Lifting Rationality Assumptions in Binary Aggregation

Umberto Grandi   Ulle Endriss

Institute for Logic, Language and Computation
University of Amsterdam

25 June 2010

Binary Aggregation

Social Choice Theory deals with the aggregation of:

Ballots in Referendum	A sequence of binary (YES/NO) answers to a set of questions
Judgments	A set of propositional formulas J represented as its characteristic function: $J(\varphi) = 1$ if $\varphi \in J$
Preferences	A linear order $a > b > c$ is a binary relation
...	

Individuals submit binary ballots of different forms and an aggregation procedure merge the individual choices into a collective one.

Lifting Rationality Assumptions

There is also a common structure underlying several theoretical studies:

Axiomatic Method	**Collective Rationality**
Independence, Neutrality, Monotonicity...	Transitivity, Completeness, Consistency of judgments...

Lifting Rationality Assumptions

There is also a common structure underlying several theoretical studies:

Axiomatic Method		**Collective Rationality**
Independence, Neutrality, Monotonicity...	$\Leftrightarrow$	Transitivity, Completeness, Consistency of judgments...

Depending on the situation at hand the collective rationality requirement changes: We describe them with a propositional language.

- AI perspective: depending on the shape of the collective rationality requirement, **which axioms are necessary to guarantee collective rationality**?

- Unifying framework for paradoxes;

An Outline

1. Introduce the framework of binary aggregation
2. Propositional languages for collective rationality assumptions
3. Results: Which axioms lift rationality assumptions of a certain shape?
4. Future directions of research

Binary Aggregation

Ingredients:

- A finite set N of individuals
- A finite set $\mathcal{I} = \{1, \ldots, m\}$ of issues
- A boolean *combinatorial domain*: $\mathcal{D} = D_1 \times \cdots \times D_m$ with $|D_i| = 2$

Definition

An aggregation procedure is a function $F : \mathcal{D}^N \to \mathcal{D}$ mapping each profile of ballots $\underline{B} = (\underline{B}_1, \ldots, \underline{B}_n)$ to an element of the domain $\mathcal{D}$.

Examples:

- Pairwise preference aggregation: issues are '$a > b$' for all alternatives a, b;
- Judgment aggregation: one issue for every formula in the agenda;
- Voting for referendums;
- etcetc...

Wilson (1975), Rubinstein and Fishburn (1986), Dokow and Holzman (2005)

Languages for Integrity Constraints

Properties of ballots can be expressed using a simple propositional language:

- One propositional symbol for every issue: $PS = \{p_1, \ldots, p_m\}$
- $\mathcal{L}_{PS}$ is the propositional language (closing under connectives $\wedge$, $\vee$,$\neg$, $\rightarrow$) generated from the set of atoms PS

An element of the domain $\mathcal{D} = \{0,1\}^m$ (a ballot) is a model for $\mathcal{L}_{PS}$.

Example: Three bills with budget constraint.

Two bills between b_1, b_2 and b_3 have to be approved/disproved
Budget constraint: $\text{IC} = \neg(p_1 \wedge p_2 \wedge p_3)$, there is budget only for two of them

Individual 1 submit $B_1 = (1,1,0)$:	it satisfies IC ✓
Individual 2 submit $B_2 = (0,1,1)$:	$B_2 \models$ IC ✓
Individual 3 submit $B_3 = (1,0,1)$:	$B_3 \models$ IC ✓

Majority aggregation approves all three bills: IC not satisfied!

Collective Rationality

Definition

A language for integrity constraints *over a domain* $\mathcal{D}$ *is a subset* $\mathcal{L} \subset \mathcal{L}_{PS}$.

IC of previous examples in the language $\mathcal{L}_{\textit{3-clauses}}$: disjunction of Lent 3.
Examples are: $\mathcal{L}_{\leftrightarrow}$ the language of positive bi-implications,
the language of conjunctions $\mathcal{L}_{cubes}$, ...

Collective Rationality

Definition

A language for integrity constraints over a domain $\mathcal{D}$ is a subset $\mathcal{L} \subset \mathcal{L}_{PS}$.

Call integrity constraint any formula IC $\in \mathcal{L}_{PS}$
$B \in \mathcal{D}$ satisfies the constraint iff $B \models$ IC

Every individual satisfies the same rationality assumption IC...
...what about the collective outcome?

Definition

Call an aggregation procedure F collectively rational for IC $\in \mathcal{L}_{PS}$ if for all profiles $\underline{B}$ such that $\underline{B}_i \models$ IC for all $i \in N$ we have that $F(\underline{B}) \models$ IC.

F is collectively rational if it lifts the rationality assumption given by the formula IC from the individual to the collective level.

Axioms for Binary Aggregation

Aggregation procedures have been studied using the axiomatic method, listing axioms as desirable properties of the functions.

Classical axioms from social choice theory can be translated in this framework:

Unanimity (U): For any profile $\underline{B} \in X^N$ and any $x \in \{0, 1\}$, if $\underline{B}_{i,j} = x$ for all $i \in N$, then $F(\underline{B})_j = x$.

Independence (I): For any issue $j \in \mathcal{I}$ and any two profiles $\underline{B}, \underline{B}' \in X^N$, if $\underline{B}_{i,j} = \underline{B}'_{i,j}$ for all $i \in N$, then $F(\underline{B})_j = F(\underline{B}')_j$.

Not all axioms have a natural translation (e.g., non-dictatorship), and new axioms are also defined like the following generalisation of May's neutrality:

Domain-Neutrality ($\mathrm{N}^{\mathcal{D}}$): For any two issues $j, j' \in \mathcal{I}$ and any profile $\underline{B} \in X^N$, if $\underline{B}_{i,j} = 1 - \underline{B}_{i,j'}$ for all $i \in N$, then $F(\underline{B})_j = 1 - F(\underline{B})_{j'}$.

Results (template)

Different lists of axioms AX define classes of functions:

$$\mathcal{F}[\mathsf{AX}] \;=\; \{F : \mathcal{D}^N \rightarrow \mathcal{D} \mid F \text{ satisfies AX}\}$$

Axioms are domain dependent, domains of interest are defines via IC:

$$\mathcal{F}_{\mathcal{L}}[\mathsf{AX}] \;=\; \{F : \mathcal{D}^N \rightarrow \mathcal{D} \mid F_{\restriction \mathrm{Mod}(\mathsf{IC})^N} \text{ sat. AX for all } \mathsf{IC} \in \mathcal{L}\}$$

Example (trivial):

If the domain X contains a single ballot
then the restricted majority rule $F^{\mathsf{maj}}_{\restriction X}$ is dictatorial.

For instance: IC $= \{\neg p_2 \wedge p_2 \wedge p_3\}$ with three issues.

Results (template)

Different lists of axioms AX define classes of functions:

$$\mathcal{F}[\mathsf{AX}] \;=\; \{F : \mathcal{D}^N \rightarrow \mathcal{D} \mid F \text{ satisfies AX}\}$$

Axioms are domain dependent, domains of interest are defines via IC:

$$\mathcal{F}_{\mathcal{L}}[\mathsf{AX}] \;=\; \{F : \mathcal{D}^N \rightarrow \mathcal{D} \mid F_{\restriction \mathrm{Mod}(\mathsf{IC})^N} \text{ sat. AX for all IC} \in \mathcal{L}\}$$

The class of procedures that lift integrity constraint in a given language is:

$$\mathcal{CR}[\mathcal{L}] \;=\; \{F : \mathcal{D}^N \rightarrow \mathcal{D} \mid F \text{ is CR for all IC} \in \mathcal{L}\}$$

What we seek are results of this form:

$$\mathcal{CR}[\mathcal{L}] = \mathcal{F}_{\mathcal{L}}[\mathsf{AX}]$$

Positive Results

Proposition

$\mathcal{CR}[\textit{cubes}] = \mathcal{F}_{\textit{cubes}}[\mathrm{U}]$.

Cubes are lifted iff the procedure satisfies unanimity.

Proof.
If a profile $\underline{B}$ is defined by a cube φ, then all individuals agree on certain issues. By unanimity so does the collective outcome, so $F(\underline{B}) \models \varphi$.

Viceversa if $\underline{B}$ is unanimous then there exists a cube φ such that for all i $B_i \models \varphi$. Therefore collective rationality for cubes implies unanimity. □

Since $\mathcal{F}_{\textit{cubes}}[\mathrm{U}] = \mathcal{F}[\mathrm{U}]$ this result can be interpreted as a characterisation of unanimity in terms of collective rationality with respect to cubes.

Positive Results

Proposition

$\mathcal{CR}[\mathit{cubes}] = \mathcal{F}_{\mathit{cubes}}[\mathsf{U}]$.

Cubes are lifted iff the procedure satisfies unanimity.

Call $\mathcal{L}_{\not\leftrightarrow}$ the language of *negative* bi-implications (i.e. of the form $p_i \leftrightarrow \neg p_j$):

Proposition

$\mathcal{CR}[\mathcal{L}_{\not\leftrightarrow}] = \mathcal{F}_{\mathcal{L}_{\not\leftrightarrow}}[\mathsf{N}^{\mathcal{D}}]$.

Analogous result for positive bi-implication:

Proposition

$\mathcal{CR}[\mathcal{L}_{\leftrightarrow}] = \mathcal{F}_{\mathcal{L}_{\leftrightarrow}}[\mathsf{N}^{\mathcal{I}}]$.

Generalised Dictatorship

How is the class of collectively rational procedures for trivial constraints?

$$\mathcal{CR}[\bot] = \mathcal{CR}[\top] = \mathcal{F}.$$

And how is a procedure that can lift any sort of propositional constraint?

$$F \in \mathcal{CR}[\mathcal{L}_{PS}] \quad \Leftrightarrow \quad \textit{F is a generalised dictatorship}$$
$$\textit{i.e., } \exists g : \mathcal{D}^N \to N \textit{ such that } F(\underline{B}) = \underline{B}_{g(\underline{B})}.$$

A procedure that is collectively rational for any integrity constraint must copy the ballot of an individual.

Negative Results

For the axioms of independence and anonymity we prove a negative result:

Proposition

There is no language $\mathcal{L} \subseteq \mathcal{L}_{PS}$ *such that* $\mathcal{CR}[\mathcal{L}] = \mathcal{F}_{\mathcal{L}}[\text{I}]$.

Proof sketch: If $\mathcal{L}$ contains a falsifiable formula, the procedure constantly giving the falsifying assignment is independent but nor collectively rational.

Proposition

There is no language $\mathcal{L} \subseteq \mathcal{L}_{PS}$ *such that* $\mathcal{CR}[\mathcal{L}] = \mathcal{F}_{\mathcal{L}}[\text{A}]$.

Proof sketch: Dictatorships are always collectively rational, but not anonymous.

Anonymity and Independence are inter-profile requirements:
collective rationality can only characterise intra-profile axioms.

The Majority Rule

A more concrete questions: which IC are lifted by the majority rule F_{maj}?

- F_{maj} satisfies U, $N^{\mathcal{D}}$ and $N^{\mathcal{I}}$ $\overset{\text{by Prop...}}{\Rightarrow}$ F_{maj} lifts *cubes*, $\mathcal{L}_{\leftrightarrow}$ and $\mathcal{L}_{\not\leftrightarrow}$.
- F_{maj} satisfies Independence and Anonymity (and Monotonicity).

  Using results on quota rules by Dietrich and List (2007) we can prove that:

F_{maj} lifts every clause of size two.
"3 bills with budget constraint" is a counterexample for size 3.

As a corollary: if there are only two issues then
majority rule is always collectively rational.

Conclusion and Future Work

In this work we have presented:

- a language to express rationality assumptions as integrity constraints IC over domains in binary aggregation;
- the concept of collective rationality of an aggregator wrt. a constraint IC;
- characterisation results for different propositional languages $\mathcal{L}$:

  Which properties of the aggregator guarantee that a certain IC is lifted.

This work can be extended in a number of ways:

1. using logic not only as a language but also as a tool to derive (im)possibility theorems for different set of axioms;
2. extend the language for full combinatorial domains;
3. characterise classical axioms in terms of collective rationality;
4. study aggregation of more complex logical structures.

More on Future Work

1. If $\mathcal{L}_{\text{pref}}$ is the language of linear orders, Arrow's Theorem takes the following form:

$$\mathcal{CR}[\mathcal{L}_{pref}] \cap \mathcal{F}_{\mathcal{L}_{pref}}[\text{U,I,NDIC}] = \emptyset,$$

2.

3. Is it true that every intra-profile axiom can be written as a collective rationality requirement (i.e., there exists a $\mathcal{L}$ such that $\mathcal{CR}[L] = \mathcal{F}_{\mathcal{L}}[\text{AX}]$)?

4. Rationality requirements IC written in FO, SO, Modal Logic (logical omniscience, introspection...):

$$\text{If } \mathcal{M}_i \models \text{IC for } i \in I \text{ does } F(\mathcal{M}_1, \ldots, \mathcal{M}_n) \models \text{IC?}$$