Exercise 1 [8 points]

State and prove Liouville Theorem.

Exercise 2 [12 points]

(i) Classify the singularities of the function of complex variable

\[ f(z) = \frac{e^{2iz}(e^{iz} - 1)}{z(z^2 + 4)^2}. \]

(ii) Find the singular part of \( f \) in a ball \( B(c,r) \) contained in the domain, where \( c \) is a singularity and \( \text{Im } c > 0 \).

(iii) Prove that \( g(x) = \frac{\sin 3x - \sin 2x}{x(x^2 + 4)^2} \in L^1(\mathbb{R}) \).

(iv) Compute the integral

\[ \int_{-\infty}^{+\infty} \frac{\sin 3x - \sin 2x}{x(x^2 + 4)^2} \, dx. \]

Exercise 3 [8 points]

(i) Prove that the function

\[ u(x,y) = \sin (\pi x)e^{-\pi y}, \quad (x,y) \in \mathbb{R}^2 \]

is harmonic.

(ii) Find an harmonic conjugate of \( u \).

(iii) Which is the underlying holomorphic function?

Exercise 4 [4 points]

Classify the singularity at 0 of the function of complex variable

\[ f(z) = \frac{e^{1/z}}{\sin z}. \]