Decision Making and Social Networks

Lecture 3: Understanding the structure of a network

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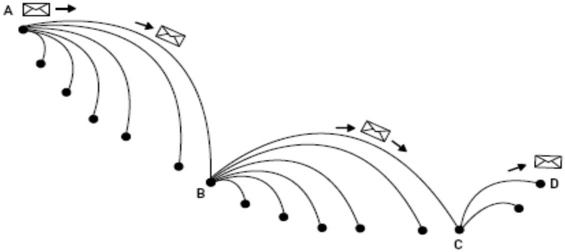
An interesting experiment

In 1967, Stanley Milgram from Harvard wanted to measure the "distance" between two random persons in the United States:

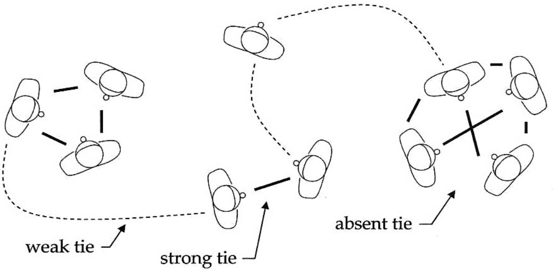
How many acquaintances does it take to connect them?



Two cities selected: Wichita, Kansas and Omaha, Nebraska. Intuitively? Around 100 steps? No...25 % of the letters made it to the recipient, and the median number of steps required was 5.5: the famous six degrees of separation.

A second interesting experiment

"The strength of weak ties", influential paper by Mark Granovetter in 1973



Reporting on experiments on labour market. How did people got their jobs?

- Mostly through their network of friends (Myers and Shultz, 1951, Rees and Shultz, 1970...)
- Not exactly through friends but through acquaintances

Weak ties are very important in small world networks!

A nice graph

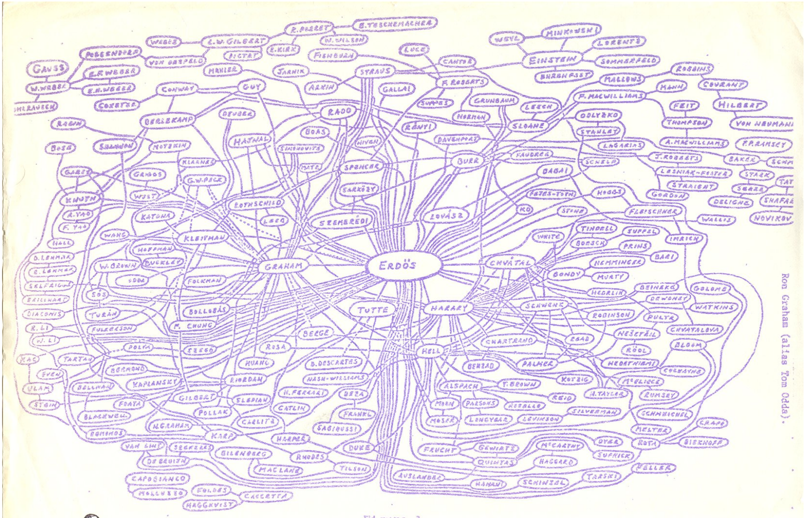


Figure 1
To appear in Topics in Graph Theory (F. Harary, ed.) New York Academy of Sciences (1979).

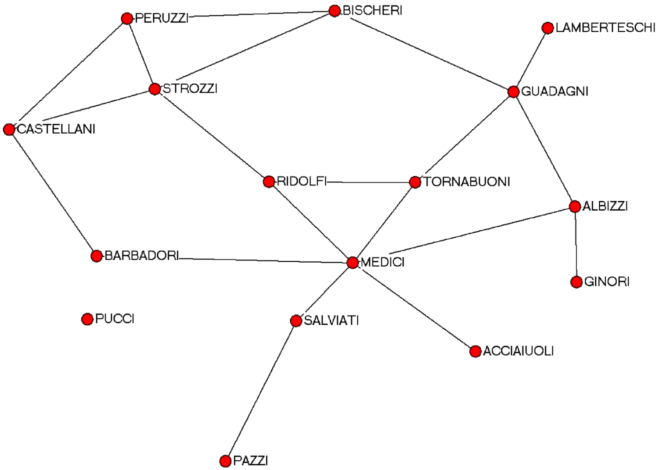
Overview

1. Understanding the structure of a network:
   A lot of definitions to identify important features of networks.
2. How networks form and evolve:
   Networks are not given, how do they grow?
3. How networks influence decision processes:
   Your presentations.

Introduction:

Understanding the structure of a network

Power structure: the Medici family



Padget and Ansell, Robust action and the rise of the Medici. *American Journal of Sociology*, 1993.
Image from M. Jackson, Social and Economic Networks. Princeton, 2008.

Power structure: the Medici family

- How popular are them? They are connected with 6 families. Strozzi: 4, Guadagni: 4. Not enough as explanation of their rise to power...
- How connected are them?
- How tight are them?
- How important are them?

Let's inspect the power structure...

Power structure: the Medici family

- $P(ij)$ number of shortest paths connecting family i to family j
- $P_k(ij)$ number of shortest paths between i and j including k
- Barbadori-Guadagni: 2, Barbadori-Guadagni including Medici: 2, Barbadori-Guadagni including Strozzi: 0

One possible definition of power

The average fraction of shortest path between two families including the Medici (M) can be expressed as follows:

$$\sum_{\{i,j|i\neq j, M\notin\{i,j\}\}} \frac{P_M(ij)/P(ij)}{(n-1)(n-2)/2}$$

Power of the Medici: 0.522. Strozzi: 0.103, Guadagni: 0.255.

L.C. Freeman, A set of measures of centrality based on betweenness, *Sociometry*, 1977.

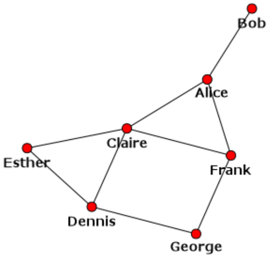
Basic Definitions:

Understanding the structure of a network

Basic definitions: Network

Definition

A network is given by a set of nodes (agents, vertices...) $N = \{1, \ldots, n\}$ *and an adjacency matrix* g.



$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

- Default is undirected: g is symmetric
- Default is irreflexive: $g_{ii} = 0$
- Weighted networks are modelled by matrixes of real numbers

Basic Definitions: Notation

- A subnetwork $g' \subset g$ if $g'_{ij} \Rightarrow g_{ij}$
- $ij \in g$ stands for $g_{ij} = 1$
- g and g' are isomorphic if there exists a relabelling ρ (i.e., bijection) of N that brings g to g', i.e., such that $g'_{\rho(i)\rho(j)} = g_{ij}$
- $g_{\restriction S}$ is the restriction of g to $S \subseteq N$
- The neighbourhood of $i \in N$ is $N_i(g) = \{j \mid ij \in g\}$.
- The $k+1$-neighbourhood of i is defined (recursively) as:

$$N_i^{k+1}(g) = N_i^k(g) \cup \bigcup_{\{j \in N_i^k(g)\}} N_j(g)$$

Basic Definitions: Paths and Cycles

- A path between i and j is a sequence of links $i_1 i_2 \dots i_k$ between distinct nodes with $i_1 = i$ and $i_k = j$ (formal: such that $i_t i_{t+1} \in g$ for all $t < k$).
- A geodesic between i and j is the shortest path connecting i and j.
- A walk between i and j is a sequence of links $i_1 i_2 \dots i_k$ with $i_1 = i$ and $i_k = j$ (nodes can repeat in a walk).
- A cycle is a walk that starts and end at the same node i.
- g is connected if there is a path between each pair of nodes. A connected component of g is a maximal connected subgraph of g.

Exercise: how to calculate the number of walks of lenght k between i and j? What is the lenght of the shortest path between i and j?

Basic Definitions: Last Slide

- A tree is:

Basic Definitions: Last Slide

- A tree is: a connected graph with no cycles.
- A forest is:

Basic Definitions: Last Slide

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- A star is: a network with one $i \in N$ such that $g_{ij} = 1$ for all j and $g_{jj'} = 0$ otherwise.
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Basic Definitions: Last Slide

- A tree is: a connected graph with no cycles.
- A forest is: such that each connected component is a tree.
- A star is: a network with one $i \in N$ such that $g_{ij} = 1$ for all j and $g_{jj'} = 0$ otherwise.
- A circle is: a network with a single cycle and such that each node has exactly two neighbours.

Exercise: how many graphs with 30 nodes?

Understanding the structure of a network: Questions

Let (N, g) be a network, and i a node:

- How popular is i? $\rightarrow$ Degree
- How large is a network? $\rightarrow$ Diameter
- How tightly connected is i? $\rightarrow$ Clustering
- How important is i? $\rightarrow$ Centrality

Degree and Degree Distribution

Definition

The degree of node i is the number of nodes that are connected to i

$$d_i(g) = |N_i(g)|$$

Exercise: how to count this on the adjacency matrix?

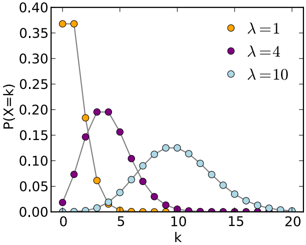
Definition

The degree distribution $p(d)$ of a network (N, g) is the frequency (listed values or probability distribution) of nodes with degree d.

This is a very important description of the network!

The Poisson Distribution

Assume links form randomly with probability p.



Degree distribution for large $|N|$ approximates:

$$p(d) = \frac{e^{-(n-1)p}((n-1)p^d)}{d!}$$

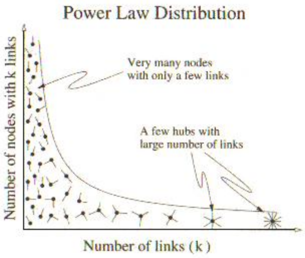
P. Erdös and A. Rényi. On Random Graphs, *Publicationes Mathematicae Debrecen*, 1959.

Image from wikipedia.

The Scale-free Distribution

Degree is $p(d) = cd^{-\gamma}$

- Scale-free because $\frac{p(kd)}{p(d)} = \frac{p(kc)}{p(c)}$.
- Linear if plotted on a log-log scale.
- Examples: WWW, collaboration networks...($2 < \gamma < 3$ usually)
- Typically organised into hubs!



Dates back to V. Pareto. *Cours d'Economie Politique*, 1896.

http://www.macs.hw.ac.uk/ pdw/topology/ScaleFree.html

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The Diameter

The distance between i and j is the length of the geodesic between i and j.

Definition

The diameter of a network (N, g) *(also applies to connected components or subgraphs) is the maximum distance between two nodes in* N*.*

Exercise: what is the diameter of a tree? of a circle?

Other interesting measures: average path length, minimal path length...

Exercise: how to compute minimal path length between i and j?

Average Path Length

Historically interesting question! Remember the 6 degrees of separation...

Here are some observations:

- Friendship network (Milgram, 1967, letter experiment): median 5.5 on 25% of letters that made it
- Math collaborations network (Grossmann, 2002): mean 7.6 max 27
- The internet (Adamic, Pitkow, 1999): mean 3.1
- Facebook (Backstrom Et Al, 2012): mean 4.74

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Clustering: possible definitions

Are my connections connected between each other? And in average?

Definition

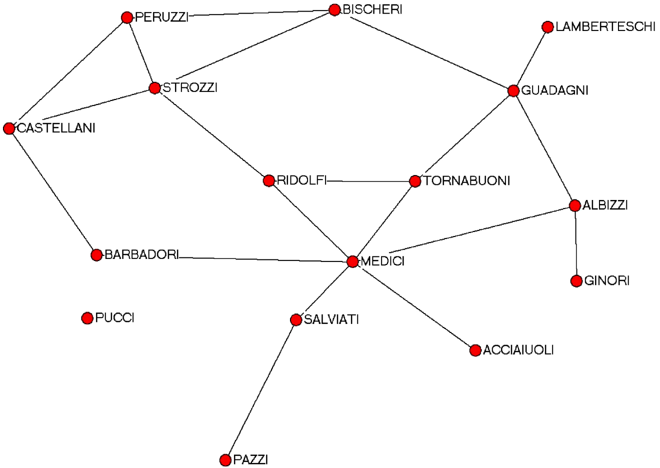
The clustering of node i is the ratio of pairs of nodes connected to i that are also connected between each other:

$$Cl(g) = \frac{\sum_i |\{jk \in g \mid k \neq j, j, k \in N_i(g)\}|}{\sum |\{jk \mid k \neq j, j.k \in N_i(g)\}|}$$

- average clustering coefficient can be measured (two different formulas).
- cliques and transitive triples are the main example of clusters.
- useful to detect small worlds networks

M.E.J. Newman. *The structure and function of complex networks.* Siam review 2003

Clustering: example



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Measures of Centrality I

There are many notions to measure how central a node is (i.e., in a decision network, how powerful):

Degree Centrality

The degree centrality of node i can be measured by its discounted degree:

$$C_i^D(g) = \frac{d_i(g)}{|n-1|}$$

Closeness Centrality

The closeness centrality of node i is the inverse of the average shortest distance between i and any other node:

$$C_i^C(g) = \frac{n-1}{\sum_{i \neq j} \ell(i,j)}$$

L.C. Freeman, A set of measures of centrality based on betweenness, *Sociometry*, 1977.

Measures of Centrality II

Betweenness Centrality

The average fraction of shortest path between two arbitrary nodes including i:

$$C_i^B(g) = \sum_{\{k,j|k\neq j, i\notin\{i,j\}\}} \frac{P_M(kj)/P(kj)}{(n-1)(n-2)/2}$$

An elegant centrality measure: the Katz measure of prestige of node i is the sum of the prestige of the nodes connected to i discounted by their degree.

Exercise 1: show that the vector of Katz prestige of all nodes is an eigenvector of the adjacency matrix discounted by the degrees.
Exercise 2: show that this is equivalent up to a scalar to degree centrality.

Eigenvector Centrality

The eigenvector centrality of node i is the i-th coordinate of the eigenvector associated to the largest eigenvalue of the adjacency matrix g.

P. Bonacich. *Power and centrality: a family of measures*, American J. of Sociology, 1987.

Last Slide

In this lecture we have defined interesting features of networks:

- Networks are represented as adjacency matrices, and this representation is very useful to compute the basic features of a network.
- There are several notions to characterise a node and describe features of a network: how popular (degree), how tightly connected (clustering), how important (centrality) ...

In the next lecture we are going to study how networks form and grow:

- Erdös-Rényi random graphs
- Preferential attachment and scale-free networks
- Strategic network formation