Binary Aggregation with Integrity Constraints: From Individual to Collective Rationality

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Everything Starts From a Paradox

In 1785 Monsieur le Marquis de Condorcet pointed out that:

$$\frac{\begin{array}{c}\triangle \succ_{\mathbf{1}} \bigcirc \succ_{\mathbf{1}} \square \\ \square \succ_{\mathbf{2}} \triangle \succ_{\mathbf{2}} \bigcirc \\ \bigcirc \succ_{\mathbf{3}} \square \succ_{\mathbf{3}} \triangle\end{array}}{\triangle \succ \bigcirc \succ \square \succ \triangle}$$

???

Everything Starts From a Paradox

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???

- Why is this a paradox?
- Why does this happen?

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???

- Why is this a paradox?
- Why does this happen?

My answer: "Because the majority rule lifts a rationality assumption if and only if it is expressible as conjunctions of 2-clauses"

Given a set of alternatives $\mathcal{X}$, a choice function is a $C : \mathcal{P}(\mathcal{X}) \rightarrow \mathcal{P}(\mathcal{X}) \setminus \emptyset$

Revealed preference (Sen, 1969)

A choice function is defined by a weak order over $\mathcal{X}$ if and only if it satisfies property α and β.

Weak order? A transitive, complete and reflexive binary relation.

Individual Rationality in Decision Theory

Who am I? An Order.

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Revealed preference (Sen, 1969)

A choice function is defined by a weak order over $\mathcal{X}$ if and only if it satisfies property α and β.

Weak order? A transitive, complete and reflexive binary relation.

More? Decision theory under uncertainty (Von Neumann, Morgenstern, Savage...)

Many Rationalities?

Judges in a court (cf. judgment aggregation):

"Captain Schettino is guilty" "The captain abandonded the ship"
"If he abandoned the ship then he is guilty"

Rational judges?
Consistent and complete judgment sets

Many Rationalities?

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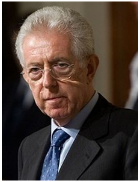
Committee deciding over multiple issues:



"Cut pensions" "Cut the number of MPs"
"Cut funding to local provinces"

Rational members?
No political power to enforce all three austerity measures:
Ballots with at most 2 yes

Integrity Constraint - Rationality Assumptions

Who am I? 01010101.

"Everything is binary"

- A finite set $\mathcal{I} = \{1, \ldots, m\}$ of issues
- Individuals express ballots in $\{0, 1\}^{\mathcal{I}}$

Integrity Constraint - Rationality Assumptions

Who am I? 01010101.

"Everything is binary"

- A finite set $\mathcal{I} = \{1, \dots, m\}$ of issues
- Individuals express ballots in $\{0,1\}^{\mathcal{I}}$

For every issue i introduce a propositional symbol p_i to form the language $\mathcal{L}_{PS}$
A rationality assumption/integrity constraint is a formula IC $\in \mathcal{L}_{PS}$

A ballot $B \in \{0,1\}^{\mathcal{I}}$ (i.e., an individual) is rational if $B \models$ IC

Social Choice Theory

Individuals interact? NO! That's game theory.

Social Choice Theory studies methods for
collective choice based on individual expressions

A number of topics: voting, fair division, measures of fairness, matching....
A number of scholars: Arrow, Sen, Myerson, Lewis Carroll...

(ps. we do Computational Social Choice, i.e., use these models for multi-agent systems, and use AI techniques to study these problems.)

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Ingredients (binary aggregation, n pers.):

- A society: a set $\mathcal{N}$ of individuals
- Individual expressions: ballots $B \in \mathcal{D} = \{0,1\}^{\mathcal{I}}$
- An aggregation procedure $F : \mathcal{D}^{\mathcal{N}} \to \mathcal{D}$

Paradoxes of Aggregation

Every individual satisfies the same rationality assumption IC...
...what about the collective outcome?

Definition

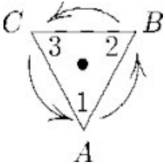
A paradox is a triple $(F, \mathbf{B}, IC)$, where:

- *F is an aggregation procedure*
- *$\mathbf{B} = (B_1, \ldots, B_n)$ a profile*
- *$IC \in \mathcal{L}_{PS}$ an integrity constraint*

such that $B_i \models IC$ for all $i \in \mathcal{N}$ but $F(\mathbf{B}) \not\models IC$.

Condorcet Paradox Revisited

	ab	bc	ac
Agent 1	1	1	1
Agent 2	0	1	0
Agent 3	1	0	0
Majority	1	1	0



Our definition of paradox:

- F is issue by issue majority rule
- the profile is the one described in the table
- IC that is violated is $p_{ab} \wedge p_{bc} \rightarrow p_{ac}$

Doctrinal Paradox

	α	$\alpha \to \beta$	β
Agent 1	1	1	1
Agent 2	0	1	0
Agent 3	1	0	0
Majority	1	1	0!!

Our definition of paradox:

- F is issue by issue majority rule
- profile described in the table
- IC that is violated is $\neg(p_\alpha \wedge p_{\neg\beta} \wedge p_{(\alpha\to\beta)})$

Common feature: clauses of size 3

Kornhauser and Sager. Unpacking the Court. *Yale Law Journal*, 1986.

The Answer to M. le Marquis

Proposition

The majority rule generates a paradox with respect to IC if and only if IC is NOT equivalent to a conjunction of clauses of size $\leqslant 2$.

Common feature of all paradoxes:
clauses of size 3 are not lifted by majority

More complex ICs? Other aggregation procedures?

Toward General Results: Collective Rationality

Definition

F is collectively rational (CR) for IC $\in \mathcal{L}_{PS}$ if for all profiles $\mathbf{B}$ such that $\mathbf{B}_i \models$ IC for all $i \in N$ then $F(\mathbf{B}) \models$ IC.

F lifts the rationality assumption given by IC from the individual to the collective level.

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Classical axioms from social choice theory can be translated in this framework:

Unanimity (U): For any profile $\mathbf{B} \in X^N$ and any $x \in \{0, 1\}$, if $\mathbf{B}_{i,j} = x$ for all $i \in N$, then $F(\mathbf{B})_j = x$.

Independence, Neutrality...

General Characterisation Results

Several languages for integrity constraints:

- *cubes*: conjunctions
- *k-pclauses*: positive disjunctions of size $\leqslant k$
- *XOR*: conjunctions of $p \leftrightarrow \neg q$
- ...

Several axioms to classify aggregation procedures:

- unanimity, independence, monotonicity, anonymity...

What we want:

$$\mathcal{CR}[\mathcal{L}] = \mathcal{F}_{\mathcal{L}}[\mathrm{AX}]$$

Characterisation Results: Examples

Cubes (conjunctions of literals) are lifted iff the procedure satisfies unanimity:

Proposition

$\mathcal{CR}[cubes] = \mathcal{F}_{cubes}[Unanimity]$.

For the axioms of independence and anonymity we prove a negative result:

Proposition

There is no language $\mathcal{L} \subseteq \mathcal{L}_{PS}$ *such that* $\mathcal{CR}[\mathcal{L}] = \mathcal{F}_{\mathcal{L}}[\mathrm{I}]$.

Quota rules and clauses:

Proposition

A quota rule is CR for a k-pclause IC if and only if $\sum_j q_j < n + k$, *with* j *ranging over all issues that occur in IC and* n *being the number of individuals, or* $q_j = 0$ *for at least one issue* j *that occurs in IC.*

Are these results useful for social choice theorists?

Yes, we can talk about orders:

Call an aggregation procedure imposed if there are two alternatives x and y such that x is collectively preferred to y in every profile:

Proposition

Any anonymous, independent and monotonic aggregation procedure for more than 3 alternatives and 2 individuals is imposed.

Proof sketch:

- Translate preference aggregation into BA with IC
- Study the syntactic property of the $\text{IC}_<$
- Use a characterisation result!
- Go back to preference aggregation

Are these results useful for computer scientists?

Yes, we can prove some complexity results.

Are these results useful for computer scientists?

Yes, we can talk about multi-agent systems:

Systems of automatic agents embedded with preferences or judgments to control and perform actions: need systematic theory of consistent aggregation!

Several applications:

- Preferential dependencies in elections
- Combinatorial vote
- Belief merging
- Distance-based procedures

Airiau et al., Aggregating dependency graphs into voting agendas in multi-issue elections, *IJCAI-11*

Are these results useful for philosophers?

A question: can we model a group of rational agents as a rational agent itself?

Proposition

F is CR with respect to all IC in $\mathcal{L}_{PS}$ if and only if F copies the ballot of a (possibly different) individual in every profile.

This class includes:

- Classical dictatorships $F(B_1, \ldots, B_n) = B_i$ for $i \in \mathcal{N}$
- Distance-based generalised dictatorship: map $(B_1, \ldots, B_n)$ to the ballot B_i that minimises the sum of the Hamming distance to the others (a sort of "median voter"...). An interesting procedure!

List and Pettit. *Group Agency*. Oxford University Press, 2011.

(ps. I do not agree with their thesis)

Conclusions

Many notions of individual rationality:

- binary issues as a general model of individual expressions;
- rationality assumptions as propositional formulas;
- focus on syntactic properties of the rationality assumption.

Collective rationality:

- Unifying framework for paradoxes;
- Systematic study of collective rationality;
- Application in preference/judgment aggregation & co.

Thanks for your attention!

U. Grandi. Binary Aggregation with Integrity Constraints. PhD Thesis, ILLC, Amsterdam 2012.

U. Grandi and U. Endriss. Binary Aggregation with Integrity Constraints, *IJCAI-2011*.