Binary Aggregation with Integrity Constraints

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A General Framework for Aggregation

Preference Aggregation
Condorcet Paradox
Arrow's Theorem

Judgment Aggregation
Doctrinal paradox
Agenda properties

Voting in Combinatorial Domains
Paradox of multiple elections
...

Binary Aggregation with Integrity Constraints
Paradox of collective rationality
Characterisation results
(from Grandi and Endriss AAAI-2010)

Binary aggregation with integrity constraints constitutes a general framework for the study of paradoxes and (im)possibility results in aggregation.

An Outline

1. Introduce the framework of binary aggregation with integrity constraints (BA with IC): definition of paradox and collective rationality.
2. Preference aggregation as an instance of BA with IC: Condorcet paradox and new impossibility result.
3. Judgment aggregation as an instance of BA with IC: doctrinal paradox and characterisation result.
4. How to avoid paradoxes: class of generalised (positional) dictatorships.

Binary Aggregation

Ingredients:

- A finite set N of individuals
- A finite set $\mathcal{I} = \{1, \ldots, m\}$ of issues
- A boolean *combinatorial domain*: $\mathcal{D} = D_1 \times \cdots \times D_m$ with $|D_i| = 2$

Definition

An aggregation procedure is a function $F : \mathcal{D}^N \to \mathcal{D}$ *mapping each profile of ballots* $\underline{B} = (\underline{B}_1, \ldots, \underline{B}_n)$ *to an element of the domain* $\mathcal{D}$.

Dokow and Holzman (2005), Grandi and Endriss (AAAI-2010)

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Example: Science Park

- $N = \{1, 2, 3\}$
- $\mathcal{I} = \{\text{University}, \text{Sportcentrum}, \text{Food supply}\}$
- Individuals submit ballots in $\mathcal{D} = \{0, 1\}^3$

$B_1 = (0, 1, 1)$ the first individual wants to have a good meal after the gym.

Integrity Constraints

A propositional language $\mathcal{L}$ to express integrity constraints on $D = \{0,1\}^m$:

- One propositional symbol for every issue: $PS = \{p_1, \ldots, p_m\}$
- $\mathcal{L}_{PS}$ closing under connectives $\wedge$, $\vee$,$\neg$, $\rightarrow$ the set of atoms PS

Given an integrity constraint IC $\in \mathcal{L}_{PS}$, a rational ballot is $B \in \mathrm{Mod}(\mathrm{IC})$

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Example: Science Park (the true story)

If there is both a university and a sport center then food supply is necessary
Propositional constraint: $\mathrm{IC} = (p_U \wedge p_S) \rightarrow p_F$

Individual 1 submits $B_1 = (1,0,0)$: B_1 satisfies IC ✓
Individual 2 submits $B_2 = (1,1,1)$: $B_2 \models \mathrm{IC}$ ✓
Individual 3 submits $B_3 = (0,1,0)$: $B_3 \models \mathrm{IC}$ ✓

Majority aggregation outputs $(1,1,0)$: IC not satisfied (as are all employees)

Paradoxes of Aggregation

Every individual satisfies the same rationality assumption IC...
...what about the collective outcome?

Definition

A paradox is a triple $(F, \underline{B}, IC)$, where:

- F *is an aggregation procedure*
- $\underline{B} = (B_1, \ldots, B_n)$ *a profile*
- $IC \in \mathcal{L}_{PS}$ *an integrity constraint*

such that $B_i \models IC$ *for all* $i \in \mathcal{N}$ *but* $F(\underline{B}) \not\models IC$.

Collective Rationality

Definition

F is collectively rational (CR) for IC $\in \mathcal{L}_{PS}$ if for all profiles $\underline{B}$ such that $\underline{B}_i \models$ IC for all $i \in N$ then $F(\underline{B}) \models$ IC.

F lifts the rationality assumption given by IC from the individual to the collective level.

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Classical axioms from social choice theory can be translated in this framework:

Unanimity (U): For any profile $\underline{B} \in X^N$ and any $x \in \{0,1\}$, if $\underline{B}_{i,j} = x$ for all $i \in N$, then $F(\underline{B})_j = x$.

Independence, Neutrality...

Characterisation Results

How can we enforce collective rationality given a class of integrity constraints?

Proposition (Grandi and Endriss, AAAI-2010)

An aggregation procedure F is collectively rational with respect to any cube (i.e. conjunction of literals) if and only if it is unanimous.

A quota rule accepts an issue j if it is accepted by more than q_j individuals. A k-pclause is a disjunction of positive literals of size $\leqslant k$.

Proposition

A quota rule is CR for a k-pclause IC if and only if $\sum_j q_j < n + k$, with j ranging over all issues that occur in IC and n being the number of individuals, or $q_j = 0$ for at least one issue j that occurs in IC.

Preference Aggregation

If we represent preferences with linear orders, a social welfare function aggregates every profile of linear orders $(<_1, \ldots, <_n)$ into a collective order.

Linear order $<$ over alternatives $\mathcal{X}$	$\Leftrightarrow$	Ballot $B_<$ over issues $\mathcal{I} = \{ab \mid a \neq b \in \mathcal{X}\}$

Property of linear orders enforced with $\mathrm{IC}_<$:

Completeness and antisymmetry: $p_{ab} \leftrightarrow \neg p_{ba}$ for $a \neq b \in \mathcal{X}$

Transitivity: $(p_{ab} \wedge p_{bc}) \rightarrow p_{ac}$ for $a, b, c \in \mathcal{X}$ pairwise distinct

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Social welfare function	$\Leftrightarrow$	Binary aggregation proc. CR with respect to IC$_<$

Axioms are preserved: unanimity, IIA, neutrality...

Condorcet Paradox

	ab	bc	ac
Agent 1	1	1	1
Agent 2	0	1	0
Agent 3	1	0	0
Majority	1	1	0

Our definition of paradox:

- F is issue by issue majority rule
- the profile is the one described in the table
- IC that is violated is $p_{ab} \wedge p_{bc} \rightarrow p_{ac}$

A New Impossibility Result

Call a SWF imposed if there are two alternatives x and y such that x is collectively preferred to y in every profile:

Proposition

Any anonymous, independent and monotonic SWF for more than 3 alternatives and 2 individuals is imposed.

Proof sketch:

- To any A, I, M social welfare function corresponds an A, I, and M binary aggregation procedure that are CR wrt $\text{IC}_<$
- An aggregation procedure satisfies A, I and M iff it is a quota rule (Dietrich and List, 2007)
- Characterisation result: quota rules lift IC iff satisfy property on quotas or the quota is zero for at least one issue (i.e., the procedure is imposed)
- the integrity constraints $\text{IC}_<$ for preference aggregation do not satisfy this property

Judgment Aggregation

JA studies the aggregation of judgments over sets of correlated propositions:

Judgment sets J over agenda Φ	$\Leftrightarrow$	Ballot B_J over issues $\mathcal{I} = \Phi$

Property of judgment sets enforced with IC_Φ:

Completeness: $p_\alpha \vee p_{\neg\alpha}$ for all $\alpha \in \Phi$

Consistency: $\neg(\bigwedge_{\alpha \in S} p_\alpha)$ for every mi-set $S \subseteq \Phi$

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Complete and consistent JA procedures for Φ	$\Leftrightarrow$	Binary aggregation proc. CR with respect to IC_Φ

Different from Dokow and Holzmann: no focus on models of judgment sets.

Doctrinal Paradox

	α	β	$\alpha \wedge \beta$
Agent 1	1	1	1
Agent 2	0	1	0
Agent 3	1	0	0
Majority	1	1	0

Our definition of paradox:

- F is issue by issue majority rule
- profile described in the table
- IC that is violated is $\neg(p_\alpha \wedge p_\beta \wedge p_{\neg(\alpha \wedge \beta)})$

An old Characterisation Result

Characterisation result in JA can be obtained with easier proofs:

Proposition (Endriss, Grandi and Porello, 2010)

An agenda Φ does not generate a paradox (Φ is safe) for neutral judgment aggregation procedures if and only if Φ satisfies the syntactic median property (i.e., only inconsistencies between equivalent formulas).

Proof sketch:

- Correspondence between neutral JA procedures and neutral BA procedures
- Characterisation result: a neutral binary aggregation procedure is collectively rational with respect to IC iff it is of the form $p_i \leftrightarrow p_j$.
- IC_Φ is a set of bi-implications iff Φ satisfies the simplified median property.

Endriss, Grandi and Porello. Complexity of JA: Safety of the Agenda (AAMAS-2010).

Characterisation Result: The Majority Rule

Proposition

The majority rule is CR with respect to IC if and only if IC is equivalent to a conjunction of clauses of size $\leqslant 2$.

Proof sketch:

- BA procedures are JA procedures with a "law" accepted by all individuals
- Result by Nehring and Puppe: majority rule is consistent iff there are no minimally inconsistent subsets of size less than 2 in the agenda

Common feature of previous paradoxes:

Condorcet: $p_{ab} \wedge p_{bc} \rightarrow p_{ac}$
Doctrinal: $\neg(p_{\alpha} \wedge p_{\beta} \wedge p_{\neg(\alpha\wedge\beta)})$
Science Park: $(p_U \wedge p_S) \rightarrow p_F$

Clauses of size 3 are not lifted by majority

Avoid paradoxes? Generalised Dictatorship

A generalised dictatorship copies the ballot of a (possibly different) individual (aka local dictatorships, positional dictatorships, rolling dictatorships):

Proposition (Grandi and Endriss, AAAI-2010)

F is collectively rational with respect to all IC in $\mathcal{L}_{PS}$ if and only if F is a generalised dictatorship

This class includes:

- Classical dictatorships $F(B_1, \ldots, B_n) = B_i$ for $i \in \mathcal{N}$
- Distance-based generalised dictatorship

Distance-based Generalised dictatorship

Choose the individual whose ballot is closest to the ballots of the others:

$$\mathrm{DBGD}(\underline{B}) = \operatorname*{argmin}_{\{B_i | i \in \mathcal{N}\}} \sum_{i' \in \mathcal{N}} H(B_i, B_{i'}),$$

where $H(B, B') = \sum_{j \in \mathcal{I}} |b_j - b'_j|$ is the Hamming distance.

A good compromise between paradoxes and complexity?

- Coincide with issue-by-issue majority when consistent
- Satisfies neutrality, anonymity and monotonicity (adapted)
- Is computationally tractable

Conclusion

Binary aggregation with integrity constraints:

- Language to express rationality assumptions
- Concept of collective rationality with respect to an IC
- General framework for paradoxes (e.g., Condorcet and doctrinal)
- New proof method for impossibility results: clash between axioms and IC
- Interesting procedures: distance-based generalised dictatorship

Bigger picture:

- Axiomatic Method: derive (im)possibility results for specific domains and specific axioms
- "AI approach": devise a machinery to reason about different application-specific domains, assumptions, axioms

Ideas for the future

- More on characterisation results and (im)possibility theorems
- Other paradoxes in the literature
- More complex combinatorial domains
- Finer logical languages