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Asymmetric norms and the Banach space theory

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An asymmetric norm is a nonnegative and subadditive positively homogeneous real function q defined on a linear space X satisfying that $q(x) = q(-x) = 0$ implies $x = 0$. An asymmetric normed linear space is a pair (X, q) such that X is a linear space and q is an asymmetric norm on X. Such a function q induces a quasi-metric d_q by mean of the formula:

$$d_q(x, y) = q(y - x), \quad x, y \in X.$$

Thus, the study of the structure of asymmetric normed linear spaces can be done from the topological point of view as a part of the so called asymmetric topology. Such kind of spaces are interesting because of their applications to the Computational Complexity Theory.

In our lecture we present other point of view for the study of the asymmetric normed linear spaces. Our framework is the Banach space theory. Although an asymmetric norm does not define in general a normed space, it is possible to apply certain techniques of the Banach space theory to several problems. For instance, many examples of asymmetric normed linear spaces are constructed using lattice norms on sequence spaces or function spaces. Thus, the theory of the Banach spaces provides good tools to study the following problems:

(1) *When an asymmetric linear space is Hausdorff.* Following classical techniques of the Banach space theory we can give a characterization of those asymmetric normed linear spaces that satisfies the T_2 separation axiom. Moreover, a reasonable generalization of the concept of complemented subspaces of Banach spaces leads to a canonical decomposition of an asymmetric normed linear space as a direct sum of a Hausdorff space and what we call a purely non Hausdorf asymmetric normed linear space.

(2) *The definition of the dual of an asymmetric normed linear space and the weak topologies.* A great part of the properties of the normed and Banach spaces are related to the weak topologies that can be defined on them in a natural way. The set of continuous linear functions on normed spaces gives a Banach space with its natural norm. Although in the case of asymmetric normed linear spaces this set does not define in general a linear space, it defines a semilinear space, and we can define in a natural way the dual of an asymmetric normed linear space. Thus, several (weak) topologies can be induced by the relation between a space and its dual. For instance, we can prove an asymmetric version of the Alaoglu Theorem on the compactness of the unit ball of the dual space with respect to the corresponding weak* topology. This is one of the main tools on Banach space theory. It is also possible to give, for a restricted class of asymmetric normed linear spaces, a version of the Hahn Banach Theorem.

(3) *Extensions of asymmetric norms defined on semilinear spaces to their linear covers.* Some of the most interesting applications of the theory of asymmetric normed linear spaces actually depends on the properties of semilinear spaces that are subsets of these spaces (for instance, the dual spaces). Therefore, the natural question that arises what are the conditions that are required to assure that there exists an extension of an asymmetric norm from a semilinear subspace to its linear cover.