Complexity of Judgment Aggregation: Safety of the Agenda

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Computational Complexity and Judgment Aggregation

Judgment aggregation is relevant for multi-agent systems

JA $\Longleftarrow\ \Longrightarrow$ **MAS**

Societies of agents have to aggregate their opinions/judgments

Is it feasible?

Computational complexity provides a first measure.

Feedback for JA:

- inspired by work on complexity of manipulation in voting theory;
- precise formal framework;
- new problems arise: safety of the agenda.

Outline

1. Basic framework of judgment aggregation in propositional logic.
2. Safety of the agenda: is consistency guaranteed for all aggregators?
3. Characterisation results for agendas: when is safety guaranteed?
4. Complexity results: checking safety is Π_p^2-complete.

Basic Definitions

A set N of individuals expressing judgments on a set of correlated propositions:

Definition

An agenda is a finite subset of propositional formulas $\Phi \subseteq \mathcal{L}_{PS}$ closed under complementation and not containing double negations.

A judgment set on an agenda Φ is a subset $J \subseteq \Phi$. Call J:

Complete: if for all $\alpha \in \Phi$ either α or its complement is in J.

Complement-free: α and its complement are never both in J.

Consistent: there exists an assignment that makes all formulas in J true.

Call $J(\Phi)$ the set of all consistent and complete judgment sets.

Definition

An aggregation procedure for agenda Φ and a set N of n individuals is a function $F : J(\Phi)^n \to 2^\Phi$.

Axioms for Aggregation Procedures I

Aggregation procedures have been studied using the axiomatic method, listing axioms as desirable properties of the functions:

Weak Rationality (WR): $F(\mathbf{J})$ is complete and complement-free.

Other standard axioms are Anonymity (A), Neutrality (N), Independence (I), Systematicity (S)=(N)+(I).

We introduce two distinct forms of monotonicity:

I-Monotonicity (M^{I}): For any φ in the agenda Φ and profiles $\mathbf{J} = (J_1, \ldots, J_i, \ldots, J_n)$ and $\mathbf{J}' = (J_1, \ldots, J_i', \ldots, J_n)$ in $J(\Phi)$, if $\varphi \notin J_i$ and $\varphi \in J_i'$, then $\varphi \in F(\mathbf{J}) \Rightarrow \varphi \in F(\mathbf{J}')$.

N-Monotonicity (M^{N}): For any φ, ψ in the agenda Φ and profile $\mathbf{J}$ in $J(\Phi)$, if $\varphi \in J_i \Leftrightarrow \psi \in J_i$ for all $i \neq k$ and $\varphi \notin J_k$ and $\psi \in J_k$, then $\varphi \in F(\mathbf{J}) \Rightarrow \psi \in F(\mathbf{J})$.

Classes of Aggregation Procedures

Such axioms can be used to define different classes of aggregation procedures:

$$\begin{array}{c}\text{Set of axioms AX}\\ \text{Agenda } \Phi\end{array} \quad \Rightarrow \quad \begin{array}{c}\text{Class of functions}\\ \mathcal{F}_\Phi[\mathrm{AX}]\end{array}$$

Proposition (representation result)

$\mathcal{F}_\Phi[\mathrm{WR}, \mathrm{A}, \mathrm{S}, \mathrm{M}^\mathrm{I}] = \mathcal{F}_\Phi[\mathrm{WR}, \mathrm{A}, \mathrm{N}, \mathrm{M}^\mathrm{N}] = \mathcal{F}_\Phi[\mathrm{WR}, \mathrm{A}, \mathrm{I}, \mathrm{M}^\mathrm{I}]$ *is the majority rule.*

We will concentrate on various weakening of the axiomatisation of majority:

$$\mathcal{F}_\Phi[\mathrm{WR}, \mathrm{A}, \mathrm{S}] \qquad \mathcal{F}_\Phi[\mathrm{WR}, \mathrm{A}, \mathrm{N}]$$

$$\mathcal{F}_\Phi[\mathrm{WR}, \mathrm{A}, \mathrm{I}] \qquad \mathcal{F}_\Phi[\mathrm{A}, \mathrm{S}, \mathrm{M}^\mathrm{I}]$$

Similar representation results can be proved for these classes.

E.g., $\mathcal{F}_\Phi[\mathrm{A}, \mathrm{S}, \mathrm{M}^\mathrm{I}]$ is the class of uniform quota rules, (Dietrich and List, 2007)

Safety of the Agenda I

Call a function consistent if $F(\mathbf{J})$ is consistent for all profiles $\mathbf{J}$ in the domain.

Possibility (characterisation) results:

There exists a consistent F in $\mathcal{F}_\Phi$[AX] $\Leftrightarrow$ Φ has a certain property.

These results characterise agendas for which consistent aggregation is possible.

E.g., Nehring and Puppe (2006), Dietrich and List (2007), Dokow and Holzman (2005).

Safety (characterisation) results:

All F in $\mathcal{F}_\Phi$[AX] are consistent $\Leftrightarrow$ Φ has a certain property.

These results characterise agendas for which consistent aggregation is guaranteed.

Safety of the Agenda II

Definition

An agenda Φ *is* *safe* *with respect to a class of aggregation procedures* $\mathcal{F}_\Phi$ *if every function in* $\mathcal{F}_\Phi$ *is consistent.*

A discursive dilemma is a counterexample for the safety of an agenda.

Proposition

Φ *is safe for* $\mathcal{F}_\Phi[\mathrm{WR}, \mathrm{A}, \mathrm{S}, \mathrm{M}^\mathrm{I}]$ *iff* Φ *satisfies the median property* MP.

Proof.

Majority rule is the only function in $\mathcal{F}_\Phi[\mathrm{WR}, \mathrm{A}, \mathrm{S}, \mathrm{M}^\mathrm{I}]$, therefore proving the safety of Φ equals proving possibility of consistent aggregation. □

Agenda Properties

An agenda Φ satisfies:

- the k-median property (kMP) for $k \geqslant 2$, if every inconsistent subset of Φ has itself an inconsistent subset of size at most k (2MP=MP).
- the simplified median property (SMP), if every nontrivially inconsistent subset of Φ (i.e., not containing any contradictory formula) has itself an inconsistent subset of the form $\{\varphi, \psi\}$ with $\models \varphi \leftrightarrow \neg\psi$.
- the syntactic simplified median property (SSMP), if every nontrivially inconsistent subset of Φ has an inconsistent subset of the form $\{\varphi, \neg\varphi\}$.

$$\text{SSMP} \Rightarrow \text{SMP} \Rightarrow \text{MP} \Rightarrow \text{kMP}$$

Safety Characterisation Results

We prove the following:

Proposition

Φ is safe for $\mathcal{F}_\Phi[\mathrm{WR}, \mathrm{A}, \mathrm{S}]$ if and only Φ satisfies the SMP.

Φ is safe for $\mathcal{F}_\Phi[\mathrm{WR}, \mathrm{A}, \mathrm{N}]$ if and only Φ satisfies the SMP *and does not contain a contradictory formula.*

Φ is safe for $\mathcal{F}_\Phi[\mathrm{WR}, \mathrm{A}, \mathrm{I}]$ if and only Φ satisfies the SSMP.

Reformulating work proved by Dietrich and List (2007) we obtain:

Proposition

*Let $k \geqslant 2$. An agenda Φ is safe for the class of uniform quota rules F_m for n individuals satisfying $m > n - \frac{n}{k}$ if and only if Φ satisfies the k*MP

.

Computational Complexity: what is a problem?

Computational Complexity provides a measure of how difficult a *problem* is, by measuring the amount of resources needed to solve it: space or time.

What is a *problem*?

Instance/Input (size)

$\Downarrow$

Question

$\Downarrow$

Answer: Yes/No (in a certain time)

Important remarks:

- Complexity is not a property of a particular procedure to solve a problem.
- Complexity is not a property of an instance of a problem: it is a function that links the size of an input to the time required to compute the answer.

Safety of the Agenda: Identifying the problem

Our problem:
Is an agenda Φ safe for the class $\mathcal{F}_\Phi[\mathsf{AX}]$?

We can define the following problem:

SAFETY[AX]	
Instance:	Agenda Φ of size n
Problem:	Is Φ safe for $\mathcal{F}_\Phi[\mathsf{AX}]$?

Using our characterisation, e.g. for $\mathcal{F}_\Phi[\mathsf{WR}, \mathsf{A}, \mathsf{I}]$, this boils down to checking the agenda property:

SSMP	
Instance:	Agenda Φ of size n
Problem:	Does Φ satisfy SSMP?

Complexity Classes: Reductions

How to compare two problems?

If P_1 can be "easily" translated into problem P_2
then P_2 is at least as difficult as P_1.

Idea: there exists a reduction from $P_1 \leqslant_r P_2$
s.t. $x \in P_1 \Leftrightarrow r(x) \in P_2$

Every problem P defines the class C_P of problems that can be reduced to it.

Famous complexity classes:

- P is the class of problems solvable in polynomial time.
- NP is the class of problems solvable in non-deterministic polynomial time.

Complexity Classes: Complete Problems

Every (interesting) class C has complete problems.

P is complete for C if C=C$_P$
They are reference points that fully characterise their class.

SAT	
Instance:	Propositional formula φ
Problem:	Is φ satisfiable?

SAT is **NP**-complete (Cook, 1971).

We are interested in Π_2^p, the class defined by the following satisfaction problem:

coQSAT$_2$	
Instance:	Quantified boolean formula (QBF) $\forall x_1, \ldots, x_r . \exists y_1 \ldots, y_s . \varphi(x_1, \ldots, x_r, y_1, \ldots, y_s)$
Problem:	Is the QBF satisfiable (i.e., true)?

Computational Complexity of SSMP

Checking safety turns out to be quite hard:

Theorem

SAFETY[WR, A, I] *is* Π_2^p*-complete.*

Proof.

- By characterisation this problem is equivalent to SSMP;
- SSMP is Π_2^p-complete: reduction from co-QSAT$_2$.

□

Complexity results

The following theorem is an overview of our complexity results:

Theorem

Checking the safety of an agenda is Π_2^p-complete for any of these classes of aggregation procedures:

1. *the majority rule, corresponding to* $\mathcal{F}_\Phi[\mathsf{WR}, \mathsf{A}, \mathsf{S}, \mathsf{M}^{\mathsf{I}}]$, $\mathcal{F}_\Phi[\mathsf{WR}, \mathsf{A}, \mathsf{I}, \mathsf{M}^{\mathsf{I}}]$, *and* $\mathcal{F}_\Phi[\mathsf{WR}, \mathsf{A}, \mathsf{N}, \mathsf{M}^{\mathsf{N}}]$*;*
2. *systematic rules:* $\mathcal{F}_\Phi[\mathsf{WR}, \mathsf{A}, \mathsf{S}]$*;*
3. *neutral rules:* $\mathcal{F}_\Phi[\mathsf{WR}, \mathsf{A}, \mathsf{N}]$*;*
4. *independent rules:* $\mathcal{F}_\Phi[\mathsf{WR}, \mathsf{A}, \mathsf{I}]$*;*
5. *for any $k \geqslant 2$, the class of uniform quota rules F_m with $m > n - \frac{n}{k}$, where n is the number of individuals.*

Proof.

SSMP, SMP, kMP for $k \geqslant 2$, are all Π_2^p-complete. □

Conclusions and Future Work

In conclusion:

- To the best of our knowledge the safety of the agenda is a new problem relevant to an application-driven study of JA.
- We have proved several characterisation results to recognise safe agendas for different classes of functions.
- We have proved that this problem is Π_2^p-complete for all classes of functions we considered.

Guaranteeing safety for various classes of functions is a very restrictive requirement, and checking these simple properties is computationally hard.

Future work includes:

- Study the complexity of the safety problem for other classes of functions.
- Introduce and study the safety of a profile.
- Study the complexity of strategic manipulation in JA.