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Commutative algebra for the rings of continuous functions

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One of the most usual functors on the category of topological spaces is the set $C(X)$ of all real-valued continuous functions on a topological space X. We can consider on $C(X)$ different algebraic structures (under the possible pointwise operations), vector space, lattice, ring, ..., and for each structure we have the classic problem of expressing topological properties of X and of a continuous map $X \xrightarrow{f} Y$ in terms of algebraic properties of $C(X)$ and of the morphism $C(Y) \xrightarrow{\circ f} C(X)$. The results obtained, so far, do not indicate that there is a more "natural" structure than the others, but if we want to make a development of General Topology parallel to the experimented by Algebraic Geometry (using the concepts and methods of Commutative Algebra), then it seems clear that $C(X)$ must be regarded as a $\mathbb{R}$-algebra.

Now, it is known that complete regularity of X is a necessary condition to recover X from the $\mathbb{R}$-algebra $C(X)$, and that this condition is not sufficient. In this point it is important to observe that $C(X)$ endowed with the topology of compact convergence is a locally m-convex algebra that allows us to recover the space X when it is completely regular. Moreover, the topology of compact convergence is the unique locally m-convex topology on the algebras of continuous functions so that: (i) when X is compact it coincides with the supremum topology, (ii) for completely regular spaces X and Y, there is a one-to-one correspondence between continuous maps $X \to Y$ and continuous morphisms of $\mathbb{R}$-algebras $C(Y) \to C(X)$.

What we are saying in this abstract justifies that we assume that all topological spaces X are completely regular (and Hausdorff), and that the $\mathbb{R}$-algebra $C(X)$ is endowed with the topology of compact convergence. It is possible to characterize when a locally m-convex algebra is $C(X)$ for some space X, and in this way it is obtained an equivalence between the category of topological spaces and the category of a certain class of locally m-convex algebras. The above mentioned development of General Topology will consist of using this equivalence of categories in order to:

- translate topological properties and methods into the algebraic language; for example, the topological operation of restricting to an open (closed) subset corresponds with the algebraic operation of constructing a "ring of fractions" (quotient ring);
- apply algebraic methods to obtain new notions and results; for instance, in Algebraic Geometry, the dimension of an affine algebraic variety V is defined to be the Krull dimension of the ring A of all algebraic functions on V, i.e., $\dim V$ is the supremum of the lengths of all chains of prime ideals in A; this definition, as it stands, is useless for topological spaces ($\dim C(X) = \infty$ with excessive frequency), but it may be translated into topology in the following way: we will define the dimension of X to be the minimum of the Krull dimensions of all dense subalgebras of $C(X)$; this "algebraic" notion of dimension has remarkable properties.