Computational Social Choice - Autumn 2011
Belief Merging

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Introduction

The situation: several agents holding beliefs about the state of the world

States are described using a finite number of propositional symbols, there is no hierarchy, nor priority, nor any difference in reliability of the sources.

The problem: how to define the beliefs of the group?
How to merge the individual belief sets into a collective one?

Example:
$K_1 = \{a, b \rightarrow c\}$
$K_2 = \{a, b\}$
$K_3 = \{\neg a\}$

$K = ?$ a candidate set could be: $\{b \rightarrow c, b, a\}$

Overview

Plan for today:

- Logic-based belief merging
- Axioms for IC merging operators and representation result
- Examples: Sum operator, Gmax operator, DA^2 operators
- Related work: Strategy-proof operators, belief revision, belief merging in judgment aggregation...

Based on the following survey paper:

Konieczny and Pino Pérez. Logic Based Merging. JPL, 2011.

Basic Definitions

Ingredients:

- A finite set of individuals $\mathcal{N}$
- A propositional language $\mathcal{L}_{PS}$ defined from a finite set of atoms PS

A knowledge base is a finite set of formulas $K \subseteq \mathcal{L}_{PS}$

Each individual submits a consistent knowledge base K_i

A profile is a multi-set of knowledge bases

$$E = \{K_1, \cdots, K_{|\mathcal{N}|}\}$$

Remark: Given a knowledge base K, denote with $\bigwedge K$ the conjunction of the formulas in K. Given a profile E denote with $\bigwedge E$ the conjunction of all the bases in E. A profile E is consistent if $\bigwedge E$ is consistent.

Belief Merging

Call:

- $\mathcal{E}$ the set of all profiles
- $\mathcal{K}$ the set of all knowledge bases
- integrity constraint any formula $\mu \in \mathcal{L}$

A merging operator is a function

$$\Delta : \mathcal{E} \times \mathcal{L} \to \mathcal{K}$$

The idea is to start from a profile E and an integrity constraint μ and get as an outcome $\Delta_\mu(E)$ the closest knowledge base to E satisfying the constraint μ.

How to formalise these properties?

Technicalities

The following remarks and notation:

- K and $\bigwedge K$ will be interchangable unless it creates ambiguity
- Two profiles are equivalent $E_1 \equiv E_2$ iff $\vdash \bigwedge E_1 \leftrightarrow \bigwedge E_2$
- $E_1 \sqcup E_2$ is the multi-set union
- $E^n = E \sqcup \cdots \sqcup E$ n-times

Axioms for IC merging operators

Definition

An operator $\Delta : \mathcal{E} \times \mathcal{L} \to \mathcal{K}$ is said to be an integrity constraint merging operator if it satisfies the following axioms **(IC0)-(IC8)**.

(IC0): $\Delta_\mu(E) \vdash \mu$

(IC1): If μ is consistent then $\Delta_\mu(E)$ is consistent

(IC2): If $\bigwedge E$ is consistent with μ then $\Delta_\mu = \bigwedge E \wedge \mu$

(IC3): If $E_1 \equiv E_2$ and $\mu_1 \equiv \mu_2$ then $\Delta_{\mu_1}(E_1) \equiv \Delta_{\mu_2}(E_2)$

Some of the axioms already proposed by P. Z. Revesz, On the Semantics of Arbitration (1997).

Axioms II

(IC4): If $K_1 \vdash \mu$ and $K_2 \vdash \mu$ then $\Delta_\mu(\{K_1, K_2\}) \wedge K_1$ is consistent if and only if $\Delta_\mu(\{K_1, K_2\}) \wedge K_2$ is consistent

This axioms is called fairness by Konieczny and Pérez (2011). It resembles a neutrality axiom with respect to consistency: When merging two knowledge bases we shall not give preference to neither of the two.

(IC5): $\Delta_\mu(E_1) \wedge \Delta_\mu(E_2) \vdash \Delta_\mu(E_1 \sqcup E_2)$

(IC6): If $\Delta_\mu(E_1) \wedge \Delta_\mu(E_2)$ is consistent, then $\Delta_\mu(E_1 \sqcup E_2) \vdash \Delta_\mu(E_1) \wedge \Delta_\mu(E_2)$

These two axioms together state that if we can subdivide the group in two subgroups that separately agree on certain beliefs, then the result is the conjunction of the aggregated beliefs of the two groups.

Axioms III

Last two axioms:

(IC7): $\Delta_{\mu_1}(E) \wedge \mu_2 \vdash \Delta_{\mu_1 \wedge \mu_2}(E)$

(IC8): If $\Delta_{\mu_1}(E) \wedge \mu_2$ is consistent, then $\Delta_{\mu_1 \wedge \mu_2}(E) \vdash \Delta_{\mu_1}(E) \wedge \mu_2$

These two axioms come from belief revision (see 2nd part of lecture)

The idea behind **(IC7)** and **(IC8)** is that when an alternative/belief is chosen in a certain set of feasible ones then if we restrict this set further and this alternative remains available it will still be chosen (strong similarity with properties of social choice functions).

Examples: Merging Operators That DO NOT Satisfy the Axioms

Meet revision:

$$\Delta_\mu(E) = \begin{cases} \bigwedge E \wedge \mu & \text{if consistent} \\ \mu & \text{otherwise} \end{cases}$$

Does not satisfy axiom **(IC6)**!!

Take the conjunction of the belief bases if consistent, the disjunction otherwise:

$$\Delta_\mu(E) = \begin{cases} \bigwedge E \wedge \mu & \text{if consistent} \\ \bigvee E \wedge \mu & \text{otherwise, if consistent} \\ \mu & \text{otherwise} \end{cases}$$

Does not satisfy axiom **(IC6)**!!

More Axioms...

There are other properties we might want to require for an IC merging operator:

Majority: $\exists n \ \Delta_\mu(E_1 \sqcup E_2^n) \vdash \Delta_\mu(E_2)$

Strong similarity with some axioms of Social Choice Theory

Majority Independence: $\forall n \ \Delta_\mu(E_1 \sqcup E_2^n) \leftrightarrow \Delta_\mu(E_1 \sqcup E_2)$

The result is independent from the "popularity": impossible!!

Konieczny and Pino Pérez. Merging information under constraints: a logical framework, 2002.

Technicalities

The following remarks and notation:

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More Axioms...

Arbitration: complicated... cf. Konieczny and Pino Pérez (2011)

Idea: if the alternatives/belief preferred by two groups over two different integrity constraints are the same, and if the whole group is indifferent with respect to what satisfies one constraint but not the other, then the result of merging the belief of the two groups together is the initial set of preferred alternatives. (In accordance to one of the first papers on belief merging by Liberatore and Schaerf, 1995.)

Liberatore and Schaerf. Arbitration: a commutative operator for belief revision, 1995.

Syncretic Assignments

Definition

A syncretic assignment is a function mapping each profile E

$$E \quad \rightarrow \quad \leqslant_E \quad \text{over } \rho : PS \rightarrow \{0,1\}$$

to a total preorder $\leqslant_E$ over assignments, satisfying the following conditions:

- If $\omega \models E$ and $\omega' \models E$ then $\omega \sim_E \omega'$
- If $\omega \models E$ and $\omega' \not\models E$ then $\omega <_E \omega'$
- If $E_1 \equiv E_2$ then $\leqslant_{E_1} = \leqslant_{E_2}$
- ...

Syncretic Assignments II

- $\forall \omega \models K_1 \exists \omega' \models K_2$ such that $w' \leqslant_{K_1 \sqcup K_2} w$

A version of "fairness": no priority given to any knowledge base K

- If $\omega \leqslant_{E_1} \omega'$ and $\omega \leqslant_{E_2} \omega'$ then $\omega \leqslant_{E_1 \sqcup E_2} \omega'$
- If $\omega <_{E_1} \omega'$ and $\omega \leqslant_{E_2} \omega'$ then $\omega <_{E_1 \sqcup E_2} \omega'$

If an assignment is at least as plausible as another for two profiles, then it continue to do so if we join the two profiles: similarity with reinforcement axiom in social choice theory.

Representation Theorem

Theorem (Konieczny and Pino Pérez, 2002)

An operator Δ is an IC merging operator if and only if there exists a syncretic assignment that maps each profile E to a total preorder $\leqslant_E$ such that

$$\mathrm{Mod}(\Delta_\mu(E)) = \min_{\mathrm{Mod}(\mu)} \leqslant_E$$

Other properties can be provided to represent majoritarian operators and arbitration.

Example: the Sum Operator

Let d be a counting distance bewteen assignments $d : \mathcal{W} \times \mathcal{W} \to \mathbb{N}$

If K is a knowledge base, define $d(K, \rho) = \min_{\varphi \in K} d(\varphi, \rho)$

Definition

Call *sum operator* *the following:*

$$\mathrm{Mod}(\Delta_\mu^\Sigma(E)) = \arg \min_{\rho \in \mathrm{Mod}(\mu)} \sum_{K \in E} d(K, \rho)$$

Proposition

$\Delta_\mu^\Sigma(E)$ *is a majority IC merging operator.*

Example: the GMax Operator

Definition

Let $E = \{K_1, \ldots, K_n\}$ *and let* $d_j^\rho = d(K_j, \rho)$. *Let* $L_E^\rho = (d_{i_1}^\rho, \ldots, d_{i_n}^\rho)$ *sorted in descending order. Using the lexicographic ordering over vectors of integers we can define the ordering* $\leqslant_E^{Gmax}$ *over assigments and the Gmax operator*

$$\mathrm{Mod}(\Delta_\mu^{Gmax}(E)) = \min_{\mathrm{Mod}(\mu)} \leqslant_E^{Gmax}$$

Proposition

$\Delta_\mu^{Gmax}(E)$ *is an arbitration IC merging operator.*

A "Concrete" Example

At a meeting of flat co-owners, the chairman proposes the construction of a swimming-pool, a tennis-court and a private-car-park. If two of these three items are build the rent will increase. Integrity Constraint:

$$\mu = ((S \wedge T) \vee (S \wedge P) \vee (T \wedge P)) \rightarrow I$$

Hamming distance: cardinality of the symmetric difference. Distance between (1, 0, 0,1) and (1, 1, 0,1) is 1 since they differ on only one issue.

Individual goals:
$K_1 = K_2 = S \wedge T \wedge P$
$K_3 = \neg S \wedge \neg T \wedge \neg P \wedge \neg I$
$K_4 = T \wedge P \wedge I$

(see some calculation on blackboard...)

The result:

- $\mathrm{Mod}(\Delta_\mu^{\Sigma}(E)) = \{(1,1,1,1)\}$
- $\mathrm{Mod}(\Delta_\mu^{Gmax}(E)) = \{(0,0,1,0),(0,1,0,0)\}$

Model-Based IC Merging Operators

Definition

Given:

- *A pseudo-distance* $d : \mathcal{W} \times \mathcal{W} \to \mathbb{R}^+$*, defining* $d(K, \rho) = \min_{\varphi \in K} d(\varphi, \rho)$
- *An aggregator* $f : \mathbb{R}^n \to \mathbb{R}^+$ *(satisfying certain basic properties)*

We can define the following operator:

$$\mathrm{Mod}(\Delta_{\mu}^{d,f}(E)) = \arg \min_{\rho \in \mathrm{Mod}(\mu)} f(d(K_1, \rho), \ldots, d(K_n, \rho))$$

Proposition

$\Delta_{\mu}^{d,f}(E)$ *is an IC merging operator for all* f *and* d.

DA^2 and Formula-Based Operators

Definition

Given:

- *A pseudo-distance* $d : \mathcal{W} \times \mathcal{W} \to \mathbb{R}^+$
- *Two aggregators* $f, g : \mathbb{R}^n \to \mathbb{R}^+$ *(satisfying certain basic properties)*

We can define the following operator:

$$\mathrm{Mod}(\Delta_\mu^{d,f}(E)) = \arg \min_{\rho \in \mathrm{Mod}(\mu)} f(d^g(K_1, \rho), \ldots, d^g(K_n, \rho))$$

Where $d^g(\rho, K) = g(d(\rho, \varphi_1), \ldots, d(\rho, \varphi_m))$ *with* $\varphi_j \in K$.

Complexity results for various distances can be found in:

Konieczny, Lang and Marquis. DA^2 merging operators. AI, 2004.

Strategy-Proofness and Belief Merging I

A satisfaction index i is a total function $i : \mathcal{L} \times \mathcal{L} \rightarrow \mathbb{R}$

Definition

- *A profile E is manipulable by a base K for i given Δ iff there exists a base K' such that $i(K, \Delta_\mu(E \sqcup \{K'\})) > i(K, \Delta_\mu(E \sqcup \{K\}))$*
- *Δ is strategy-proof for i iff there is no IC μ and profile E such that E is manipulable for i.*

Everaere, Konieczny and Marquis. The Strategy-Proofness Landscape of Merging. JAIR, 2007.

Strategy-Proofness and Belief Merging II

Examples of satisfaction index:

Weak drastic index

$$i_{dw}(K, K_\Delta) = \begin{cases} 1 & K \wedge K_\Delta \text{is consistent} \\ 0 & \text{otherwise} \end{cases}$$

Probabilistic index

$$i_p(K, K_\Delta) = 0 \text{ if } |\mathrm{Mod}(K_\Delta)| = 0 \text{ otherwise}$$

$$i_p(K, K_\Delta) = \frac{|(\mathrm{Mod}(K) \cap \mathrm{Mod}(K_\Delta)|}{|\mathrm{Mod}(K_\Delta)|}$$

Proposition (Everaere et Al. 2004)

Let f be an aggregation function, $\Delta_\mu^{H,\Sigma}$ is stragegy proof for i_p if and only if the initial knowledge base K is complete (i.e., it has only one model).

Belief Revision (AGM Approach)

Given a knowledge base K (be it a scientific theory, a set of beliefs, a corpus of norms...), there are three operations that can be performed when confronted with new information:

- Expansion $K \oplus \varphi$: adding the new information to the knowledge base (possibly getting an inconsistent outcome).
- Contraction $K \ominus \varphi$: deleting a previously kept belief, obtaining something from which φ does not follow.
- Revision $K \circledast \varphi = (K \ominus \neg\varphi) \oplus \varphi$
  Firts make space for φ by deleting everything that contradicts it, and then expand the knowledge base safely in order to mantain consistency.

AGM give axioms to norm the behaviour of these three operators:

Alchourrón, Gärdenfors and Makinson. On The Logic of Theory Change... JSL, 1985.

For finite propositional languages these are equivalent axioms for an AGM revision operator (Katsuno and Mendelson, 1991):

R1: $\varphi \circ \mu$ implies μ

R4: If $\varphi_1 \leftrightarrow \varphi_2$ and $\mu_1 \leftrightarrow \mu_2$ then $\varphi_1 \circ \mu_1 \leftrightarrow \varphi_2 \circ \mu_2$

R2: If $\varphi \wedge \mu$ is consistent then $\varphi \circ \mu \leftrightarrow \varphi \wedge \mu$

R5: $(\varphi \circ \mu) \wedge \psi$ implies $\varphi \circ (\mu \wedge \psi)$

R3: If μ is consistent then $\varphi \circ \mu$ is consistent

R6: If $(\varphi \circ \mu) \wedge \psi$ is consistent then $\varphi \circ (\mu \wedge \psi)$ implies $(\varphi \circ \mu) \wedge \psi$

Theorem

If Δ is an IC merging operator then the operator $\circ$ defined as $K \circ \mu = \Delta_\mu(K)$ is an AGM revision operator.

The other way round (from belief revision to merging) is not as easy:

Konieczny and Pino Pérez. Confluence operators. JELIA 2008.

Belief Merging and Judgment Aggregation

The two frameworks have many similarities, but at the moment there has not been any systematic study that attempted at combining them:

Judgment Aggregation	**Belief Merging**
Finite agenda	Infinite number of formulas $\mathcal{L}_{PS}$ (easy to fix)
Non-anonymous profiles	Anonymous profiles (multi-sets)
Complete judgment sets	No assumption on knowledge bases
No IC (easy to fix)	IC
Constraints of rationality	Constraints of feasibility (and rationality)
Focus on existing rules and on consistency	Focus on new rules while consistency is enforced by definition

Can we import results from one framework to the other?

Belief Merging in JA and SCT

Distance based aggregation procedures were introduced in the literature on judgment aggregation in the following paper:

Pigozzi. Belief merging and the discursive dilemma: an argument-based account to paradoxes of judgment aggregation. Synthese, 2006

Followed by several other papers:

Miller and Osherson. Methods for Distance-Based Judgment Aggregation. SCW, 2009.

Lang et Al. Judgment aggregation rules based on minimization. TARK XIII, 2011.

Rules based on distances were already studied in classical SCT (Kemeny rule, distance rationalizability...). But belief merging might play an interesting role if we try to combine beliefs and preferences into a collective action:

Hylland and Zeckhauser. The Impossibility of Bayesian Group Decision Making with Separate Aggregation of Beliefs and Values. Econometrica, 1979.

Related Work

Syntax-Based merging operators:

Baral et Al. Combining Knowledge Bases Consisting of First-Order Theories. Computational Intelligence, 1992.

Konieczny. On the difference between merging knowledge bases and combining them. KR, 2000.

Merging of weighted goal/belief bases, belief merging and negotiation...
References in the following survey papers:

Grégoire and Konieczny. Logic-Based Approaches to Information Fusion. Information Fusion, 2006.

Konieczny and Pino Pérez. Logic Based Merging. JPL, 2011.

Summary

During today's lecture we have seen:

- a model of multi-agents beliefs based on propositional logic
- an axiomatisation for belief merging operators based on this model
- a representation result for this axiomatisation: syncretic assignments
- examples of operators: Sum, Gmax, DA^2...

A number of related topics:

- Judgment aggregation: similar but different
- Belief revision: single-agent version of belief merging
- Strategy-proofness: negative results