

Binary Aggregation with Integrity Constraints

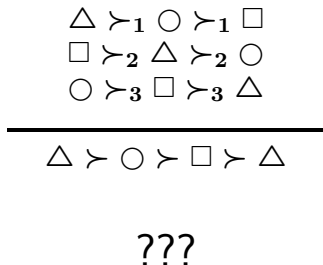
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Everything Starts From a Paradox

In 1785 Monsieur le Marquis de Condorcet pointed out that:



- Why is this a paradox?
- Why does this happen?

Outline

1. What is a paradox?

- Various notions of individual rationality
- A **propositional language** for rationality assumptions
- Binary aggregation with integrity constraints
- General definition of paradox

2. Why do paradoxes come about?

- Languages for integrity constraints
- **Collective rationality** and axiomatic properties
- Characterisation Results
- An answer to M.Condorcet: the **majority rule**

3. Conclusions and related work

Part I:
Individual Rationalities and Paradoxes

Individual Rationality in Decision Theory

The problem: Individuals choosing over a set of alternatives \mathcal{X}
Rational behaviour: Maximise a **weak order** over \mathcal{X}
(transitive, complete and reflexive binary relation)

- Linear orders to avoid ties
- Partial orders over large domains
- Acyclic relations defined from choice functions

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Remark: we do **not** talk about uncertainties.



Many Rationalities?

Judges in a court (cf. judgment aggregation):



“O.J.Simpson is guilty” “O.J.Simpson wore bloody gloves”
“If O.J.Simpson wore the glove then he is guilty”

Rational judges?

Consistent and complete **judgment sets**

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Consistent and complete **judgment sets**

Committee deciding over multiple issues:



“Cut pensions” “Cut the number of MPs”
“Cut funding to local provinces”

Rational members?

No political power to enforce all three austerity measures:

Ballots with at most 2 yes

Binary Aggregation with Integrity Constraints



"Everything is binary"

- Individuals express yes/no ballots over a finite set of issues \mathcal{I}
- A propositional language can be interpreted over ballots
- **Rationality assumptions/integrity constraints** are formulas in this language

Binary Aggregation

Ingredients:

- A finite set N of individuals
- A finite set $\mathcal{I} = \{1, \dots, m\}$ of **issues**
- A boolean *combinatorial domain*: $\mathcal{D} = D_1 \times \dots \times D_m$ with $|D_i| = 2$

Definition

An aggregation procedure is a function $F : \mathcal{D}^N \rightarrow \mathcal{D}$ mapping each profile of ballots $\mathbf{B} = (\mathbf{B}_1, \dots, \mathbf{B}_n)$ to an element of the domain \mathcal{D} .

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Example: Science Park

- $N = \{1, 2, 3\}$
- $\mathcal{I} = \{\text{University, Sportcentrum, Food supply}\}$
- Individuals submit ballots in $\mathcal{D} = \{0, 1\}^3$

$B_1 = (0, 1, 1)$ the first individual wants to have a good meal after the gym.

Integrity Constraints

A **propositional language** \mathcal{L} to express integrity constraints on $D = \{0, 1\}^m$

- One propositional symbol for every issue: $PS = \{p_1, \dots, p_m\}$
- \mathcal{L}_{PS} closing under connectives $\wedge, \vee, \neg, \rightarrow$ the set of atoms PS

Given an integrity constraint $IC \in \mathcal{L}_{PS}$, a **rational** ballot is $B \in \text{Mod}(IC)$

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Given an integrity constraint $IC \in \mathcal{L}_{PS}$, a **rational** ballot is $B \in \text{Mod}(IC)$

Example: Science Park (the true story)

If there is both a university and a sport center then food supply is necessary

Propositional constraint: $IC = (p_U \wedge p_S) \rightarrow p_F$

Individual 1 submits $B_1 = (1, 0, 0)$: B_1 satisfies IC ✓

Individual 2 submits $B_2 = (1, 1, 1)$: $B_2 \models IC$ ✓

Individual 3 submits $B_3 = (0, 1, 0)$: $B_3 \models IC$ ✓

Majority aggregation outputs $(1, 1, 0)$: IC **not** satisfied (as are all employees)

Paradoxes of Aggregation

Every individual satisfies the **same** rationality assumption IC...
...what about the collective outcome?

Definition

A **paradox** is a triple (F, \mathbf{B}, IC) , where:

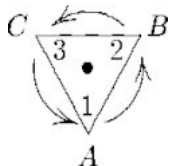
- F is an aggregation procedure
- $\mathbf{B} = (B_1, \dots, B_n)$ a profile
- $IC \in \mathcal{L}_{PS}$ an integrity constraint

such that $B_i \models IC$ for all $i \in \mathcal{N}$ but $F(\mathbf{B}) \not\models IC$.

Condorcet Paradox Revisited



	<i>ab</i>	<i>bc</i>	<i>ac</i>
Agent 1	1	1	1
Agent 2	0	1	0
Agent 3	1	0	0
Majority	1	1	0



Our definition of paradox:

- F is issue by issue majority rule
- the profile is the one described in the table
- **IC that is violated** is $p_{ab} \wedge p_{bc} \rightarrow p_{ac}$

Doctrinal Paradox

	α	$\alpha \rightarrow \beta$	β
Agent 1	1	1	1
Agent 2	0	1	0
Agent 3	1	0	0
Majority	1	1	0!!

Our definition of paradox:

- F is issue by issue majority rule
- profile described in the table
- IC that is violated is $\neg(p_\alpha \wedge p_{\neg\beta} \wedge p_{(\alpha \rightarrow \beta)})$

Common feature: **clauses of size 3**

Ostrogorski Paradox

	E	V	I	P
Agent 1	0	1	0	0
Agent 2	0	1	0	0
Agent 3	1	0	0	0
Agent 4	1	1	1	1
Agent 5	1	1	0	1
Majority	1	1	0	0

Our definition of paradox:

- F is issue by issue majority rule
- **IC that is violated** is $P \leftrightarrow [(E \wedge V) \vee (V \wedge I) \vee (I \wedge E)]$

After some calculation IC is equivalent to:

$$(P \vee \neg E \vee \neg V) \wedge (P \vee \neg E \vee \neg I) \wedge (P \vee \neg I \vee \neg V) \wedge (\neg P \vee E \vee V) \wedge (\neg P \vee E \vee I) \wedge (\neg P \vee I \vee V)$$

Common feature: **clauses of size 3**

Part II:

Characterisation Results for Collective Rationality

Collective Rationality

Definition

F is *collectively rational* (CR) for $IC \in \mathcal{L}_{PS}$ if for all profiles \mathbf{B} such that $\mathbf{B}_i \models IC$ for all $i \in N$ then $F(\mathbf{B}) \models IC$.

F *lifts* the rationality assumption given by IC from the individual to the *collective* level.

$$\mathcal{CR}[\mathcal{L}] = \{F : \mathcal{D}^N \rightarrow \mathcal{D} \mid \mathcal{N} \text{ is finite and } F \text{ is CR for all } IC \in \mathcal{L}\}$$

where $\mathcal{L} \subseteq \mathcal{L}_{PS}$ is a sublanguage

Languages for Integrity Constraints

Definition

A *language for integrity constraints* is a subset $\mathcal{L} \subseteq \mathcal{L}_{PS}$ that is closed under conjunctions, logical equivalence and contains \top and \perp .

- if F is CR wrt. φ and to ψ then is CR wrt. $\varphi \wedge \psi$
- if F is CR wrt. φ then it is so for every equivalent formulas
- \top and \perp are trivial requirements for CR

Lemma

If \mathcal{L}_1 and \mathcal{L}_2 are languages for IC, then if $\mathcal{L}_1 \neq \mathcal{L}_2$ then $CR[\mathcal{L}_1] \neq CR[\mathcal{L}_2]$

Languages and Axioms

Several **languages for integrity constraints**:

- *cubes*: conjunctions
- *k-pclauses*: positive disjunctions of size $\leq k$
- *XOR*: conjunctions of $p \leftrightarrow \neg q$
- ...

Several **axioms** to classify aggregation procedures:

- **Unanimity (U)**: For any profile $\mathbf{B} \in X^N$ and any $x \in \{0, 1\}$, if $\mathbf{B}_{i,j} = x$ for all $i \in N$, then $F(\mathbf{B})_j = x$.
- Independence, Neutrality...

Characterisation Results

Different lists of **axioms** AX define classes of functions:

$$\mathcal{F}_{\mathcal{L}}[AX] = \{F: \mathcal{D}^{\mathcal{N}} \rightarrow \mathcal{D} \mid F|_{\text{Mod}(\text{IC})^{\mathcal{N}}} \text{ sat. } AX \text{ for all } \text{IC} \in \mathcal{L}\}$$

Recall that the class of CR procedures for a **language** \mathcal{L} is:

$$\mathcal{CR}[\mathcal{L}] = \{F: \mathcal{D}^{\mathcal{N}} \rightarrow \mathcal{D} \mid F \text{ is CR for all } \text{IC} \in \mathcal{L}\}$$

What we want:

$$\mathcal{CR}[\mathcal{L}] = \mathcal{F}_{\mathcal{L}}[AX]$$

Characterisation Results: Examples

Cubes (conjunctions of literals) are lifted iff the procedure satisfies unanimity:

Proposition

$$\mathcal{CR}[\text{cubes}] = \mathcal{F}_{\text{cubes}}[\text{Unanimity}].$$

Similar results can be proven for language of **equivalences** (issue-neutrality), **XOR formulas** (domain-neutrality), **positive implications** (neutral-monotonicity).

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Similar results can be proven for language of **equivalences** (issue-neutrality), **XOR formulas** (domain-neutrality), **positive implications** (neutral-monotonicity).

For the axioms of independence we prove instead a negative result:

Proposition

There is no language $\mathcal{L} \subseteq \mathcal{L}_{PS}$ such that $\mathcal{CR}[\mathcal{L}] = \mathcal{F}_{\mathcal{L}}[\text{I}]$.

The same result holds for the axiom of **anonymity** and **monotonicity**

Committee and Quota Rules

Interesting classes of procedures:

- Independence → Committee rules

Every issue j a **decisive** committee $N_j \subseteq \mathcal{N}$

$$\mathcal{F}[I] = \text{Committee rules}$$

- Anonymity → Only size matter

Committee of same size have same decision power

- Monotonicity → Quota rules

Every issue j a **quota** q_j

$$\mathcal{F}[I, A, M] = \text{Quota rules}$$

Example: The **majority** rule has uniform quota $\lceil \frac{n+1}{2} \rceil$

Quota Rules and Languages of Clauses

Interesting results concerning positive/negative clauses, and a general equation:

Proposition

A quota rule is CR with respect to a k -clause IC iff

$$\sum_{j \text{ negative}} q_j + \sum_{j \text{ positive}} (n - q_j + 1) > n(k - 1) \quad (1)$$

for issues j that occur positive or negative in IC, or $q_j = 0$ for some issue j that occurs positive in IC, or $q_j = n + 1$ for issue j negative in IC.

Yet no characterisation result for general languages of clauses (for $k > 2$):

$$CR[k\text{-clauses}] \cap QR = \emptyset$$

My Answer to M. Condorcet: The Majority Rule

Proposition

The majority rule is collectively rational with respect to IC if and only if IC is equivalent to a conjunction of clauses of size ≤ 2 .

$$\mathcal{IC}(\text{Maj}) = \text{2-clauses}$$

Common feature of all paradoxes:
clauses of size 3 are not lifted by majority

Part III: Related Work and Conclusion

Are These Results Useful for Social Choice Theorists?

Yes, we can talk about orders:

Call an aggregation procedure **imposed** if there are two alternatives x and y such that x is collectively preferred to y in every profile:

Proposition

Any anonymous, independent and monotonic aggregation procedure for more than 3 alternatives and 2 individuals is imposed.

Proof sketch:

- Translate preference aggregation into BA with IC
- Study the syntactic property of the $IC_{<}$
- Use a characterisation result!
- Go back to preference aggregation

Are These Results Useful for Philosophers?

A question: can we model a group of rational agents as a **rational agent itself**?

Proposition

F is CR with respect to all IC in \mathcal{L}_{PS} if and only if F copies the ballot of a (possibly different) individual in every profile.

This class includes:

- Classical dictatorships: copy the ballot of the same individual.
- Distance-based generalised dictatorships: copy the ballot of the average individual.

Does this tell us anything about the problem of **group agency**?

Conclusions

Many notions of individual rationality:

- binary issues as a general model of individual expressions;
- rationality assumptions as propositional formulas;
- focus on syntactic properties of the rationality assumption.

Collective rationality:

- Unifying framework for paradoxes;
- Systematic study of collective rationality;
- Application in preference/judgment aggregation & co.

Thanks for your attention!

Grandi and Endriss. Binary Aggregation with Integrity Constraints, *IJCAI-2011*, *LIRA Yearbook-10*.
Grandi and Endriss. Lifting Rationality Assumptions in Binary Aggregation, *AAAI-2010*.