On Compatible Multi-issue Group Decisions

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20 June 2012

LOFT 2012 - University of Sevilla, Spain

Multiple Election Paradox (MEP)

	A	B	C
Voter 1	yes	yes	no
Voter 2	yes	yes	no
Voter 3	yes	no	yes
Voter 4	yes	no	yes
Voter 5	no	yes	yes
Voter 6	no	yes	yes
Voter 7	no	yes	yes
Voter 8	no	yes	yes
Voter 9	yes	no	no
Voter 10	yes	no	no
Majority	yes	yes	yes

The outcome is not supported by any of the individuals!

Brams, Kilgour, and Zwicker. The paradox of multiple elections. *Social Choice and Welfare*, 1998.

Examples

Similar situations can happen:

- When voters do not agree on the correlation between issues: e.g., when voting for a committee, different combinations of people may work well together and others not.
- When voters have different preferential dependencies between issues: e.g., when deciding a menu individuals may start from the drink, while others from the dish.
- In general, when voters have full preferences over combinations of issues, and have to submit only their top combination, the collective outcome could be not ranked first by any of the voters or even ranked last by all voters (see Lacy and Niou, 2000).

Consistency...and?

In designing collective decision making mechanisms, consistency or collective rationality was the first problem (see judgment and preference aggregation)

Given a rationality assumption, is the outcome rational?

	α	$\alpha \to \beta$	β
Agent 1	Yes	Yes	Yes
Agent 2	No	Yes	No
Agent 3	Yes	No	No
Majority	Yes	Yes	No!!

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Majority	Yes	Yes	No!!

But: agents/individuals may resists to take actions that they do not recognise as compatible/representative of the individual opinions.

We want to provide a first investigation of such a problem:

How to measure the compatibility of an aggregation procedure?
How to quantify the relation between collective outcome and individual ballots?

Outline

1. Definitions of the framework:
    - Binary aggregation: vote on sequences of yes/no questions;
    - Generalise it to allow for abstentions.

2. Several definitions of compatibility:
   how to quantify the relation between individual and collective ballots?

3. Definition of some old and new aggregation procedures:
   can we enforce compatibility?

4. Conclusions and future work.

Binary Aggregation

Ingredients:

- A finite set N of individuals
- A finite set $\mathcal{I} = \{1, \dots, m\}$ of issues
- Complete binary ballots are elements of $\mathcal{D} = \{0, 1\}^{\mathcal{I}}$
- Allow abstentions: domain of aggregation is $\mathcal{D} = \{0, 1, A\}^{\mathcal{I}}$

Definition

A non-resolute aggregation procedure is a function $F : \mathcal{D}^N \to 2^{\mathcal{D}}$ *mapping each profile of ballots* $\boldsymbol{B} = (\boldsymbol{B}_1, \dots, \boldsymbol{B}_n)$ *to a subset element of the domain* $\mathcal{D}$.

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Binary aggregation can be used to model:

- Multiple elections
- Voting for committees/candidates
- Preference and judgment aggregation (see Grandi and Endriss, 2011)
- ...

Axiomatic Properties for Non-Resolute Procedures

Standard axioms work well with abstentions: A is the same as a 0 or a 1.

Exception: Unanimity on A does not entails collective abstention!

Adaptation is required for non-resolute procedures:

Unanimity* (U*): For any profile $\boldsymbol{B}$ and any $x \in \{0, 1\}$, if $b_{i,j} = x$ for all $i \in \mathcal{N}$, then $F(\boldsymbol{B})_j = x$ for all $B \in F(\boldsymbol{B})$.

Independence* (I*): For any issue $j \in \mathcal{I}$, $x \in \{0, 1, A\}$ and profiles $\boldsymbol{B}, \boldsymbol{B}' \in X^{\mathcal{N}}$, if $b_{i,j} = b'_{i,j}$ for all $i \in \mathcal{N}$, then there exists a $B \in F(\boldsymbol{B})$ such that $B_j = x$ iff there exists $B' \in F(\boldsymbol{B}')$ such that $B'_j = x$.

Monotonicity* (M*): ...

Compatibility I

Avoid instances of MEP in which the selected outcome has been submitted by the fewest individuals:

Definition

An aggregation procedure satisfies strong compatibility if on every profile each ballot in the outcome is supported by at least one individual:

$$F(\boldsymbol{B}) \subseteq \{B_1, \ldots, B_n\} \text{ for all profiles } \boldsymbol{B} = (B_1, \ldots, B_n).$$

Easy generalisation (e.g., majority consistency, unanimity rule...):

Definition

An aggregation procedure satisfies k-strong compatibility ($k \leqslant |\mathcal{N}|$) if on every profile each ballot in the outcomes is supported by at least k individuals.

Compatibility II

Two ballots are compatible if there is no disagreement on any issue.
Abstentions are O.K. (i.e., only problem if $b_j = 1 - b'_j$)

Definition

An incomplete aggregation procedure satisfies weak-compatibility if each ballot in the outcome is compatible with all the individual ballots $B_1, \ldots, B_n$.

For larger groups we can generalise to the following:

Definition

An incomplete aggregation procedure satisfies k*-weak compatibility* ($k \leqslant |\mathcal{N}|$) *if each ballot in the outcome is compatible with at least* k *individual ballots.*

Caution: the last notion is weaker than the first.

The Average Voter Rule

To obtain strong compatible rules, select one of the individual ballots:

Definition

The average voter rule (AVR) *chooses the individual ballot that minimises the sum of the Hamming distance* $H(B, B') = \sum_{j \in \mathcal{I}} |b_j - b'_j|$ *to all other individual ballots:*

$$\mathrm{AVR}(\boldsymbol{B}) \quad = \quad \operatorname*{argmin}_{\{B_i | i \in \mathcal{N}\}} \sum_{s \in \mathcal{N}} H(B_i, B_s),$$

Proposition (trivial)

AVR *satisfies strong compatibility.*

The Average Voter Rule II

The average voter rule does not coincide with proposition-wise majority:

	p_1	p_2	p_3	p_4	p_5
B_1	1	1	0	1	1
B_2	0	1	1	0	1
B_3	1	0	1	1	0
Maj	1	1	1	1	1
AVR	1	1	0	1	1

Proposition

AVR *satisfies U*, A and M*.* AVR *does not satisfy I*.*

The Conflict-Free Rule

Conflict-free rule (CFR)

For every $j \in \mathcal{I}$*, let* b_j^c *be the* j*th element of the collective outcome* $\mathrm{CFR}(\boldsymbol{B})$*:*

$$b_j^c = \begin{cases} 1 & \exists i \in \mathcal{N} \ s.t. \ b_{i,j} = 1 \ and \ \nexists l \in \mathcal{N} \ s.t. \ b_{l,j} = 0 \\ 0 & \exists i \in \mathcal{N} \ s.t. \ b_{i,j} = 0 \ and \ \nexists l \in \mathcal{N} \ s.t. \ b_{l,j} = 1 \\ A & otherwise \end{cases}$$

	p	q	r
B_1	1	0	A
B_2	A	A	1
B_3	A	0	1
CFR	1	0	1

	p	q	r
B_1'	1	1	A
B_2'	1	0	0
B_3'	A	0	1
CFR	1	A	A

The CFR associates with every profile $\boldsymbol{B}$ of incomplete ballots, the most committed ballot (i.e., the ballot with less abstentions as possible) in the set of compatible outcomes.

k-Conflict Free Rule

The k-Conflict Free Rule (k-CFR)

The k-CFR associates with every profile $\boldsymbol{B}$ the outcome of the CFR rule on each subprofile $\boldsymbol{B}_K$ of $\boldsymbol{B}$ of size k.

If F is the 2-CFR:

	p	q	r
B_1	1	0	A
B_2	A	A	1
B_3	A	0	1
$F(\boldsymbol{B}_{2,3})$	A	0	1
$F(\boldsymbol{B}_{1,2})$	1	0	1
$F(\boldsymbol{B}_{1,3})$	1	0	1

	p	q	r
B'_1	1	1	A
B'_2	1	0	0
B'_3	A	0	1
$F(\boldsymbol{B}'_{2,3})$	1	0	A
$F(\boldsymbol{B}'_{1,2})$	1	A	0
$F(\boldsymbol{B}'_{1,3})$	1	A	1

One possibility is to output the most committed outcome: $(1, 0, 1)$ in the first case, all three ballots in the second case.

Compatibility and Axiomatic Properties

Proposition (trivial)

The CFR *satisfies weak compatibility. The* k-CFR *satisfies* k*-compatibility*

Proposition

The CFR *is a resolute procedure and satisfies U, I, A and M.*

Proposition

The k-CFR *is a non-resolute procedure and satisfies U*, I*, A* and M*.*

Bigger groups

So far we have considered toy examples to illustrate our definitions and rules.

What about bigger groups? Do compatible rules collapse to all-abstention in case of heterogeneous groups?

An Example from "The Paradox of Multiple Elections"

Brams, Kilgour and Zwicker consider 52 voters who cast the following numbers of votes for the 27 combinations:

YYY:0 YYN:4 YNY:4 NYY:4 YYA:4 YAY:4 AYY:4 YNN:1 NYN:1 NNY:1 YAA:1 AYA:1 AAY:1 NAA:1 NAN:1 NNA:1 NYA:1 ANY:1 YAN:1 AAA:5 ANN:1 ANA:1 AAN:1 AYN:1 NAY:1 YNA:1 NNN:5

Complete-reversal paradox: proposition-wise majority voting outputs YYY that received 0 votes. But AAA and NNN have the most votes (i.e., 5).

The CFR returns AAA, a ballot submitted by 5 individuals.
The k-CFR gives NNN with $k = 16$, compatible with a sub-profile of 16 of the 52 voters, or YYY with $k = 20$.

CFR ensures a compatible decision for all the individuals but collapses to an abstention on all the issues. k-CFR guarantees compatible outcomes for a representative subset of the profile resulting in outcomes with fewer abstentions.

k, s-compatibility (idea)

In the previous example, the k-CFR cannot guarantee a compatible and fully committed (i.e., no abstention) outcome with the majority of the group. We may **compromise on the compatibility level**. We identity two parameters:

Group ratio (k)

Commitment (s)

k, s-compatibility: an outcome B is k, s-compatible iff B does not contain more than s abstentions and is compatible with respect to k individual ballots.
$\Rightarrow$ B is a 'less compatible' outcome than the one with all abstentions, but it is more committed.

Conclusions and Future Work

We provided a first study of compatibility in binary aggregation, studying how to quantify the relation between the collective and the individual ballots:

- Defined four notions of compatibility: strong, k-strong, weak, k-weak.
- Studied three rules based on those definitions: AVR, CFR, k-CFR.

A lot of work still to be done:

- More refined notions of compatibility: k, s approach.
- Investigate the compatibility of existing rules.
- Very related work: distance-based rationalizability.
- Combine compatibility and consistency (e.g., in judgment aggregation).