is equal to zero for any piece-wise closed loop $\gamma : [a, b] \to D$, the fluid is called **irrotational**. The notion of irrotationality is equivalent to the fluid to be *potential*.

Definition 2.1 *A stationary fluid of velocity $v \in C(A)$ is called* **potential** *if there exists a (necessarily C^1) potential function $\Phi : A \to I\!R$, which means*

$$\nabla \Phi(x) = v(x) \quad \textit{for all} \ \ x \in A \tag{2.14}$$

eeq (In the frame-work of 1-forms one says that the form $\omega = v_1 dx_1 + v_2 dx_2 + v_3 dx_3$ is exact and that Φ is a primitive)

Theorem 2.1 *Let A be open and let $v \in C(A)$ be a velocity fields. Then*

- *v is irrotational if and only if it is potential;*
- *if v is potential (i.e., irrotational) and of class C^1, then the vector $\operatorname{curl} v$ vanishes everywhere, i.e.*[1]

$$\operatorname{curl} v(x) = 0 \qquad \textit{for all} \ \ x \in A. \tag{2.15}$$

In coordinates, this means

$$v_{2z} - v_{3y} = 0 \quad v_{3x} - v_{1z} = 0 \quad v_{1x} - v_{2x} = 0 \quad {}^{2} \tag{2.16}$$

[1] The "curl" of v is often denoted as $\nabla \times v$, and sometimes, as *rot v*

[2] In the 1-forms's framework this reads "the form $\omega \doteq \sum_{i=1}^{i} v_i dx_3$ is *closed*", or $d\omega = 0$ where d denotes *external differentiation*, see e.g. [1] or [2]

- If A is simply connected [3] *then condition (2.15) is also sufficient for v to be irrotational. Namely, iv v is C^1 and A is simply connected*

$$\operatorname{curl} v \equiv 0 \Leftrightarrow v \text{ is irrotational} \tag{2.17}$$

It is trivial to verify that:

Proposition 2.1 *If a fluid has velocity v of class C^1, irrotational (=potential) and incompressible, then any potential Φ of v is a harmonic function.*

Proof. *It is a trivial computation:*

$$\Delta\Phi = \operatorname{div}(\nabla u) = \operatorname{div}(v) = 0.$$

2.1 Streamlines and the Complex potential

We have seen that a two dimensional irrotational (potential) incompressible fluid has a potential Φ [4] which is an harmonic function on A. Hence (see ...) there exists a conjugate harmonic $\Psi : A \to I\!R$ to Ψ, i.e. the complex valued map

$$F \doteq \Phi + i\Psi \tag{2.18}$$

is a holomorphic map (as soos as A is regarded as a subset of $\mathbb{C}$).

The holomorphic map F is called a **complex potential** of the fluid. How F is actually related to the fluid kinematics?

Proposition 2.2 *Let us consider a irrotational, incompressible, two dimensional fluid with C^1 velocity field v. Let $F = \Phi + i\Psi$ be a complex potential for this fluid, constructed as above. Then*

(1) $v = \nabla\Psi$;

(2) The level set of the (harmonic) function Ψ are the streamlines of the fluid, run with velocity $|F'| = |\nabla\Phi| = |\nabla\Psi|$ in the direction of $\nabla\Phi$ (which coincides with the direction which "sees $\nabla\Psi$ pointing on the left").

(3) The complex derivative of the conjugate $\bar{F} = \Phi - i\Psi$, coincides with the velocity vector field:

$$\bar{F}' = \Phi_x + i\Phi_y = v_1 + iv_2.$$

Proof. There is nothing to prove for (1). As for (2), assume that $\nabla\Phi(x, y) \neq 0$ (which implies that $\nabla\Psi(x, y) \neq 0$ as well) . Then $v = \nabla\Phi$ is orthogonal to $\nabla\Psi(x, y)$ which in turn is orthogonal to the level sets of Ψ. Hence the latter are tangent to $v = \nabla\Phi$, i.e. they coincide with streamlines. The other statements are nothing the repetitions of what we have already proved for the real and imaginary parts of every holomorphic function. Finally, (3) is just a consequence of Cauchy-Riemann conditions:

$$v_1 + iv_2 = \Phi_x + i\Phi_y = \Phi_x - i\Psi_x = \bar{F}'.$$

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[3] i.e. every curve is homotopic to every point

[4] If A is connected, the potential is defined up to addition of a real number

3 Harmonic invariance via holomorphic maps

Theorem 3.1 *Let $\Omega, A \subset \mathbb{C}$ be open subset, and let $f : \Omega \to \mathbb{C}$ be a holomorphic function such that $A \subset f(\Omega)$. Let $u : A \to I\!R$ be a harmonic map. Then the composed map $U \doteq u \circ f : \Omega \to I\!R$ is harmonic as well.*

Proof. There is a direct way of proving this result, as done in the textbook [**?**]. It simply consists in verifying that U is a harmonic function, i.e. $\Delta U = 0$. equations. (In particular one uses the fact that if $f = \alpha + i\beta$ is holomorphic then $\nabla\alpha$ and $\nabla\beta$ are orthogonal, and $|\nabla\alpha| = |\nabla\beta| = |f'(z_0)|$).

Alternatively, one can argue as follows. Let $z_0 \in \Omega$ and $w_0 \doteq f(z_0)$, so the harmonic map u is defined in a whole disc $D_r \doteq \{w \mid |w - w_0| < r\}$ for an enough small $r > 0$. In particular, D_r is convex, hence simply connected. Hence by Theorem 1.1 there exists a conjugate harmonic $v : D \to I\!R$ to u. This means that the complex valued map $g \doteq u + iv : D \to \mathbb{C}$ is holomorphic. Let $\delta > 0$ be so small that $f(D_\delta(z_0)) \subset D_r$. Therefore the map $G = U + iV : D_\delta \to \mathbb{C}$ defined as

$$G(x + iy) = U(x, y) + iV(x, y) \doteq g \circ f(x + iy) = u \circ f(x + iy) + iv \circ f(x + iy)$$

is holomorphic (recall that compositions of holomorphic functions are holomorphic, see Proposition 1.4). Hence $U = u \circ f$ is harmonic on D_δ, in that it is the real part of a holomorphic map (see Proposition 1.5).

3.1 Application to Dirichlet problems for the Laplace equation

The above result can be useful in the solution of boundary value problem. Indeed assume that $f : \Omega \to A$ is holomorphic and such that boundary $\partial\Omega$ is mapped in to ∂A : $f(\partial\Omega) = \partial A$. Let $u : A \to I\!R$ be a solution of

$$\begin{cases} \Delta u = 0 & \text{on} \quad A \\ \\ u = \phi & \text{in} \quad \partial A \end{cases} \tag{3.19}$$

Then the map $U = u \circ f$ is a solution of the boundary value problem

$$\begin{cases} \Delta U = 0 & \text{on} \quad \Omega \\ \\ U = \phi \circ f & \text{in} \quad \partial\Omega \end{cases} \tag{3.20}$$

It might happen that problem (3.19) is easier then problem (3.20), e.g. because the form of A is particularly simple (say, A is a rectangle), and that one is able to find a solution u of (3.19). Then $U = u \circ f$ is a solution of problem (3.20).

Example. Let $\Omega \subset \mathbb{C}$ be the intersection the external of the unit circle with the first quadrant, namely $\Omega \doteq \{z = x + iy \ |z| \geq 1 \ x > 0, \ y > 0\}$. Moreover, set

$$\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$$

$$\Gamma_1 \doteq e^{i[0,\pi/2]}, \quad \Gamma_2 \doteq [1, +\infty[, \quad \Gamma_3 \doteq i[1, +\infty[,$$

and consider the Dirichlet problem

$$\begin{cases} \Delta U = 0 & \text{on} \quad \Omega \\ \\ U = 0 & \text{in} \quad \Gamma_2 \\ U = 17 & \text{in} \quad \Gamma_3 \end{cases} \tag{3.21}$$

Let $f(z) = \log(z)$ denote the principle value of the logarithm. We know that it is holomorphicthat it maps Ω onto the strip $A =]0, +\infty[+ i]0, \pi/2[$. Furthermore, one has $f(\partial\Omega) = \partial A$. More closely,

$$f(\Gamma_1) = G_1 \doteq 1 + i[0, \pi/2], \quad f(\Gamma_2) = G_2 \doteq [1, +\infty[, \quad f(\Gamma_2) = G_3 \doteq [1, +\infty[+ i\pi/2.$$

In view of Theorem 3.1, problem (3.21) is transformed by the holomorphic map f into the new boundary value problem

$$\begin{cases} \Delta u = 0 & \text{on} \quad A \\ \\ u = 0 & \text{in} \quad G_2 \\ \\ u = 17 & \text{in} \quad G_3. \end{cases} \tag{3.22}$$

Now, it is trivial to verify that the linear map $u(\alpha, \beta) = \dfrac{34\beta}{\pi}$ is a solution of (3.22). It follows that

$$U(x, y) = u \circ f(x + iy) = u \circ (\log \sqrt{x^2 + x^2} + \arctan\left(\frac{y}{x}\right)) = \frac{34 \arctan\left(\frac{y}{x}\right)}{\pi}$$

is a solution of the boundary value problem (3.21).

3.2 Application to Neumann problems for the Laplace equation

A Neumann Boundary Value problem consists in specifying the value of the *normal derivative* of the solution on the boundary of the domain. If $n(x)$ is the *outer normal* at a point $x \in \partial\Omega$, the the normal derivative of a function u is nothing but the directional derivative $\partial_n u(x)$ of u in the direction of n. If u is differentiable:

$$\partial_n u(x) = \nabla u(x) \times n.$$

There is an important feature of normal derivatives:

Lemma 3.1 *Let $f : \Omega \to A$ be holomorphic, and suppose that Ω has a regular boundary $\partial\Omega$. Let $z_0 = x_0 + iy_0 \in \partial\Omega$ such that $f'(z_0) \neq 0$. Then:*

- $w_0 \doteq f(z_0) \in \partial A$
- *The normal $N(w_0)$ to A at w_0 is given by*

$$N(w_0) = \frac{f'(x_0)}{|f'(x_0)|} n(z_0)$$

This result can be useful in the solution of boundary value problem of Neumann type.

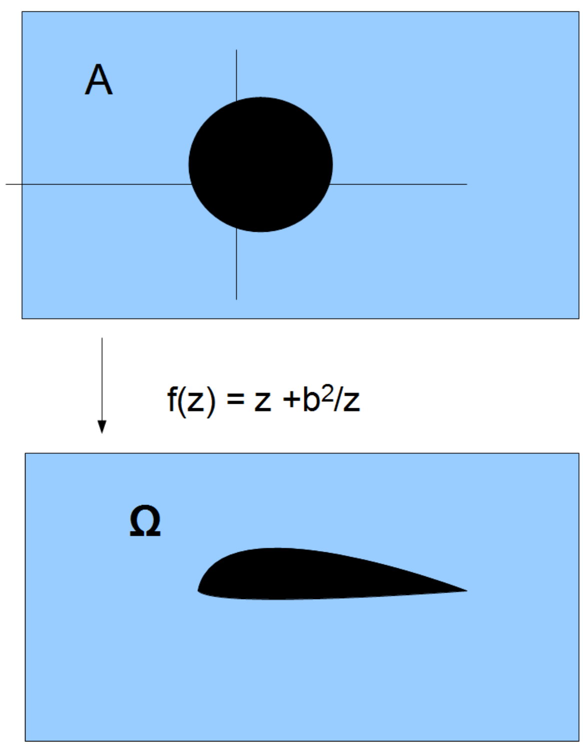
Theorem 3.2 *Assume that $f : \Omega \to A$ is a holomorphic conformal* [5] *and such that boundary $\partial\Omega$ is mapped into ∂A : $f(\partial\Omega) = \partial A$. Let $u : A \to \mathbb{R}$ be a solution of*

$$\begin{cases} \Delta u = 0 & on \quad A \\ \partial_n u = \eta & in \quad \partial A \end{cases} \tag{3.23}$$

Then the map $U = u \circ f$ is a solution of the boundary value problem

$$\begin{cases} \Delta U = 0 & on \quad \Omega \\ \partial_N U = \eta \cdot f' & in \quad \partial\Omega \end{cases} \tag{3.24}$$

In particular, it is interesting, e.g. in aerodynamics, the case when $\eta = 0$, meaning that in the orthogonal direction to the boundary the derivative of u must be zero. In fluid dynamics this means that the velocity $v = \nabla u$ has to be *tangential* to the boundary. In view of Theorem 3.2, one can study the problem on the domain A (supposed "simpler") and then transfer the results on the "more complicated" domain Ω. This is what done, for instance in the design of airfoils -in absence of viscosity-, where the airfoil shape $\mathbb{C}\backslash\Omega$ is obtained as the image of a circle $\mathbb{C}\backslash A$. See...



[5] This assumptions ensures that $f' \neq 0$ everywhere.

References

[1] G. De Marco, *ANALISI DUE*, Zanichelli.

[2] V. Arnold, Mathemathical methods of classical mechanics