The Common Structure of Aggregation Paradoxes (and how to avoid them)

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In 1785 Monsieur le Marquis de Condorcet pointed out that:

$$\triangle \succ_{\mathbf{1}} \bigcirc \succ_{\mathbf{1}} \square$$
$$\square \succ_{\mathbf{2}} \triangle \succ_{\mathbf{2}} \bigcirc$$
$$\bigcirc \succ_{\mathbf{3}} \square \succ_{\mathbf{3}} \triangle$$

$$\triangle \succ \bigcirc \succ \square \succ \triangle$$

???

- Why is this a paradox?
- Why does this happen?

Outline

1. What is a paradox?
    - Various notions of individual rationality
    - A propositional language for rationality assumptions
    - General definition of paradox
2. Why do paradoxes come about?
    - Common syntactic property of paradoxical rationality assumptions
    - Characterisation of when the majority rule generates a paradox
3. How to avoid paradoxes?
    - The average voter rule
    - Axiomatic and complexity results

Individual Rationality in Decision Theory

The problem:	Individuals choosing over a set of alternatives $\mathcal{X}$
Rational behaviour:	Maximise a weak order over $\mathcal{X}$
	(transitive, complete and reflexive binary relation)

- Linear orders to avoid ties
- Partial orders over large domains
- Acyclic relations defined from choice functions

Many Rationalities?

Judges in a court (cf. judgment aggregation):

"Captain Schettino is guilty" "The captain abandonded the ship"
"If he abandoned the ship then he is guilty"

Rational judges?
Consistent and complete judgment sets

Many Rationalities?

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Rational judges?
Consistent and complete judgment sets

Committee deciding over multiple issues:

"Cut pensions" "Cut the number of MPs"
"Liberalise the market"

Rational members?
No political power to enforce all three austerity measures:
Ballots with at most 2 yes

Binary Aggregation

Ingredients:

- A finite set N of individuals
- A finite set $\mathcal{I} = \{1, \ldots, m\}$ of issues
- A boolean *combinatorial domain*: $\mathcal{D} = D_1 \times \cdots \times D_m$ with $|D_i| = 2$

Definition

An aggregation procedure is a function $F : \mathcal{D}^N \to \mathcal{D}$ mapping each profile of ballots $\mathbf{B} = (\mathbf{B}_1, \ldots, \mathbf{B}_n)$ to an element of the domain $\mathcal{D}$.

Dokow and Holzman (2008), Grandi and Endriss (AAAI-2010)

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Example: Austerity measures

- $N = \{1, 2, 3\}$
- $\mathcal{I} = \{\text{Cut pensions}, \text{Increase taxes}, \text{Stimulate growth}\}$
- Individuals submit ballots in $\mathcal{D} = \{0, 1\}^3$

$B_1 = (1, 0, 1)$ the first individual is probably young.

Integrity Constraints and Rationality Assumptions

A propositional language $\mathcal{L}$ to express integrity constraints on $D = \{0,1\}^m$

- One propositional symbol for every issue: $PS = \{p_1, \ldots, p_m\}$
- $\mathcal{L}_{PS}$ closing under connectives $\wedge$, $\vee$,$\neg$, $\rightarrow$ the set of atoms PS

Given an integrity constraint IC $\in \mathcal{L}_{PS}$, a rational ballot is $B \in \mathrm{Mod}(\mathrm{IC})$

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Example: Austerity Measures

Economists say that IC $= (p_P \wedge p_I) \rightarrow p_G$

Individual 1 submits $B_1 = (1,0,0)$: B_1 satisfies IC ✓
Individual 2 submits $B_2 = (0,1,0)$: $B_2 \models$ IC ✓
Individual 3 submits $B_3 = (1,1,1)$: $B_3 \models$ IC ✓

Majority aggregation outputs $(1,1,0)$: IC not satisfied

Every individual satisfies the same rationality assumption IC...
...what about the collective outcome?

Definition

A paradox is a triple $(F, \mathbf{B}, IC)$, where:

- F *is an aggregation procedure*
- $\mathbf{B} = (B_1, \dots, B_n)$ *a profile*
- $IC \in \mathcal{L}_{PS}$ *an integrity constraint*

such that $B_i \models IC$ for all $i \in \mathcal{N}$ but $F(\mathbf{B}) \not\models IC$.

Preference Aggregation

Linear order $<$ over alternatives $\mathcal{X}$ $\Leftrightarrow$ Ballot $B_{\leqslant}$ over issues $\mathcal{I} = \{ab \mid a \neq b \in \mathcal{X}\}$

$\text{IC}_<$ encodes the rationality assumption for decision theory:

Irreflexivity: $\neg p_{aa}$ for all $a \in \mathcal{X}$

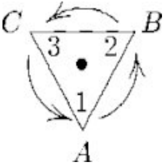
Completeness: $p_{ab} \vee p_{ba}$ for all $a \neq b \in \mathcal{X}$

Transitivity: $p_{ab} \wedge p_{bc} \rightarrow p_{ac}$ for $a, b, c \in \mathcal{X}$ pairwise distinct

Social welfare function $\Leftrightarrow$ Binary aggregation proc. CR with respect to $\text{IC}_<$

Condorcet Paradox Revisited

	△○	○□	△□
Agent 1	1	1	1
Agent 2	0	1	0
Agent 3	1	0	0
Maj	1	1	0



Our definition of paradox:

- F is issue by issue majority rule
- the profile is the one described in the table
- IC that is violated is $p_{ab} \wedge p_{bc} \rightarrow p_{ac}$

Judgment Aggregation

Judgment sets J over agenda Φ	$\Leftrightarrow$	Ballot B_J over issues $\mathcal{I} = \Phi$

IC_Φ encodes the rationality assumption of judgment aggregation:

Completeness: $p_\alpha \vee p_{\neg\alpha}$ for all $\alpha \in \Phi$

Consistency: $\neg(\bigwedge_{\alpha \in S} p_\alpha)$ for every mi-set $S \subseteq \Phi$

Complete and consistent JA procedures for Φ	$\Leftrightarrow$	Binary aggregation proc. CR with respect to IC_Φ

Doctrinal Paradox

	α	$\alpha \to \beta$	β
Agent 1	1	1	1
Agent 2	0	1	0
Agent 3	1	0	0
Majority	1	1	0!!

Our definition of paradox:

- F is issue by issue majority rule
- profile described in the table
- IC that is violated is $\neg(p_\alpha \wedge p_{\neg\beta} \wedge p_{(\alpha\to\beta)})$

Common feature: Three issues

Kornauser and Sager. Unpacking the court. Yale Law Journal, 1986.

Ostrogorski Paradox

	E	V	I	P
Agent 1	0	1	0	0
Agent 2	0	1	0	0
Agent 3	1	0	0	0
Agent 4	1	1	1	1
Agent 5	1	1	0	1
Majority	1	1	0	0

Our definition of paradox:

- F is issue by issue majority rule
- IC that is violated is $P \leftrightarrow [(E \wedge V) \vee (V \wedge I) \vee (I \wedge E)]$

After some calculation IC is equivalent to a conjunction of clauses of size 3:

$$(P\vee\neg E\vee\neg V)\wedge(P\vee\neg E\vee\neg I)\wedge(P\vee\neg I\vee\neg V)\wedge(\neg P\vee E\vee V)\wedge(\neg P\vee E\vee I)\wedge(\neg P\vee I\vee V)$$

M. Ostrogorski. La démocratie et l'organisation de partis politiques, 1902.

Bezembinder and van Acker. The Ostrogorski paradox and its relation... Math. Sociology, 1985

Multiple Election Paradox

The outcome of the majority rule is the acceptance of all three issues, even if this combination was not voted for by any of the individuals:

	A	B	C
Voter 1	1	0	1
Voter 2	0	1	1
Voter 3	1	1	0
Maj	1	1	1

- Every paradox of aggregation by our definition is an instance of the MEP - no agent votes for irrational outcomes -
- Every instance of MEP can be seen as a paradox of aggregation by devising suitable integrity constraints -e.g., disjunction of individual ballots -

Brams, Kilgour, and Zwicker. The paradox of multiple elections. *Social Choice and Welfare*, 1998.

The Common Structure of Paradoxes

Proposition (Grandi and Endriss, IJCAI-2011)

The majority rule is collectively rational with respect to IC if and only if IC is equivalent to a conjunction of clauses of size $\leqslant 2$.

$$\mathcal{IC}(Maj) = \textit{2-clauses}$$

Common feature of all paradoxes:
clauses of size 3 are not lifted by majority

More syntactical characterisation results:

Grandi and Endriss. Lifting Rationality Assumptions in Binary Aggregation. AAAI 2010.

How to Avoid all Paradoxes?

A generalised dictatorship copies the ballot of a (possibly different) individual (aka local dictatorships, positional dictatorships, rolling dictatorships):

Proposition

F is collectively rational with respect to all IC in $\mathcal{L}_{PS}$ if and only if F is a generalised dictatorship.

This class includes:

- Classical dictatorships $F(B_1, \ldots, B_n) = B_i$ for $i \in \mathcal{N}$
- Average Voter Rule: map $(B_1, \ldots, B_n)$ to the ballot B_i that minimises the sum of the Hamming distance to the others (the "average voter"). An interesting procedure!

The Average Voter Rule

Definition

The average voter rule (AVR) *chooses the individual ballot that minimises the sum of the Hamming distance* $H(B, B') = \sum_{j \in \mathcal{I}} |b_j - b'_j|$ *to all other individual ballots:*

$$\mathrm{AVR}(\mathbf{B}) = \operatorname*{argmin}_{\{B_i | i \in \mathcal{N}\}} \sum_{s \in \mathcal{N}} H(B_i, B_s),$$

The AVR shares interesting axiomatic properties:

Proposition

The AVR *satisfies a non-resolute version of anonymity, unanimity and monotonicity. It does not satisfy independence.*

Complexity of the AVR

$\textsc{WinDet}^\star(F)$	
Instance:	Integrity constraint IC, profile $\mathbf{B}$, issue j.
Question:	Is there a $B \in F(\mathbf{B})$ such that $B_j = 1$?

Fact (very easy proof)

$\textsc{WinDet}^\star(\mathrm{AVR})$ *is in P.*

Complexity of the AVR

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Instance: Integrity constraint IC, profile $\mathbf{B}$, issue j.
Question: Is there a $B \in F(\mathbf{B})$ such that $B_j = 1$?

Fact (very easy proof)

WinDet$^\star(\mathrm{AVR})$ *is in P.*

Manipulable$^\star(F)$
Instance: IC, ballot B, partial profile $\mathbf{B}_{-i} \in \mathrm{Mod}(\mathrm{IC})^{|\mathcal{N}|-1}$.
Question: Is there B' s.t. $H(B, F^\star(B', \mathbf{B}_{-i})) < H(B, F^\star(B, \mathbf{B}_{-i}))$?

Conjecture (now proven)

Manipulability*(*AVR*) is in P.*

Conclusions

Many notions of individual rationality generates many paradoxes:

- binary issues as a general model of individual expressions;
- rationality assumptions as propositional formulas;
- common syntactic property behind paradoxes: clauses of size 3.

How to avoid all paradoxes?

- Copy the ballot of a different individual;
- Do it in an intelligent way: the AVR;
- Good axiomatic properties, easy to use and (unfortunately) also easy to manipulate.

Thanks for your attention!

Grandi and Endriss. Binary Aggregation with Integrity Constraints, *IJCAI-2011*.
Grandi and Endriss. Lifting Rationality Assumptions in Binary Aggregation, *AAAI-2010*.