Binary Aggregation with Integrity Constraints

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An Outline

Introduce the framework of binary aggregation with integrity constraints, giving two crucial definitions: paradox and collective rationality.

1. BA with IC is a general framework for the study of aggregation paradoxes:
   - Preference and judgment aggregation are instances of BA with IC
   - Classical paradoxes falls under our general definition

2. BA with IC provides general impossibility results:
   - Prove characterisation results linking collective rationality and axioms
   - Obtain (im)possibility results in preference and judgment aggregation

3. Conclusion and directions for future research

Binary Aggregation

Ingredients:

- A finite set N of individuals
- A finite set $\mathcal{I} = \{1, \ldots, m\}$ of issues
- A boolean *combinatorial domain*: $\mathcal{D} = D_1 \times \cdots \times D_m$ with $|D_i| = 2$

Definition

An aggregation procedure is a function $F : \mathcal{D}^N \to \mathcal{D}$ mapping each profile of ballots $\underline{B} = (\underline{B}_1, \ldots, \underline{B}_n)$ to an element of the domain $\mathcal{D}$.

Wilson (1975), Rubinstein and Fishburn (1986), Dokow and Holzman (2005)

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Example: Three bills

- $N = \{a_1, a_2, a_3\}$
- $\mathcal{I} = \{1, 2, 3\}$ one issue for every bill
- Individuals submit ballots in $\mathcal{D} = \{0, 1\}^3$

$B_1 = (0, 1, 0)$ the first individual accepts only the second bill.

Integrity Constraints

A propositional language $\mathcal{L}$ to express integrity constraints:

- One propositional symbol for every issue: $PS = \{p_1, \ldots, p_m\}$
- $\mathcal{L}_{PS}$ closing under connectives $\wedge$, $\vee$,$\neg$, $\rightarrow$ the set of atoms PS

IC $\in \mathcal{L}$ expresses properties of ballots $B \in \mathcal{D} = \{0,1\}^m$
Given an integrity constraint IC $\in \mathcal{L}_{PS}$, a rational ballot is $B \in \mathrm{Mod}(\text{IC})$

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Example: Three bills with budget constraint.

Two bills between b_1, b_2 and b_3 have to be approved/disproved

Budget constraint: IC $= \neg(p_1 \wedge p_2 \wedge p_3)$, there is budget only for two of them

Individual 1 submits $B_1 = (1,1,0)$:	B_1 satisfies IC ✓
Individual 2 submits $B_2 = (0,1,1)$:	$B_2 \models$ IC ✓
Individual 3 submits $B_3 = (1,0,1)$:	$B_3 \models$ IC ✓

Majority aggregation approves all three bills: IC not satisfied!

Every individual satisfies the same rationality assumption IC...
...what about the collective outcome?

Definition

A paradox is a triple $(F, \underline{B}, IC)$, where F is an aggregation procedure, $\underline{B} = (B_1, \ldots, B_n)$ a profile, $IC \in \mathcal{L}_{PS}$ an integrity constraint, and $B_i \models IC$ for all $i \in \mathcal{N}$ but $F(\underline{B}) \not\models IC$.

Paradoxes and Collective Rationality

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Definition

A paradox is a triple $(F, \underline{B}, IC)$*, where* F *is an aggregation procedure,* $\underline{B} = (B_1, \ldots, B_n)$ *a profile,* $IC \in \mathcal{L}_{PS}$ *an integrity constraint, and* $B_i \models IC$ *for all* $i \in \mathcal{N}$ *but* $F(\underline{B}) \not\models IC$*.*

Definition

F *is collectively rational (CR) for* $IC \in \mathcal{L}_{PS}$ *if for all profiles* $\underline{B}$ *such that* $\underline{B}_i \models IC$ *for all* $i \in N$ *then* $F(\underline{B}) \models IC$*.*

F lifts the rationality assumption given by IC
from the individual to the collective level.

Axioms for Binary Aggregation

Classical axioms from social choice theory can be translated in this framework:

Unanimity (U): For any profile $\underline{B} \in X^N$ and any $x \in \{0, 1\}$, if $\underline{B}_{i,j} = x$ for all $i \in N$, then $F(\underline{B})_j = x$.

Independence (I): For any issue $j \in \mathcal{I}$ and any two profiles $\underline{B}, \underline{B}' \in X^N$, if $\underline{B}_{i,j} = \underline{B}'_{i,j}$ for all $i \in N$, then $F(\underline{B})_j = F(\underline{B}')_j$.

Neutrality (N$^{\mathcal{I}}$)...

New axioms can be defined like the following generalisation of May's neutrality:

Domain-Neutrality (N$^{\mathcal{D}}$): For any two issues $j, j' \in \mathcal{I}$ and any profile $\underline{B} \in X^N$, if $\underline{B}_{i,j} = 1 - \underline{B}_{i,j'}$ for all $i \in N$, then $F(\underline{B})_j = 1 - F(\underline{B})_{j'}$.

Part I:
A general definition of paradox

Preference Aggregation

Linear order $<$ over alternatives $\mathcal{X}$ $\Leftrightarrow$ Ballot $B_{\leqslant}$ over issues $\mathcal{I} = \{ab \mid a \neq b \in \mathcal{X}\}$

Property of linear orders enforced with IC$_{<}$:

Completeness and irreflexivity:
$p_{ab} \vee \neg p_{ba}$ for $a \neq b \in \mathcal{X}$ $\quad \neg p_{aa}$ for all $a \in \mathcal{X}$
Transitivity: $p_{ab} \wedge p_{bc} \rightarrow p_{ac}$ for $a, b, c \in \mathcal{X}$ pairwise distinct

Social welfare function $\Leftrightarrow$ Binary aggregation proc. CR with respect to IC$_{<}$

Axioms are preserved: unanimity, IIA, neutrality...

Condorcet Paradox

	ab	bc	ac
Agent 1	1	1	1
Agent 2	0	1	0
Agent 3	1	0	0
Majority	1	1	0

Our definition of paradox:

- F is issue by issue majority rule
- the profile is the one described in the table
- IC that is violated is $p_{ab} \wedge p_{bc} \rightarrow p_{ac}$

Judgment Aggregation

Judgment sets J over agenda Φ $\Leftrightarrow$ Ballot B_J over issues $\mathcal{I} = \Phi$

Property of judgment sets enforced with IC_Φ:

Completeness: $p_\alpha \vee p_{\neg\alpha}$ for all $\alpha \in \Phi$

Consistency: $\neg(\bigwedge_{\alpha \in S} p_\alpha)$ for every mi-set $S \subseteq \Phi$

Complete and consistent JA procedures for Φ $\Leftrightarrow$ Binary aggregation proc. CR with respect to IC_Φ

Different from Dokow and Holzmann: no focus on models of judgment sets.

Doctrinal Paradox

	α	β	$\alpha \wedge \beta$
Agent 1	1	1	1
Agent 2	0	1	0
Agent 3	1	0	0
Majority	1	1	0

Our definition of paradox:

- F is issue by issue majority rule
- profile described in the table
- IC that is violated is $\neg(p_\alpha \wedge p_\beta \wedge p_{\neg(\alpha \wedge \beta)})$

Ostrogorski Paradox

	E	V	I	P
Agent 1	0	1	0	0
Agent 2	0	1	0	0
Agent 3	1	0	0	0
Agent 4	1	1	1	1
Agent 5	1	1	0	1
Majority	1	1	0	0

Our definition of paradox:

- F is issue by issue majority rule
- IC that is violated is $P \leftrightarrow [(E \wedge V) \vee (V \wedge I) \vee (I \wedge E)]$

After some calculation IC is equivalent to:

$$(P \vee \neg E \vee \neg V) \wedge (P \vee \neg E \vee \neg I) \wedge (P \vee \neg I \vee \neg V) \wedge (\neg P \vee E \vee V) \wedge (\neg P \vee E \vee I) \wedge (\neg P \vee I \vee V)$$

Common feature: clauses of size 3

Part II:
(Im)possibility results

General Characterisation Results

Given a list of axioms AX and a language for integrity constraints $\mathcal{L} \subseteq \mathcal{L}_{PS}$ we can define classes of aggregation procedures:

The class of aggregation procedures satisfying axioms AX:

$$\mathcal{F}_{\mathcal{L}}[\mathrm{AX}] \quad = \quad \{F : \mathcal{D}^{\mathcal{N}} \to \mathcal{D} \mid F \text{ satisfies AX on all } \mathcal{L}\text{-domains}\}$$

The class of procedures that lift integrity constraint in a given language is:

$$\mathcal{CR}[\mathcal{L}] \quad = \quad \{F : \mathcal{D}^{N} \to \mathcal{D} \mid F \text{ is CR for all IC} \in \mathcal{L}\}$$

What we want:

$$\mathcal{CR}[\mathcal{L}] = \mathcal{F}_{\mathcal{L}}[\mathrm{AX}]$$

Some Characterisation Results

Cubes (conjunctions of literals) are lifted iff the procedure satisfies unanimity:

- $\mathcal{CR}[\mathit{cubes}] = \mathcal{F}_{\mathit{cubes}}[\text{Unanimity}]$.

The XOR-language is made of constraints of the form $p_i \leftrightarrow \neg p_j$:

- $\mathcal{CR}[\mathcal{L}_{\text{XOR}}] = \mathcal{F}_{\mathcal{L}_{\text{XOR}}}[\text{Domain-Neutral}]$
- There is no $\mathcal{L}$ that characterises anonymous procedures.

Other results:

Umberto Grandi and Ulle Endriss. Lifting Rationality Assumptions in Binary Aggregation. In *Proceedings of the 24th AAAI Conference on Artificial Intelligence (AAAI-2010)*, July 2010.

The Majority Rule

Proposition

The majority rule is CR with respect to IC if and only if IC is equivalent to a conjunction of clauses of size $\leqslant 2$.

Proof: Translate binary aggregation into judgment aggregation and use a theorem by Nehring and Puppe (majority rule is consistent iff there are no minimally inconsistent subsets of size less than 2 in the agenda).

Common feature of all paradoxes seen in Part 1:
clauses of size 3 are not lifted by majority

Impossibility Result in Preference Aggregation

Call a SWF imposed if there are two alternatives x and y such that x is collectively preferred to y in every profile:

Proposition

Any anonymous, independent and monotonic SWF for more than 3 alternatives and 2 individuals is imposed.

Proof sketch:

- to any A, I, M social welfare function corresponds an A, I and M binary aggregation procedure
- characterisation result: A, I, M aggregation procedures lift IC iff IC has a certain property or the acceptance quota for at least one issue is zero (i.e., the procedure is imposed)
- the integrity constraints $\text{IC}_<$ for preference aggregation do not satisfy this property

Part III:
Conclusion and future work

Generalised Dictatorship

A generalised dictatorship copies the ballot of a (possibly different) individual (aka local dictatorships, positional dictatorships, rolling dictatorships):

Proposition

F is CR with respect to all IC in $\mathcal{L}_{PS}$ if and only if F is a generalised dictatorship

This class includes:

- Classical dictatorships $F(B_1, \ldots, B_n) = B_i$ for $i \in \mathcal{N}$
- Distance-based generalised dictatorship: map $(B_1, \ldots, B_n)$ to the ballot B_i that minimises the sum of the Hamming distance to the others (a sort of "median voter"...). An interesting procedure!

Future work: how this procedure relates to existing distance based procedures?

General Combinatorial Domains

- Domain of aggregation: $\mathcal{D} = D_1 \times \cdots \times D_m$
- Language on atomic propositions $\{x_j = a \mid j = 1, \ldots, m$ and $a \in D_j\}$
- Integrity constraint for approval voting:

$$\bigwedge_{j=1,\ldots,m} \bigvee_{a \in D_j} x_j = a$$

IC for majority rule: more complex...

Examples of an interesting constraints (voting for commitees):

- $\neg(x_1 = a \wedge x_2 = c \wedge x_3 = d)$ – incompatibility of a committee
- $x_1 = a \wedge x_2 = a \rightarrow (x_3 = b \vee x_3 = c \vee x_4 = d)$ – minority representation

Future work: Characterisation results can be used for this framework to produce interesting results? What is the notion of paradox here?

Preferential Dependencies

Several representation languages for preferences over combinatorial domains express preferential dependencies:

- CP-nets (Boutilier et Al, 2004)
- Prioritized goals (Lang, 2004)
- PL (Bienvenue, Lang and Wilson, 2010)
- ...

Preferencial dependencies are binary objects (e.g., graphs)

Future work:

- Aggregate dependencies using binary aggregation (taking care of eventual ICs on individual preferencial dependencies)
- Formulate a clustering of issues or a voting agenda for sequential elections based on collective preferential dependencies

Conclusion

Binary aggregation with integrity constraints:

- language to express rationality assumptions
- concept of collective rationality with respect to an IC
- general framework for paradoxes (e.g., Condorcet and doctrinal)
- characterisation results relating languages and axioms
- new proof method for impossibility results: clash between axioms and IC

Bigger picture:

- Axiomatic Method: derive (im)possibility results for specific domains and specific axioms
- "AI approach": devise a machinery to reason about different application-specific domains, assumptions, axioms