Binary Aggregation by Selection of the Most Representative Voter

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[Joint work with Ulle Endriss]
Selection of the Closest Opinion

\[ \text{argmin} \{ d(o_i, o_1, \ldots, o_n) \mid o_i \in \mathcal{N} \} \]
Selection of the Most Representative Voter

\[ \arg\min \{ d(o_i, o_1, \ldots, o_n) \mid o_i \in \mathcal{N} \} \]
Outline

1. A general framework for aggregation problems:
   - Binary aggregation with integrity constraints
   - Preferences, judgments, multi-issue elections...
   - Generalised dictatorships

2. Distance-based rules:
   - Paradoxical profiles
   - Kemeny rule
   - Slater rule

3. Selection of the most representative voter:
   - Low computational complexity
   - Axiomatic properties
   - Approximation results
Suppose three agents in a multi-agent system need to decide whether to perform a collective decision $A$. The decision is performed if two parameters $T_1$ and $T_2$ exceed a given threshold. Consider the following situation:

<table>
<thead>
<tr>
<th></th>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agent 1</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Agent 2</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Agent 3</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>Majority</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
</tbody>
</table>

Should the agents perform action $A$ or not?

- A majority of agents think the first parameter exceeds the threshold.
- A majority of agents think the second parameter exceeds the threshold.
- But: a majority of agents think action $A$ should not be performed!!
Binary Aggregation

Ingredients:
- A finite set $\mathcal{N}$ of individuals
- A finite set $\mathcal{I} = \{1, \ldots, m\}$ of issues
- A boolean combinatorial domain: $\mathcal{D} = \{0, 1\}^\mathcal{I}$

**Definition**

An aggregation procedure is a function $F : \mathcal{D}^\mathcal{N} \rightarrow \mathcal{D}$ mapping each profile of ballots $B = (B_1, \ldots, B_n)$ to an element of the domain $\mathcal{D}$.

Wilson (1975), Dokow and Holzman (JET 2010), Grandi and Endriss (AIJ 2013)
Binary Aggregation

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Example: Three agents with sensors

- $\mathcal{N} = \{1, 2, 3\}$
- $\mathcal{I} = \{T_1, T_2, A\}$
- Individuals submit ballots in $\mathcal{D} = \{0, 1\}^3$

$B_1 = (0, 1, 0)$ the first individual think the action should not be performed.
Integrity Constraints

A propositional language $\mathcal{L}$ to define the subset of rational ballots in $\{0,1\}^\mathcal{I}$:

- One propositional symbol for every issue: $PS = \{p_1, \ldots, p_m\}$
- $\mathcal{L}_{PS}$ closed under connectives $\land, \lor, \neg, \rightarrow$ the set of atoms $PS$

Given an integrity constraint $IC \in \mathcal{L}_{PS}$, a rational ballot is $B \in \text{Mod}(IC)$
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**Example: Three agents with sensors**

Perform action $A$ if both parameters exceed the thresholds.

Propositional constraint: $IC = (p_{T_1} \land p_{T_2}) \to p_A$

- Individual 1 submits $B_1 = (1, 1, 1)$: $B_1$ satisfies $IC$ ✓
- Individual 2 submits $B_2 = (0, 1, 0)$: $B_2 \models IC$ ✓
- Individual 3 submits $B_3 = (1, 0, 0)$: $B_3 \models IC$ ✓

Majority aggregation outputs $(1, 1, 0)$: $IC$ not satisfied.
### Preference Aggregation as Binary Aggregation

<table>
<thead>
<tr>
<th>Agent 1</th>
<th>Agent 2</th>
<th>Agent 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A &gt; B &gt; C</td>
<td>B &gt; C &gt; A</td>
<td>C &gt; A &gt; B</td>
</tr>
</tbody>
</table>

| Maj | A > B > C > A !! |

#### Condorcet Paradox (1785)

Preferences as binary ballots + integrity constraint

<table>
<thead>
<tr>
<th>A &gt; B</th>
<th>B &gt; C</th>
<th>A &gt; C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agent 1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Agent 2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Agent 3</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

| Maj | 1     | 1     | 0     |
Judgment Aggregation as Binary Aggregation

JA studies the aggregation of judgments over sets of correlated propositions:

Judgment sets $J$ over agenda $\Phi$ $\iff$ Ballot $B_J$ over issues $\mathcal{I} = \Phi$

Properties of judgment sets enforced with $\text{IC}_\Phi$:

**Completeness:** $p_\alpha \lor p_{\neg \alpha}$ for all $\alpha \in \Phi$

**Consistency:** $\neg (\bigwedge_{\alpha \in S} p_\alpha)$ for every mi-set $S \subseteq \Phi$
The Doctrinal Paradox

Kornhauser and Sager (1986)

Judgments as binary ballots + integrity constraint

\[ IC = \neg(p_\alpha \land p_\beta \land p_{\neg(\alpha \land \beta)}) \]

<table>
<thead>
<tr>
<th></th>
<th>( p_\alpha )</th>
<th>( p_\beta )</th>
<th>( p_{\alpha \land \beta} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agent 1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Agent 2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Agent 3</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Maj</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Collective Rationality

Definition

$F$ is collectively rational (CR) for $IC \in \mathcal{L}_{PS}$ if for all profiles $B$ such that $B_i \models IC$ for all $i \in N$ then $F(B) \models IC$.

$F$ lifts the rationality assumption given by $IC$ from the individual to the collective level.

If you want to know more about collective rationality:

Grandi and Endriss, Lifting Integrity Constraints in Binary Aggregation. AIJ, 2013.
Avoid paradoxes? Generalised Dictatorship

A **generalised dictatorship** copies the ballot of a (possibly different) individual (aka local dictatorships, positional dictatorships, rolling dictatorships):

**Proposition**

\[ F \text{ is collectively rational with respect to all IC in } L_{PS} \text{ if and only if } F \text{ is a generalised dictatorship} \]

This class includes:

- Classical dictatorships \( F(B_1, \ldots, B_n) = B_i \) for \( i \in N \)
- Rules based on the selection of the most representative voter
Another route has been taken in the literature to avoid paradoxes:

- Given an integrity constraint, consider $\text{Mod}(IC)$
- Define a distance between a profile and a ballot
- Pick the ballot in $\text{Mod}(IC)$ which is closest to the individual inputs!

Examples of distance-based rules have been defined in preference aggregation and more recently imported in the literature on judgment aggregation:

*Konieczny and Pino Pérez, Merging information under constraints... JLC, 2002.*
*Pigozzi. Belief merging and the discursive dilemma... Synthese, 2006*
In preference aggregation we have a number of individuals submitting preferences in the form of a linear order.

**Definition**

*The Kemeny rule picks the linear orders which minimises the sum of the Kendall $\tau$ distance to the individual preferences. $\Theta^P_2$-complete*

**Definition**

*The Slater rule picks the linear orders minimising the Kendall $\tau$ distance to the outcome of the majority rule. NP-hard (at least)*

The Kendall $\tau$ distance counts the number of pairwise disagreements between two linear orders.

Hemaspaandra et Al. *The complexity of Kemeny elections. Theoretical Computer Science, 2005*

The Kemeny and Slater rule comes under different names in binary aggregation:

Definition

The DBR (aka Kemeny rule, Prototype) picks the consistent ballots minimising the sum of the Hamming distances to the individual ballots. $\Theta^P_2$-complete

Definition

The Slater rule (aka Endpoint) picks the consistent ballots minimising the Hamming distance to the outcome of the majority rule. NP-hard (at least)

The Hamming distance $H$ between an individual input and the outcome is the number of issues on which they differ.
Selection of the Most Representative Voter

Basic idea:
Restrict the search space to $\text{Supp}(B) = \{B_1, \ldots, B_n\}$

Definition
The average-voter rule is the aggregation rule that selects those individual ballots that minimise the Hamming distance to the profile:

$$\text{AVR}(B) = \arg\min_{B \in \text{Supp}(B)} \sum_{i \in \mathcal{N}} H(B, B_i)$$

Definition
The majority-voter rule is the aggregation rule that selects those individual ballots that minimise the Hamming distance to one of the majority outcomes:

$$\text{MVR}(B) = \arg\min_{B \in \text{Supp}(B)} \min\{H(B, B') | B' \in \text{Maj}(B)\}$$
An Example

The AVR and the MVR can give radically different results:

<table>
<thead>
<tr>
<th>Issue:</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 voter:</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2 voters:</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10 voters:</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10 voters:</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Maj:</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>MVR:</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>AVR:</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>AVR:</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Hamming distance of AVR from the profile: 48
Hamming distance of MVR from the profile: 65

Observation
\[ \mathcal{H}(\text{AVR}(B), B) \leq \mathcal{H}(\text{MVR}(B), B) \text{ where } \mathcal{H}(B, B) = \sum_i H(B, B_i) \]
Recall that $m$ is the number of issues; $n$ is the number of voters.

Winner determination for the AVR is in $O(mn \log n)$

- compute the vector of sums in $O(mn)$
- compute the difference between each ballot (multiplied by $n$) to the vector of sums in $O(mn \log n)$ [$O(\log n)$ because of integers up to $n$]

Winner determination for the MVR is in $O(mn)$

- compute the majority vector in $O(mn)$
- compare each ballot to the majority vector in $O(mn)$

Conclusion? Both rules are easy to compute (MVR is easier)
Axiomatic Properties

Rules based on the most representative voter satisfy interesting properties:

- No paradox ever, whatever the IC (no other rule has this property)
- Unanimity guaranteed (obvious)
- Neutrality guaranteed (less obvious)

\[ F \text{ satisfies reinforcement if for any two profiles } B \text{ and } B' \text{ such that:} \]

- \( \text{Supp}(B) = \text{Supp}(B') \)
- \( F(B) \cap F(B') \neq \emptyset \)

we have that \( F(B \oplus B') = F(B) \cap F(B') \)

If two groups independently agree that a certain outcome is the best, we would expect them to uphold this choice when choosing together.

**Theorem**

*The AVR satisfies reinforcement, but the MVR does not.*
Approximation Results

Can we compare the outcome of AVR and MVR with that of Kemeny rule?

$F$ is said to be an $\alpha$-approximation of $F'$ if for every profile $B$:

$$d(F(B), B) \leq \alpha \cdot d(F'(B), B)$$

Good in case $F'$ is intractable (like distance-based rules) and $\alpha$ is a constant.

**Theorem**

*Both the AVR and the MVR are 2-approximations of the DBR (for any IC)*.

Very likely that $\alpha$ decreases if we increase the logical complexity of IC.
Short and Long Proof

A short direct proof can be obtained using the triangular inequality:

\[ \mathcal{H}(\text{AVR}(B), B) = \sum_{i=1}^{n} H(\text{AVR}(B), B_i) \leq \sum_{i=1}^{n} \frac{1}{n} \cdot \sum_{k=1}^{n} H(B_k, B_i) \]

\[ \leq \sum_{i=1}^{n} \frac{1}{n} \cdot \sum_{k=1}^{n} [H(B_k, \text{DBR}(B)) + H(\text{DBR}(B), B_i)] \]

\[ = \frac{1}{n} \cdot [n \cdot \mathcal{H}(\text{DBR}(B), B) + n \cdot H(\text{DBR}(B), B_i)] \]

\[ = 2 \cdot \mathcal{H}(\text{DBR}(B), B) \]

But longer proofs lead us to more precise bounds:

**Theorem**

If \( n > m \), then the AVR and the MVR are \( \alpha \)-approximations of the \( \text{DBR}^{\text{IC}} \) with \( \alpha = 2 \cdot \frac{m-1}{m} \) for any integrity constraint \( \text{IC} \).
Known approximation results for the Kemeny rule in preference aggregation:

- 2-approximation by Dwork et Al. 2001
- $\frac{11}{7}$-approximation by Ailon et Al. 2008
- PTAS by Kenyon-Mathieu and Schudy, 2007

Better approximation ratio but rather "technical" rules!

**Question 1**

*Can we obtain smaller approximation bounds for AVR and MVR if we restrict to the domain of preferences?*

**Question 2**

*To the best of our knowledge AVR and MVR have not been studied as preference aggregation rules. Do they have interesting additional properties?*
Conclusions

Binary aggregation with integrity constraints is a general framework for the study of aggregation problems such as preference and judgment aggregation:

We have seen that it suggests novel simple procedures to be used in practice!

Rules based on the selection of the most representative voter:

- Outcome will never be paradoxical
- Very low complexity
- Social-choice theoretic properties (not independence!)
- 2-approximation of distance-based rule (aka Kemeny)

Future work:

- Tideman’s ranked-pairs rule as a distance-based rule
- Better approximation results on restricted domains