On Compatible Multi-issue Group Decisions

Umberto Grandi

Institute for Logic, Language and Computation
University of Amsterdam

14 March 2012

(joint work with Gabriella Pigozzi)

Multiple Election Paradox

The outcome of the majority rule is the acceptance of all three issues, even if this combination was not voted for by any of the individuals:

	A	B	C
Voter 1	1	0	1
Voter 2	0	1	1
Voter 3	1	1	0
Maj	1	1	1

Even if the outcome is consistent, it is not supported by any of the individuals!

Brams, Kilgour, and Zwicker. The paradox of multiple elections. *Social Choice and Welfare*, 1998.

Consistency, Axiomatic Properties...and?

In the design of collective decision making mechanisms, consistency may be considered the first problem (see my PhD thesis).

But: agents/individuals may resists to take actions that they do not recognise as compatible/representative of the individuals opinions.

Standard axioms in judgment/binary aggregation are not concerned with this problem (except for unanimity).

We want to provide a first investigation of such a problem:

How to measure the compatibility of an aggregation procedure?
How to quantify the relation between collective outcome and individual ballots?

Outline

- The framework of binary aggregation
  - New! Generalisation to non-resolute aggregation procedures
  - New! Generalisation to allow abstentions
- Several definitions of compatibility
- Some old and new aggregation procedures
- Another idea: combine consistency and compatibility?

Binary Aggregation

Ingredients:

- A finite set N of individuals
- A finite set $\mathcal{I} = \{1, \ldots, m\}$ of issues
- Complete binary ballots are elements of $\mathcal{D} = \{0,1\}^{\mathcal{I}}$

Definition

A complete (non-resolute) aggregation procedure is a function $F : \mathcal{D}^N \to 2^{\mathcal{D}}$ *mapping each profile of ballots* $\underline{B} = (\underline{B}_1, \ldots, \underline{B}_n)$ *to a subset element of the domain* $\mathcal{D}$.

Binary Aggregation

Ingredients:

- A finite set N of individuals
- A finite set $\mathcal{I} = \{1, \ldots, m\}$ of issues
- Complete binary ballots are elements of $\mathcal{D} = \{0, 1\}^{\mathcal{I}}$

Definition

A complete (non-resolute) aggregation procedure is a function $F : \mathcal{D}^N \to 2^{\mathcal{D}}$ *mapping each profile of ballots* $\underline{B} = (\underline{B}_1, \ldots, \underline{B}_n)$ *to a subset element of the domain* $\mathcal{D}$.

- Allow abstentions: domain of aggregation is $\mathcal{D}_A = \{0, 1, A\}^{\mathcal{I}}$

Definition

An incomplete (non-resolute) aggregation procedure is a function $F : \mathcal{D}_A^N \to 2^{\mathcal{D}_A}$ *mapping each profile of possibly incomplete ballots* $\underline{B} = (\underline{B}_1, \ldots, \underline{B}_n)$ *to a subset element of the domain* $\mathcal{D}$.

Axiomatic Properties for Non-Resolute Procedures

Standard axioms work well with abstentions: A is the same as a 0 or a 1.

Exception: Unanimity on A does not entails collective abstention!

Axiomatic Properties for Non-Resolute Procedures

Standard axioms work well with abstentions: A is the same as a 0 or a 1.

Exception: Unanimity on A does not entails collective abstention!

Adaptation is required for non-resolute procedures:

Unanimity* (U*): For any profile $\underline{B}$ and any $x \in \{0, 1\}$, if $b_{i,j} = x$ for all $i \in \mathcal{N}$, then $F(\underline{B})_j = x$ for all $B \in F(\underline{B})$.

Axiomatic Properties for Non-Resolute Procedures

Standard axioms work well with abstentions: A is the same as a 0 or a 1.

Exception: Unanimity on A does not entails collective abstention!

Adaptation is required for non-resolute procedures:

Unanimity* (U*): For any profile $\underline{B}$ and any $x \in \{0,1\}$, if $b_{i,j} = x$ for all $i \in \mathcal{N}$, then $F(\underline{B})_j = x$ for all $B \in F(\underline{B})$.

Independence* (I*): For any issue $j \in \mathcal{I}$, $x \in \{0,1,A\}$ and profiles $\underline{B}, \underline{B}' \in X^{\mathcal{N}}$, if $b_{i,j} = b'_{i,j}$ for all $i \in \mathcal{N}$, then there exists a $B \in F(\underline{B})$ such that $B_j = x$ iff there exists $B' \in F(\underline{B}')$ such that $B'_j = x$.

Axiomatic Properties for Non-Resolute Procedures

Standard axioms work well with abstentions: A is the same as a 0 or a 1.

Exception: Unanimity on A does not entails collective abstention!

Adaptation is required for non-resolute procedures:

Unanimity* (U*): For any profile $\underline{B}$ and any $x \in \{0, 1\}$, if $b_{i,j} = x$ for all $i \in \mathcal{N}$, then $F(\underline{B})_j = x$ for all $B \in F(\underline{B})$.

Independence* (I*): For any issue $j \in \mathcal{I}$, $x \in \{0, 1, A\}$ and profiles $\underline{B}, \underline{B}' \in X^{\mathcal{N}}$, if $b_{i,j} = b'_{i,j}$ for all $i \in \mathcal{N}$, then there exists a $B \in F(\underline{B})$ such that $B_j = x$ iff there exists $B' \in F(\underline{B}')$ such that $B'_j = x$.

Monotonicity* (M*): For any $j \in \mathcal{I}$ and profiles $\underline{B}$ and $\underline{B}'$, if $b_{i,j} = 1$ entails $b'_{i,j} = 1$ for all $i \in \mathcal{N}$, and for some $s \in \mathcal{N}$ we have $b_{s,j} = 0$ and $b'_{s,j} = 1$, then if $F(\underline{B})$ contains a B such that $B_j = 1$ then so does $F(\underline{B}')$.

Consistency

Collective rationality:
Collective output is of the same "form" of individual ballots.

- Integrity constraints: work well with complete ballots. F is CR with respect to IC if whenever all individuals $B_i \models$ IC then $F(\underline{B}) \models$ IC.
- Subsets for the more general case:

Definition

Given a subset $\mathcal{R} \subseteq \mathcal{D}$ (respectively, $\mathcal{R} \subseteq \mathcal{D}_A$) an aggregation procedure F is collectively rational with respect to $\mathcal{R}$ if $F(\underline{B}) \in \mathcal{R}$ for all profiles $\underline{B} \in \mathcal{R}^{\mathcal{N}}$.

Compatibility for Complete Procedures

First notion, strong (or minimal, but reviewers did not like it):

Definition

An aggregation procedure satisfies strong compatibility if the outcome $F(\underline{B})$ on every profile is supported by at least one individual, i.e., all $F(\underline{B}) \subseteq \{B_1, \ldots, B_n\}$ for all profiles $\underline{B} = (B_1, \ldots, B_n)$.

Easy generalisation (e.g., majority consistency, unanimity rule...):

Definition

An aggregation procedure satisfies k-strong compatibility ($k \leqslant |\mathcal{N}|$) if for every profile all ballots in the outcome $F(\underline{B})$ are supported by at least k individuals.

Compatibility for Incomplete Procedures

Two ballots are compatible if there is no disagreement on any issue
Abstentions are O.K. (i.e., only problem if $b_j = 1 - b'_j$)

Definition

An incomplete aggregation procedure satisfies weak-compatibility if for every profile the outcome $F(\underline{B})$ is compatible with all the individual ballots $B_1, \ldots, B_n$.

Definition

An incomplete aggregation procedure satisfies k-weak compatibility ($k \leqslant |\mathcal{N}|$) if the outcome $F(\underline{B})$ is compatible with at least k individual ballots in $B_1, \ldots, B_n$.

Caution: the last notion is less strong than the first!

One Last Definition

The following notion disregards the correlation between multiple issues:

Definition

An aggregation procedure (complete or incomplete) satisfies k-compatibility over issues ($k \leqslant |\mathcal{N}|$) if, for every profile and every issue, the outcome $F(\underline{B})_j$ is compatible with at least k individuals.

The Average Voter Rule

Definition

The average voter rule (AVR) *chooses the individual ballot that minimises the sum of the Hamming distance* $H(B, B') = \sum_{j \in \mathcal{I}} |b_j - b'_j|$ *to all other individual ballots:*

$$\mathrm{AVR}(\underline{B}) \quad = \quad \underset{\{B_i | i \in \mathcal{N}\}}{\operatorname{argmin}} \sum_{s \in \mathcal{N}} H(B_i, B_s),$$

Proposition

The AVR *satisfies* $U^\star$ *and* $A^\star$*. It does not satisfy* $I^\star$ *(nor* $M^\star$*).*

The Average Voter Rule II

The average voter rule does not coincide with majority:

	p_1	p_2	p_3	p_4	p_5
B_1	1	1	0	1	1
B_2	0	1	1	0	1
B_3	1	0	1	1	0
AVR	1	1	0	1	1

Proposition (trivial)

AVR *satisfies minimal legitimacy.*

Compatible Procedures

Conflict-free rule (CFR)

The CFR associates with every profile $\underline{B}$ of incomplete ballots, the most committed ballot (i.e., the ballot with less abstentions as possible) in the set of compatible outcomes.

Proposition

The CFR *is a resolute procedure and satisfies U, I, A and M.*

Proposition

The CFR *satisfies weak compatibility.*

k-Conflict Free Rule

The k-Conflict Free Rule (k-CFR)

The k-CFR associates with every profile$\underline{B}$ the outcome of the CFR rule on all subprofiles $\underline{B}_K$ of $\underline{B}$ of size k.

Proposition

$k\text{-CFR}$ *satisfies k-compatibility and it satisfies U*, I*, A* and M*.*

Examples

Let k=2.

	p	q	r
B_1	1	0	A
B_2	A	A	1
B_3	A	0	1
CFR	1	0	1
$F(\underline{B}_{2,3})$	A	0	1
$F(\underline{B}_{1,2})$	1	0	1
$F(\underline{B}_{1,3})$	1	0	1

	p	q	r
B'_1	1	1	A
B'_2	1	0	0
B'_3	A	0	1
CFR	1	A	A
$F(\underline{B}'_{2,3})$	1	0	A
$F(\underline{B}'_{1,2})$	1	A	0
$F(\underline{B}'_{1,3})$	1	A	1

k-Quota Rule

k-Quota Rule (k-QR)

Given a profile $\underline{B}$, the collective outcome on issue j is $x = \{0, 1, A\}$ if there are at least k individuals with $b_j = x$, otherwise the collective outcome is A.

Proposition

k-QR *satisfies k-compatibility over issues and U*, I*, M*, A* unless* $k = |\mathcal{N}|$ *or* $k = 0$.

Combine Consistency with Compatibility

Example

Three automatic trading agents have to decide on whether to buy certain stocks. The first agent thinks they should buy the stock (B) because the revenue is increasing (R) and people are selling a considerable amount of stocks (S): its conclusion function is $B \leftrightarrow (R \wedge S)$ and its ballot as in the profile. The second agent submits the rule $R \leftrightarrow \neg B$ (against speculation). The third agent's conclusion function is $S \leftrightarrow \neg B$ (avoid dangerous markets).

	B	R	S
Agent 1	1	1	1
Agent 2	0	1	0
Agent 3	0	0	1

Proposed result:

Maximal consistent sets of formulas (idea from belief merging):
$K_1 = \{S \leftrightarrow \neg B, R \leftrightarrow \neg B\}$ *and* $K_2 = \{(S \wedge R) \leftrightarrow B\}$*. The first is consistent with two individual ballots, so it is chosen as collective conclusion function. Identify a set of premises and use the AVR to get that* S *and* R *are accepted. The result on* B *is therefore 0 (like conclusion based).*

Conclusions

1. Binary Aggregation framework for non-resolute incomplete aggregation procedures

2. Various notion of compatibility: how close is the collective outcome to the individual ballots?

   What about distance rationalizability?

3. Some basic rules for each notion of compatibility

4. Next? Develop the last idea of aggregating reasons and ballots