Mathematical Methods, 18/06/13

Name ....

Note: Part B of the exam contains only exercises 1b, 4, 5.

Exercise 1 [8 points]

- **1a** Prove by the definition (not with the Theorem of residues) that for any loop $\alpha : [a,b] \to \mathbb{C}\backslash\{0\}$ then
$$\frac{1}{2\pi i}\int_\alpha \frac{dz}{z} \in \mathbb{Z}.$$
- **1b** State and prove the theorem on normal convergence of Fourier series. (Give a complete proof.) Show that if $(c_k)_{k\in\mathbb{Z}} \subset l^2(\mathbb{Z})$ then the sum of the trigonometric series
$$\sum_{k\in\mathbb{Z},\, k\neq 0} \frac{c_k}{k} e^{2\pi k i t}$$
is a continuous function.

Exercise 2 [5 points]

(i) For $m \in \mathbb{Z}$, classify the singularity at 0 of the function of complex variable
$$f(z) = \frac{\sinh\left(\frac{1}{z}\right)}{z^m}.$$
(ii) Find the Laurent series of f at $z = 0$.
(iii) Compute for $m \in \mathbb{Z}$ the value of the integral
$$\int_\gamma f(z)\, dz,$$
where $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$.

Exercise 3 [7 points]

(iv) Using the Theorem of residues, compute the integral
$$\int_0^{2\pi} \frac{dt}{1+\sin^2(t)}.$$

Exercise 4 [5 points]

Find solutions $u \in C^2(\mathbb{R})$ and $\pi-$periodic of the differential equation
$$u''(t) + u(t) = \sin^2(t).$$

Exercise 5 [7 points]

Consider the function $f(t) = (t^3 - t)\chi_{(-1,1)}(t)$.
(i) As a consequence of the theory, determine the function spaces containing $\Phi(f)$ and the speed of decay at infinity of the Fourier transform.
(ii) Compute the Fourier transform of $\chi_{(-1,1)}(t)$.
(iii) Based on (ii), compute the Fourier transform of $(3t^2 - 1)\chi_{(-1,1)}(t)$.
(iv) Based on (iii), compute the Fourier transform of f. Does it correspond to what expected?