# Complexity of Judgment Aggregation: Safety of the Agenda

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12 May 2010

# Computational Complexity and Judgment Aggregation

Judgment aggregation is relevant for multi-agent systems

**JA** $\Longleftarrow$ $\Longrightarrow$ **MAS**

Societies of agents have to aggregate their opinions/judgments

Is it feasible?
- Theoretical analysis
- Computational Complexity

| | |
|---|---|
| "Greek bonds are junk" | "$p$" |
| "If Greek bonds are junk then they should be sold" | "$p \rightarrow q$" |
| "Greek bonds should be sold" | "$q$" |

| | $p$ | $p \rightarrow q$ | $q$ |
|---|---|---|---|
| AG1: | Yes | No | No |
| AG2: | Yes | Yes | Yes |
| AG3: | No | Yes | No |
| Majority: | Yes | Yes | No |

Each agent express a consistent judgment but majority is contradictory!

# Outline

1. Basic framework of judgment aggregation for propositional logic:
   - Distinction between logical and syntactical properties of judgment sets;
2. Safety of the agenda:
   - Is consistency guaranteed for all aggregators in a certain class?
3. Characterisation results for safe agendas.
4. Complexity results: checking safety is $\Pi_p^2$-complete.
   - Reduce the problem to checking agenda properties;
   - Use quantified boolean formulas to prove membership;
   - Reduction from co-QSAT$_2$ to prove completeness.

# Basic Definitions

A set $N$ of individuals expressing judgments on a set of correlated propositions:

**Definition**

*An* agenda *is a finite subset of propositional formulas* $\Phi \subseteq \mathcal{L}_{PS}$ *closed under complementation and not containing double negations.*

A judgment set on an agenda $\Phi$ is a subset $J \subseteq \Phi$. Call $J$:

**Complete**: if for all $\alpha \in \Phi$ either $\alpha$ or its complement is in $J$. (easy)

**Complement-free**: $\alpha$ and its complement $\sim \alpha$ are never both in $J$. (easy)

**Consistent**: there is an assignment to make all formulas in $J$ true. (not easy!)

Call $J(\Phi)$ the set of all consistent and complete judgment sets.

**Definition**

*An* aggregation procedure *for agenda* $\Phi$ *and a set* $N$ *of* $n$ *individuals is a function* $F : J(\Phi)^n \rightarrow 2^{\Phi}$.

# Axioms for Aggregation Procedures I

Aggregation procedures have been studied using the axiomatic method, listing axioms as desirable properties of the functions:

**Weak Rationality** (WR): $F$ is complete and complement-free.

These properties are all computationally easy.

Other standard axioms:

**Unanimity** (U): If $\varphi \in J_i$ for all $i$ then $\varphi \in F(\mathbf{J})$.

**Anonymity** (A): $F$ is symmetric with respect to individuals.

**Neutrality** (N): If $\varphi$ and $\psi$ share the same pattern of individuals' judgments then their outcome must be the same.

**Independence** (I): The outcome of $F$ over $\varphi$ depends only on the individuals' judgments over $\varphi$.

**Systematicity** (S)=(N)+(I).

Two forms of monotonicity for independent ($\mathsf{M}^{\mathsf{I}}$) and neutral procedures ($\mathsf{M}^{\mathsf{N}}$).

## Classes of Aggregation Procedures

Such axioms can be used to define different classes of aggregation procedures:

$$\begin{array}{c}\text{Set of axioms AX}\\ \text{Agenda } \Phi\end{array} \quad \Rightarrow \quad \begin{array}{c}\text{Class of functions}\\ \mathcal{F}_\Phi[\mathsf{AX}]\end{array}$$

**Proposition (representation result)**

$\mathcal{F}_\Phi[\mathsf{WR}, \mathsf{A}, \mathsf{S}, \mathsf{M}^{\mathsf{I}}] = \mathcal{F}_\Phi[\mathsf{WR}, \mathsf{A}, \mathsf{N}, \mathsf{M}^{\mathsf{N}}]$ *is the majority rule.*

We will concentrate on various weakening of the axiomatisation of majority:

$$\mathcal{F}_\Phi[\mathsf{WR}, \mathsf{A}, \mathsf{S}] \qquad \mathcal{F}_\Phi[\mathsf{WR}, \mathsf{A}, \mathsf{N}]$$

$$\mathcal{F}_\Phi[\mathsf{WR}, \mathsf{A}, \mathsf{I}] \qquad \mathcal{F}_\Phi[\mathsf{A}, \mathsf{S}, \mathsf{M}^{\mathsf{I}}]$$

# Safety of the Agenda I

Call a procedure consistent if $F(\mathbf{J})$ is consistent for all profiles $\mathbf{J}$ in the domain.

Possibility (characterisation) results:

There exists a consistent $F$ in $\mathcal{F}_\Phi$[AX] $\Leftrightarrow$ $\Phi$ has a certain property.

These results characterise agendas for which consistent aggregation is possible.

E.g., Nehring and Puppe (2006), Dietrich and List (2007), Dokow and Holzman (2005).

Safety (characterisation) results:

All $F$ in $\mathcal{F}_\Phi$[AX] are consistent $\Leftrightarrow$ $\Phi$ has a certain property.

These results characterise agendas for which consistent aggregation is guaranteed.

# Safety of the Agenda II

**Definition**

*An agenda $\Phi$ is safe with respect to a class of aggregation procedures $\mathcal{F}_\Phi$ if every function in $\mathcal{F}_\Phi$ is consistent.*

**Characterisation Result I**

*$\Phi$ is safe for $\mathcal{F}_\Phi[\mathrm{WR}, \mathrm{A}, \mathrm{S}, \mathrm{M}^{\mathrm{I}}]$ iff every inconsistent subset of $\Phi$ contains an inconsistent subset of size 2 (median property* MP*).*

**Proof.**

Majority rule is the only function in $\mathcal{F}_\Phi[\mathrm{WR}, \mathrm{A}, \mathrm{S}, \mathrm{M}^{\mathrm{I}}]$.

Therefore proving safety $=$ proving possibility.

The proof follows from Nehring and Puppe (2006). □

# Agenda Properties

An agenda $\Phi$ satisfies:

- the $k$-median property ($k$MP) for $k \geqslant 2$, if every inconsistent subset of $\Phi$ has itself an inconsistent subset of size at most $k$ (2MP=MP).
- the syntactic simplified median property (SSMP), if every nontrivially inconsistent subset of $\Phi$ has an inconsistent subset of the form $\{\varphi, \neg\varphi\}$.

**Characterisation Result II**

$\Phi$ *is safe for* $\mathcal{F}_\Phi[\mathrm{WR}, \mathrm{A}, \mathrm{I}]$ *if and only* $\Phi$ *satisfies the* SSMP.

We have similar results for the class of systematic or neutral procedures (SMP), and a bound on the acceptance threshold for quota rules ($k$MP).

# Computational Complexity: Identifying the problem

Our problem:
Is an agenda $\Phi$ safe for the class $\mathcal{F}_\Phi[\text{AX}]$?

We can define the following problem:

| SAFETY[AX] | |
|---|---|
| **Instance:** | Agenda $\Phi$ of size $n$ |
| **Problem:** | Is $\Phi$ safe for $\mathcal{F}_\Phi[\text{AX}]$? |

Using our characterisation, e.g. for $\mathcal{F}_\Phi[\text{WR}, \text{A}, \text{I}]$, this boils down to checking the agenda property:

| SSMP | |
|---|---|
| **Instance:** | Agenda $\Phi$ of size $n$ |
| **Problem:** | Does $\Phi$ satisfy SSMP? |

# A Complexity Class: $\Pi_2^p$

A problem is in $\Pi_2^p$ if it is solved by co-NP algorithm using a polynomial number of calls to an NP-oracle. A complete problem for this class is the following:

| coQSAT$_2$ | |
|---|---|
| **Instance:** | Quantified boolean formula (QBF) $\forall x_1, \ldots, x_r. \exists y_1 \ldots, y_s. \varphi(x_1, \ldots, x_r, y_1, \ldots, y_s)$ |
| **Problem:** | Is the QBF satisfiable (i.e., true)? |

We can reformulate the SSMP in the satisfiability of a particular QBF, thus proving membership to $\Pi_2^p$.

# Computational Complexity of SSMP

**Theorem**

SAFETY[WR, A, I] *is* $\Pi_2^p$*-complete.*

**Proof.**

- By characterisation this problem is equivalent to SSMP;
- SSMP is in $\Pi_2^p$ (membership): reduction to co-$QSAT_2$.
- SSMP is $\Pi_2^p$-complete (hardness): by reduction from co-$QSAT_2$. For every QBF $\Delta$ we can build an agenda $\Phi_\Delta$ such that if $\Phi_\Delta$ satisfies the SSMP iff the initial QBF $\Delta$ is satisfiable.

□

Checking safety for the class of neutral procedures, systematic procedures, for the majority rule and for quota rules is also $\Pi_2^p$-complete.

# Conclusions and Future Work

In conclusion:

- To the best of our knowledge the safety of the agenda is a new problem relevant to an application-driven study of JA.
- We have proved several characterisation results to recognise safe agendas for different classes of functions.
- We have proved that this problem is $\Pi_2^p$-complete for all classes of functions we considered.

For more questions: Blue poster session B-85

Future work includes:

- Study the complexity of strategic manipulation in JA.
- Introduce and study the safety of a profile.