Mathematical Methods, 15/07/13

Name ....

Note: Part B of the exam contains only exercises 1b, 4, 5.

Exercise 1 [8 points]

- **1a** Define a pole for an holomorphic function. State and prove equivalent properties to characterize poles.
- **1b** Given functions $f, g \in L^1(\mathbb{R})$ define the product of convolution $f * g$. Which is the function space of $f * g$? Prove the statement.

Exercise 2 [8 points]

Consider the function $f : \mathbb{R} \to \mathbb{C}$, $f(t) = \frac{\sin(t)}{(t-i)^2}$.

- (i) Prove that $f \in L^1(\mathbb{R})$;
- (ii) As a consequence of the theory, determine the function spaces containing $\Phi(f)$ and the speed of decay at infinity of the Fourier transform.
- (iii) Compute the Fourier transform of f. (Use the Theorem of residues).

Exercise 3 [4 points]

Determine an holomorphic function $f : \mathbb{C}\backslash\{3, -i, 2+i\} \to \mathbb{C}$ such that: f has at 3 a pole of order 2, an essential singularity at $-i$, a removable singularity at $2+i$ and satisfies $\int_{\gamma_4} f(z)\, dz = 1$, where $\gamma_4(t) = 4e^{it}$, $t \in [0, 2\pi]$.

Exercise 4 [6 points]

Suppose that $u \in C^3(\mathbb{R}) \cap L^1(\mathbb{R})$ is a solution of the differential equation

$$u'''(t) - u'(t) - tu(t) = 0.$$

- (i) Find the Fourier transform $\hat{u}$; ($\hat{u}$ solves a differential equation)
- (ii) prove that such u exists (many of them) and $u \in C^\infty \cap C_o$;
- (iii) can we impose to u the condition $\int_{-\infty}^{+\infty} u(t)\, dt = 1$?
- (iv) Write a formula for such u.

Exercise 5 [6 points]

Consider the function

$$f(t) = \begin{cases} 1, & t \in (-\pi, -\pi/2), \\ 0, & t \in (-\pi/2, \pi/2); \\ 1, & t \in (\pi/2, \pi). \end{cases}$$

(i) Determine the Fourier coefficients of f.
(ii) Find the sum of the series $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1}$;
(iii) Find the sum of the series $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$.