Complexity of Judgment Aggregation: Safety of the Agenda

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Aggregating the judgments of a group of agents regarding a set of interdependent propositions can lead to inconsistent outcomes. One of the parameters involved is the agenda, the set of propositions on which agents are asked to express an opinion. We introduce the problem of checking the safety of the agenda: for a given agenda, can we guarantee that the aggregation will never produce an inconsistent outcome for any procedure satisfying certain axioms? We establish necessary and sufficient conditions for the safety of the agenda for different combinations of axioms, and we analyse the computational complexity of checking these properties.

Judgment Aggregation: Discursive Dilemma

JA was developed to generalise and study paradoxical situations that arises when a collective judgment has to be made on a set of correlated propositions.

Discursive Dilemma

	p	$p \to q$	q
Judge 1:	Yes	Yes	Yes
Judge 2:	No	Yes	No
Judge 3:	Yes	No	No
Majority:	Yes	Yes	No

Kornhauser and Sager, Unpacking the court. Yale Law Journal, 1986.

List and Pettit, Aggregating sets of judgments. Economics and Philosophy, 2002

Each individual is rational (i.e., has a consistent judgment) but the majority is contradictory!

Definition 1. *An agenda is a finite subset of propositional formulas $\Phi \subseteq \mathcal{L}_{PS}$ closed under complementation and not containing double negations. A judgment set is a subset $J \subseteq \Phi$. Call J:*

- **Complete**: *if for all $\alpha \in \Phi$ either α or its complement is in J.* (checking is easy)
- **Complement-free**: *if α and its complement $\sim\alpha$ are never both in J.* (checking is easy)
- **Consistent**: *if there is an assignment to make all formulas in J true.* (checking is hard!)

Judgment Aggregation: Axioms

Definition 2. *Call $J(\Phi)$ the set of all consistent and complete judgment sets. An aggregation procedure for agenda Φ and a set N of n individuals is a function $F : J(\Phi)^n \to 2^\Phi$.*

Weak Rationality (WR): F is non-null, complete and complement-free. (checking is easy!)

Unanimity (U): If $\varphi \in J_i$ for all i then $\varphi \in F(\mathbf{J})$.
Anonymity (A): F is symmetric with respect to individuals.
Neutrality (N): F is symmetric with respect to formulas.
Independence (I): The outcome of F over φ depends only on the individuals' judgments on φ.
Systematicity (S)=(N)+(I). Two forms of **monotonicity**: $(\mathrm{M^I})$ and $(\mathrm{M^N})$.

Such axioms can be used to define different classes of aggregation procedures:

Set of axioms AX Agenda Φ	$\Longrightarrow$	Class of procedures $\mathcal{F}_\Phi[\mathrm{AX}]$

Proposition 1. $\mathcal{F}_\Phi[\mathrm{WR, A, S, M^I}]$=$\mathcal{F}_\Phi[\mathrm{WR, A, N, M^N}]$ *is the majority rule.*

Various weakenings of majority: $\mathcal{F}_\Phi[\mathrm{WR, A, S}]$, $\mathcal{F}_\Phi[\mathrm{WR, A, N}]$, $\mathcal{F}_\Phi[\mathrm{WR, A, I}]$ and $\mathcal{F}_\Phi[\mathrm{A, S, M^I}]$.

Safety of the Agenda

Call a procedure consistent if $F(\mathbf{J})$ is consistent for all profiles $\mathbf{J}$ in the domain.

Results from the literature are of the following form:

Possibility (characterisation) results:
There exists a consistent F in $\mathcal{F}_\Phi[\mathrm{AX}]$ $\Leftrightarrow$ Φ has a certain property.

These results characterise agendas for which consistent aggregation is possible.

Results that we are looking for are instead of the form:

Safety (characterisation) results:
All F in $\mathcal{F}_\Phi[\mathrm{AX}]$ are consistent $\Leftrightarrow$ Φ has a certain property.

These results characterise agendas for which consistent aggregation is guaranteed.

We prove several safety characterisation results:

Definition 3. *An agenda Φ is safe with respect to a class of aggregation procedures $\mathcal{F}_\Phi$ if every procedure in $\mathcal{F}_\Phi$ is consistent.*

Proposition 2. Φ *is safe for $\mathcal{F}_\Phi[\mathrm{WR, A, S, M^I}]$ iff every inconsistent subset of Φ contains an inconsistent subset of size 2 (median property MP).*

Majority rule is the only function in $\mathcal{F}_\Phi[\mathrm{WR, A, S, M^I}]$.

Therefore proving safety = proving possibility.

(Possibility result is known from the literature.)

An agenda Φ satisfies:

- k-median property (kMP), if every inconsistent subset of Φ has an inconsistent subset of size at most k (2MP=MP).
- simplified MP (SMP), if every non-trivially inconsistent subset of Φ has itself an inconsistent subset of the form $\{\varphi, \psi\}$ with $\models \varphi \leftrightarrow \neg\psi$.
- syntactic SMP (SSMP), if every non-trivially inconsistent subset of Φ has an inconsistent subset of the form. $\{\varphi, \neg\varphi\}$.

SSMP $\Rightarrow$ SMP $\Rightarrow$ MP $\Rightarrow$ kMP

Proposition 3. Φ *is safe for $\mathcal{F}_\Phi[\mathrm{WR, A, S}]$ iff Φ satisfies the* SMP.

Proposition 4. Φ *is safe for $\mathcal{F}_\Phi[\mathrm{WR, A, N}]$ iff Φ satisfies the* SMP *and does not contain a contradictory formula.*

Proposition 5. Φ *is safe for $\mathcal{F}_\Phi[\mathrm{WR, A, I}]$ if and only Φ satisfies the* SSMP.

Reformulating work proved by Dietrich and List (2007) we obtain:

Proposition 6. *Let $k \geqslant 2$. An agenda Φ is safe for the class of uniform quota rules F_m for n individuals satisfying $m > n - \frac{n}{k}$ if and only if Φ satisfies the* $k\mathrm{MP}$.

Complexity of SoA: Identifying the Problem

Is an agenda Φ safe for the class $\mathcal{F}_\Phi[\mathrm{AX}]$? Problem description:

SAFETY[AX]
Instance: Agenda Φ of size n
Problem: Is Φ safe for $\mathcal{F}_\Phi[\mathrm{AX}]$?

Using our characterisation, e.g. for $\mathcal{F}_\Phi[\mathrm{WR, A, I}]$, this boils down to:

SSMP
Instance: Agenda Φ of size n
Problem: Does Φ satisfy the SSMP?

The same can be done for the remaining problems using SMP, MP and $k\mathrm{MP}$.

Complexity of SoA: The class Π_2^p

Logical satisfaction problems play an important role in defining several complexity classes. The most well-known is SAT (which is **NP**-complete):

SAT
Instance: Propositional formula φ
Problem: Is φ satisfiable?

Above **P** and **NP** sits the polynomial hierarchy **PH**.
We are interested in the second layer Π_2^p, the class defined by the following satisfaction problem:

$\mathrm{coQSAT_2}$
Instance: Quantified boolean formula (QBF) of the form $\forall x_1, \ldots, x_r. \exists y_1 \ldots, y_s. \varphi(x_1, \ldots, x_r, y_1, \ldots, y_s)$
Problem: Is the QBF satisfiable (i.e., true)?

Complexity of SoA: Π_2^p-Completeness

We can reformulate the SSMP in such a way that it can easily be translated into the satisfiability of a particular QBF. An agenda satisfies this property iff every subset of positive formulas φ_i^+ (identified by an assignment to the propositional variables y_i) is satisfiable (i.e., there exists an assignment to the variables x_j in φ_i^+ that makes these formulas true):

$$[\forall y_1 \cdots \forall y_{n^+} \exists x_1 \cdots \exists x_m : (y_1 \leftrightarrow \varphi_1^+) \wedge \cdots \wedge (y_{n^+} \leftrightarrow \varphi_{n^+}^+)]$$

This proves membership to Π_2^p.

The following theorem is an overview of our complexity results:

Theorem 1. *Checking the safety of an agenda is Π_2^p-complete for any of these classes of procedures:*

1. *the majority rule, corresponding to $\mathcal{F}_\Phi[\mathrm{WR, A, S, M^I}]$ and $\mathcal{F}_\Phi[\mathrm{WR, A, N, M^N}]$;*
2. *systematic rules: $\mathcal{F}_\Phi[\mathrm{WR, A, S}]$;*
3. *neutral rules: $\mathcal{F}_\Phi[\mathrm{WR, A, N}]$;*
4. *independent rules: $\mathcal{F}_\Phi[\mathrm{WR, A, I}]$;*
5. *for any $k \geqslant 2$, the uniform quota rules F_m with $m > n - \frac{n}{k}$, where n is the number of individuals.*

We sketch the proof for the class $\mathcal{F}_\Phi[\mathrm{WR, A, I}]$:

- By characterisation (Proposition 5), $\mathrm{SAFETY[WR, A, I]}$ is equivalent to SSMP;
- SSMP is in Π_2^p (membership): reduction to $\mathrm{coQSAT_2}$.
- SSMP is Π_2^p-complete (hardness): by reduction from $\mathrm{coQSAT_2}$. For every QBF Δ we can build an agenda Φ_Δ such that Φ_Δ satisfies the SSMP iff the initial QBF Δ is satisfiable.

The proof is analogous for the other classes, since SSMP, SMP, $k\mathrm{MP}$ for $k \geqslant 2$, are all Π_2^p-complete.

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