Iterated Majority Voting

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Workshop on Iterative Voting and Voting Games
Departement of Mathematics, Padova
Introduction

**Goal:** Study a generic model of sequential decision making

- Set of $n$ agents $N$.
- Set of $m$ alternatives (or outcomes, or states of the world) $X$.
- There is a current alternative $x(t) \in X$.
- An agent proposes a different alternative $x^* \in X$.
- The agents vote between $x(t)$ and $x^*$.
  - $x^*$ wins: update of the current state $x(t+1) \leftarrow x^*$.
  - $x(t)$ wins: the current state remains the same: status quo $x(t+1) \leftarrow x(t)$.

- Can this process lead to a “good” outcome?
  - Communication may be reduced (no need to submit the entire preferences).
  - Decision may be easier to make?
Issues

- What voting rule?
  - majority?
  - unanimity?
- Are some properties guaranteed?
  - Pareto Optimality?
  - Fairness?
  - Termination?
  - Cycles?
Related topic: tournaments
voting rules based on the majority graph
### Solution Concepts

<table>
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<th>Method</th>
<th>Description</th>
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<td>Copeland solution (C)</td>
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<td>Markov solution (MA)</td>
<td>Methods for ranking</td>
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<td>Slater solution (SL)</td>
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<td>Tournament equilibrium set (TEQ)</td>
<td>Based on Contestation</td>
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Markov solution

- Random walk in the majority graph.
- Set of winners is the set of outcomes that have a positive probability to be the current outcome in the limit.
- The Markov winners do not depend on the initial outcome (some level of fairness).
Another solution: using elimination trees

- ex: knockout tournaments (tennis tournament, soccer cups)
- Form an *agenda*, i.e. set up the order at which each issue will face another issue.
- Given the structure of a tournament (i.e. the complete majority graph), the agent that forms the agenda can manipulate the winner.

⇒ Justify our choice of using agent for proposing an alternative
⇒ The proposing agent is randomly selected: the agenda is probabilistic
Related work in Political science

Situations where a policy remains in effect until replaced by a new legislation.

- proposal is
  - made by an agent (endogenous, natural as it is part of the problem, but makes a more complex process to analyse) [Baron 96, Kalandrakis 06]
  - provided by the environment (exogenous, e.g. policy is drawn from probability density, easier to interpret as there is no decision on which proposal to make) [Penn 08]

- every voters receives a utility for winning policy
  - agents are maximizing a discounted sum (utility they have now with the current policy, plus what they will have in the future)

- study equilibrium strategies
- ex: divide-a-dollar game
Example showing existence of cycles
(the alternatives are allocation of goods)

<table>
<thead>
<tr>
<th>agent</th>
<th>$A$</th>
<th>$A'$</th>
<th>$A''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\emptyset$</td>
<td>${r_1}$</td>
<td>${r_1, r_4}$</td>
</tr>
<tr>
<td>2</td>
<td>$\emptyset$</td>
<td>${r_2}$</td>
<td>${r_2, r_5}$</td>
</tr>
<tr>
<td>3</td>
<td>$\emptyset$</td>
<td>${r_3}$</td>
<td>${r_3, r_6}$</td>
</tr>
<tr>
<td>4</td>
<td>${r_1}$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>5</td>
<td>${r_2, r_3}$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>6</td>
<td>${r_4}$</td>
<td>${r_4}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>7</td>
<td>${r_5, r_6}$</td>
<td>${r_5, r_6}$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>
Escaping cycles

- Using restriction on the valuation function (Sen’s triplewise value function)
- Using restriction on the protocol:
  - Do not allow an outcome to be proposed twice
    (May model the process of making a law, and adding amendment)
    may require large memory space ✗
  - Using different voting rule
    but this may not always guarantee the absence of cycles ✗
  - Adding a bound on the length of the decision sequence ✔
Assumptions

- Agents may be indifferent between two outcomes, ties are possible.
- We allow strategic choice for proposing an outcome.
- We allow strategic voting.
- Each agent $i$ has a utility function $u_i : \mathcal{X} \rightarrow \mathbb{R}$.
- Utility matrix $U_0$ of size $m \times n$ with $U_0(x, i) = u_i(x)$.
- Utility of two agents may not be comparable.
- Utility functions are common knowledge.
Game and Protocol for a round $t$

**Definition** A *game* is $\langle N, \mathcal{X}, U_0, q, T, x_0 \rangle$ where

- $N$ is the set of agents
- $\mathcal{X}$ is the set of alternatives
- $U_0$ is the matrix of utility for each agent and each alternative
- $q \in [0,1]$ is the quota of the voting rule
- $T$ is the deadline, i.e., the number of rounds played
- $x_0 \in \mathcal{X}$ is the initial alternative

- The current alternative is $x(t)$.
- An agent is randomly selected and proposes an alternative $x^*$ (including the status quo).
- Agents vote between $x(t)$ and $x^*$.
- The winner of the election is the current alternative for the next round.
Backward Induction

- $W_t(x, y)$: probability that $y$ becomes the current alternative at time $t + 1$ when $x$ was the current alternative at time $t$
- $W_t$ is the transition matrix at time $t$
- $U_t(x, i)$: expected utility of alternative $x$ for agent $i$ at $t$.
- $U_{t+1} = W_{t+1} W_t W_{t-1} \ldots W_1 \cdot U_0$

**How to vote?** $i$ votes for current alternative when $U_t(x^*, i) < U_t(x(t), i)$

**What to propose?**
1. compute the set $X^w$ of winning alternatives against $x(t)$
2. form the set of proposals $P_i = \arg\max_{x \in X^w} U_t(x, i)$
3. if the expected utility of a proposal in $P_i$ is greater than the expected utility of the current alternative, pick with equi-probability a proposal in $P_i$
   otherwise, propose the status quo.
Example

\[ U_0 = \begin{pmatrix} 4 & 1 & 2 \\ 2 & 4 & 1 \\ 1 & 3 & 4 \end{pmatrix} \]

\[ W_1 = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \\ 0 & 2/3 & 1/3 \end{pmatrix} \]

\[ U_1 = \frac{1}{3} \begin{pmatrix} 6 & 7 & 10 \\ 10 & 6 & 5 \\ 5 & 11 & 6 \end{pmatrix} \]

... \[ U_\infty = \begin{pmatrix} 2.0795 & 2.6549 & 2.7560 \\ 2.0795 & 2.6549 & 2.7560 \\ 2.0795 & 2.6549 & 2.7560 \end{pmatrix} \]
Definition: a game is said to be **intra-state convergent** when $\forall i \in \mathbb{N}, \forall x \in X \lim_{t \to \infty} [U_t(x, i) - U_{t+1}(x, i)] = 0$

→ expected value converges

Definition: a game is said to be **inter-state convergent** when $\forall i \in \mathbb{N}, \forall (x, y) \in X^2 \lim_{t \to \infty} [U_t(x, i) - U_{t+1}(y, i)] = 0$

→ all expected values converges to the same value

inter-state:

\[
\begin{cases}
  \text{fair with respect to the initial outcome.} \\
  \text{not guaranteed (indifference between outcomes)}
\end{cases}
\]

Definition: a game is said to be **fundamentally convergent** when the limit of the product of the transition matrix $\lim_{t \to \infty} W_{\tau=t}^1 W_{\tau}$ is a matrix with identical rows.

Proposition:

**fundamentally convergence $\Rightarrow$ inter-state $\land$ intra-state convergence**
Sufficient conditions for convergence

- $q \approx 0$ convergence is guaranteed, but prediction inaccurate
- $q \approx 1$ the final outcome is Pareto efficient.
  
  When multiple Pareto optimal outcomes exist, the game is \textbf{not} inter-state convergence.

\textbf{Proposition:} A two-outcome game is intra-state convergent.
\textbf{Proposition:} A two-outcome game with $q < 50\%$ is inter-state convergent.
\textbf{NB:} Existence of weak Condorcet winners is not a sufficient condition (it is possible that even if a unique Condorcet winner exist, it is not chosen as final outcome)
Varying utility range

- $u_i(x)$ is drawn from a uniform distribution either
  - continuous in $[0,1]$
  - discrete in $\{0,1,\ldots,u_{max}\}$

- 15 alternatives, 1000 utility matrices, $q = 50\%$
Varying the quota

15 alternatives, 100 agents, 1000 utility matrices

15 alternatives, 100 agents, averaged over 1000 samples

dichotomous utility
utility values in \{0, \ldots, 4\}
continuous in \([0,1]\)

Stéphane Airiau & Ulle Endriss (Dauphine) - Iterated Majority Voting
Conclusion and future work

- Study of a generic iterated negotiation framework
- Convergence results for 2-alternative games
- The likelihood of ties affects convergence properties
- Future work:
  - In case of convergence, can we predict the deadline to have fairness?
  - Variation of the protocols
  - Proof for more any number of alternatives (at least for intra-state convergence).
  - Scenario where convergence is guaranteed.