Mathematical Methods, 05/09/13

Name ....

Note: Part B of the exam contains only exercises 1b, 2(ii), 3, 4.

Exercise 1 [8 points]

- **1a** Define the order of a zero for an holomorphic function. Discuss the structure of the set of zeroes of holomorphic functions.
- **1b** State and prove the duality Lemma in the theory of the Fourier transform.

Exercise 2 [10 points]

Consider the function $f : \mathbb{R} \to \mathbb{C}$, $f(t) = \frac{e^t}{1+e^{2t}}$.

- (i) Prove that $f \in L^1(\mathbb{R})$;
- (ii) As a consequence of the theory, determine the function spaces containing $\Phi(f)$ and the speed of decay at infinity of the Fourier transform.
- (iii) Compute the Fourier transform of f. (In the complex plane integrate on rectangles of vertices $\{\pm r, \pm r + i\pi\}$, use the Theorem of residues and let $r \to +\infty$.) Explain carefully the details.

Exercise 3 [7 points]

Suppose that $u \in C^2(\mathbb{R}) \cap L^2(\mathbb{R})$ is such that $u'' \in L^2(\mathbb{R})$ and u is a solution of the differential equation

$$(u'' - u) * f\ (t) = g(t).$$

Here $f(t) = e^{-|t-3|}$ and $g(t) = \mathrm{sinc}(t)$.

- (i) Find the Fourier transform $\hat{u}$;
- (ii) prove that such u exists and compute it.

(Carefully explain the steps in the arguments with the theory.)

Exercise 4 [7 points]

Consider the 2π-periodic and even function such that $f(t) = te^{-t}$, as $t \in [0, \pi]$.

(i) Before computing the Fourier coefficients, discuss the convergence of the Fourier series of f and the speed of decay of its coefficients in real form, as determined by the theory.

(ii) Compute the Fourier coefficients of f. (Compute them in complex form.)

(iii) Is the decay of the coefficients as expected? Write the Fourier series in real form.