Iterated Majority Voting

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Workshop on Iterative Voting and Voting Games
Departement of Mathematics, Padova

Introduction

Goal: Study a generic model of sequential decision making

- Set of n agents N.
- Set of m alternatives (or outcomes, or states of the world) $\mathcal{X}$
- There is a current alternative $x(t) \in \mathcal{X}$
- An agent proposes a different alternative $x^\star \in \mathcal{X}$
- The agents vote between $x(t)$ and $x^\star$

  $x^\star$ wins: update of the current state $x(t+1) \leftarrow x^\star$

  $x(t)$ wins: the current state remains the same: status quo: $x(t+1) \leftarrow x(t)$

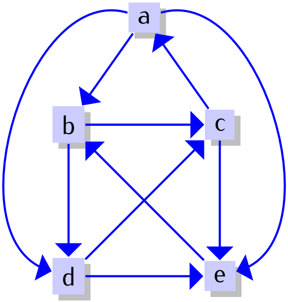
➩ Can this process lead to a "good" outcome?

- communication may be reduced (no need to submit the entire preferences)
- decision may be easier to make?

Issues

- What voting rule?
  - majority?
  - unanimity?
- Are some properties guaranteed ?
  - Pareto Optimality?
  - Fairness?
  - Termination?
  - Cycles?

Related topic: tournaments
voting rules based on the majority graph



Solution Concepts

- Copeland solution (C)
- the Long Path (LP)
- Markov solution (MA)
- Slater solution (SL)

Methods for ranking

- Uncovered set (UC)
- Iterations of the Uncovered set (UC^{∞})
- Dutta's minimal covering set (MC)

Based on the notion of covering

- Bipartisan set (BP)

Game theory based

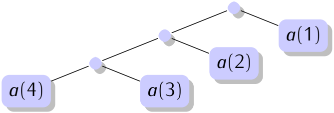
- Bank's solution (B)
- Tournament equilibrium set (TEQ)

Based on Contestation

Markov solution

- Random walk in the majority graph.
- Set of winners is the set of outcomes that have a positive probability to be the current outcome in the limit
- The Markov winners do not depend on the initial outcome (some level of fairness)

Another solution: using elimination trees



- ex: knockout tournaments (tennis tournament, soccer cups)
- Form an ***agenda***, i.e. set up the order at which each issue will face another issue.
- Given the structure of a tournament (i.e. the complete majority graph), the agent that forms the agenda can manipulate the winner.

➾ Justify our choice of using agent for proposing an alternative

➾ The proposing agent is randomly selected: the agenda is probabilistic

Related work in Political science

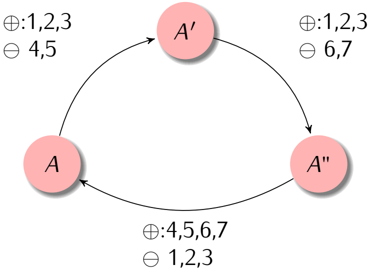
Situations where a policy remains in effect until replaced by a new legislation.

- proposal is
  - made by an agent (endogenous, natural as it is part of the problem, but makes a more complex process to analyse) [Baron 96, Kalandrakis 06]
  - provided by the environment (exogenous, e.g. policy is drawn from probability density, easier to interpret as there is no decision on which proposal to make) [Penn 08]
- every voters receives a utility for winning policy
  agents are maximizing a discounted sum (utility they have now with the current policy, plus what they will have in the future)
- ⇨ study equilibrium strategies
- ex: divide-a-dollar game

Example showing existence of cycles

(the alternatives are allocation of goods)

agent	A	A'	A''
1	$\emptyset$	$\{r_1\}$	$\{r_1, r_4\}$
2	$\emptyset$	$\{r_2\}$	$\{r_2, r_5\}$
3	$\emptyset$	$\{r_3\}$	$\{r_3, r_6\}$
4	$\{r_1\}$	$\emptyset$	$\emptyset$
5	$\{r_2, r_3\}$	$\emptyset$	$\emptyset$
6	$\{r_4\}$	$\{r_4\}$	$\emptyset$
7	$\{r_5, r_6\}$	$\{r_5, r_6\}$	$\emptyset$



Escaping cycles

- Using restriction on the valuation function (Sen's triplewise value function)
- Using restriction on the protocol:
  - Do not allow an outcome to be proposed twice
    (May model the process of making a law, and adding amendment)
    may require large memory space ✘
  - Using different voting rule
    but this may not always guarantee the absence of cycles ✘
  - Adding a bound on the length of the decision sequence ✔

Assumptions

- Agents may be indifferent between two outcomes, ties are possible.
- We allow strategic choice for proposing an outcome.
- We allow strategic voting.
- Each agent i has a utility function $u_i : \mathcal{X} \to \mathbb{R}$.
- Utility matrix U_0 of size $m \times n$ with $U_0(x,i) = u_i(x)$.
- Utility of two agents may not be comparable.
- Utility functions are common knowledge.

Game and Protocol for a round t

Definition A ***game*** is $\langle N, \mathcal{X}, U_0, q, T, x_0 \rangle$ where

N is the set of agents

$\mathcal{X}$ is the set of alternatives

U_0 is the matrix of utility for each agent and each alternative

$q \in [0,1]$ is the quota of the voting rule

T is the deadline, i.e., the number of rounds played

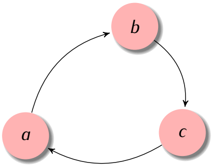
$x_0 \in \mathcal{X}$ is the initial alternative

- The current alternative is $x(t)$.
- An agent is randomly selected and proposes an alternative $x^\star$ (including the status quo).
- Agents vote between $x(t)$ and $x^\star$.
- The winner of the election is the current alternative for the next round.

Backward Induction

- $W_t(x, y)$: probability that y becomes the current alternative at time $t+1$ when x was the current alternative at time t
- ⇨ W_t is the transition matrix at time t
- $U_t(x, i)$: expected utility of alternative x for agent i at t.
- ⇨ $U_{t+1} = W_{t+1} W_t W_{t-1} \dots W_1 \cdot U_0$
- *How to vote?* i votes for current alternative when $U_t(x^\star, i) < U_t(x(t), i)$
- *What to propose?*
  1. compute the set X^w of winning alternatives against $x(t)$
  2. form the set of proposals $P_i = arg\max_{x \in X_w} U_t(x, i)$
  3. ***if*** the expected utility of a proposal in P_i is greater than the expected utility of the current alternative, pick with equi-probability a proposal in P_i
     otherwise, propose the status quo.

Example



	1	2	3	
$U_0 =$	4	1	2	a
	2	4	1	b
	1	3	4	c

	a	b	c	
$W_1 =$	1/3	0	2/3	a
	2/3	1/3	0	b
	0	2/3	1/3	c

$$U_1 = \frac{1}{3}\begin{pmatrix} 6 & 7 & 10 \\ 10 & 6 & 5 \\ 5 & 11 & 6 \end{pmatrix}$$

$\dots$

$$U_\infty = \begin{pmatrix} 2.0795 & 2.6549 & 2.7560 \\ 2.0795 & 2.6549 & 2.7560 \\ 2.0795 & 2.6549 & 2.7560 \end{pmatrix}$$

Convergence

Definition: a game is said to be ***intra-state convergent*** when $\forall i \in N$, $\forall x \in \mathcal{X}$ $\lim_{t\to\infty} [U_t(x,i) - U_{t+1}(x,i)] = 0$

➡ expected value converges

Definition: a game is said to be ***inter-state convergent*** when $\forall i \in N$, $\forall (x,y) \in \mathcal{X}^2$ $\lim_{t\to\infty} [U_t(x,i) - U_{t+1}(y,i)] = 0$

➡ all expected values converges to the same value

inter-state:

- *fair* with respect to the initial outcome.
- *not guaranteed* (indifference between outcomes)

Definition: a game is said to be ***fundamentally convergent*** when the limit of the product of the transition matrix $\lim_{t\to\infty} W^1_{\tau=t} W_\tau$ is a matrix with identical rows.

Proposition:
fundamentally convergence $\Rightarrow$ *inter-state* $\wedge$ *intra-state* convergence

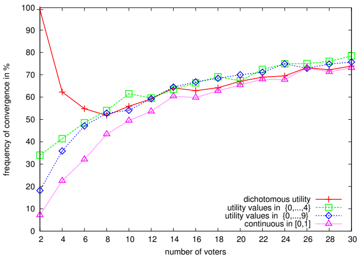
Sufficient conditions for convergence

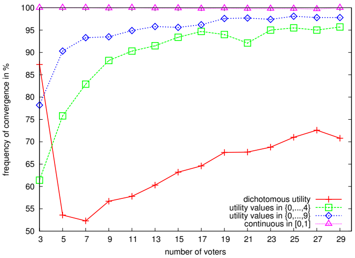
- $q \approx 0$ convergence is guaranteed, but prediction inaccurate
- $q \approx 1$ the final outcome is Pareto efficient.
  When multiple Pareto optimal outcomes exist,
  the game is **not** inter-state convergence.

Proposition: A two-outcome game is intra-state convergent.
Proposition: A two-outcome game with $q < 50\%$ is inter-state convergent.
NB: Existence of weak Condorcet winners is not a sufficient condition (it is possible that even if a unique Condorcet winner exist, it is not chosen as final outcome)

Varying utility range

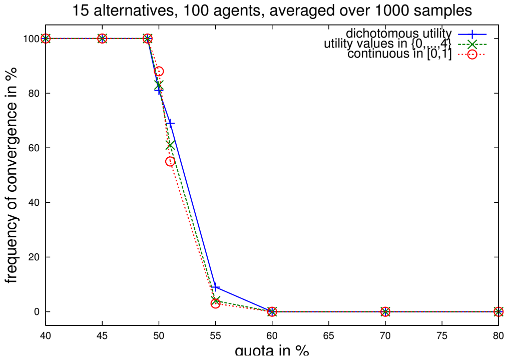
- $u_i(x)$ is drawn from a uniform distribution either
  - continous in $[0,1]$
  - discrete in $\{0,1,\ldots,u_{max}\}$
- 15 alternatives, 1000 utility matrices, $q = 50\%$





Varying the quota

15 alternatives, 100 agents, 1000 utility matrices



Conclusion and future work

- Study of a generic iterated negotiation framework
- Convergence results for 2-alternative games
- The likelihood of ties affects convergence properties
- Future work:
  - In case of convergence, can we predict the deadline to have fairness?
  - Variation of the protocols
  - Proof for more any number of alternatives (at least for intra-state convergence).
  - Scenario where convergence is guaranteed.