Optimal portfolio for HARA utility functions in a pure jump multidimensional incomplete market

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Abstract: In this paper we analyse a pure jump incomplete market where the risky assets can jump upwards or downwards. In this market we show that, when an investor wants to maximise a HARA utility function of his/her terminal wealth, his/her optimal strategy consists of keeping constant proportions of wealth in the risky assets, thus extending the classical Merton result to this market. Finally, we compare our results with the classical ones in the diffusion case in terms of scalar dependence of portfolio proportions on the risk-aversion coefficient.

Keywords: convex duality; duality gap; equivalent martingale measures; HARA utility functions; HJB equation; incomplete markets; Poisson processes; portfolio optimisation.

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1 Introduction

The problem of maximisation of a utility function of the wealth of an investor operating in financial markets is a classical one. Very early on (even before modern finance saw the light with the celebrated Black and Scholes formula), it was discovered that, when the investor maximises a utility function u belonging to a 'good' class, then his optimal portfolio consists of allocating his wealth in constant proportion between the assets in the market: this result holds both in discrete (Samuelson, 1969) as well as in continuous time (Merton, 1969). These 'good' functions are the so-called HARA (hyperbolic absolute risk aversion) utility functions, characterised by the fact that $-u''(x)/u'(x) = \delta/x$, that is the so-called Pratt's absolute risk aversion (Pratt, 1964), is an hyperbole. These utility functions are, thus, $u(x) = \log x$ and $u(x) = x^\gamma/\gamma$, with $\gamma = 1 - \delta < 1$ (the case $\gamma = 0$ corresponding to the logarithm). Notice that this result does not necessarily hold true if the utility function has another generic form. These results, originally proven by the use of dynamic programming, have been recently generalised using convex duality, even in incomplete diffusion markets (Karatzas et al., 1991).

In this paper, we solve the same problem in a market where a riskless asset and n risky assets are present and the risky assets, driven by an m-dimensional Poisson process with independent components and constant intensities, can jump upwards or downwards in continuous time. This is an extension of the multinomial model, in the sense that the price of the risky assets can increase or decrease for fixed factors but, in this model, the instants of these changes are not fixed but random. While market models which include jumps already appeared in the literature in the field of utility maximisation problems dealing with only one risky asset (Bellamy, 2001; Jeanblanc Picqué and Pontier, 1990) and in other contexts (Korn et al., 2003; Lim and Zhou, 2004; Runggaldier, 2003), to the authors' knowledge this is the first time that the utility maximisation problem is explicitly solved in a multidimensional model which includes jumps. Also, this model can be obtained if one assumes that the price vector of the risky assets is driven by a multivariate Poisson process (Runggaldier, 2003). This model can be significant when dealing (for example) with high-frequency data, where the evolution of the prices has not a continuous evolution but all the price movements are due to jumps. Also, by assigning to some of the intensities values much smaller or much greater than the others, one can represent models where different time scales play different roles. As these assets are modelled via $m > n$ independent Poisson processes with constant intensities, this market is incomplete. The final result is that the optimal strategy of the investor is to keep constant proportions of his wealth in the market's assets, thus, it is similar to the previous strategies already present in literature about HARA utility functions.

In order to derive our results, we use the by-now classical method of convex duality (see Schachermayer (2004) for a survey). This method consists of transforming the original 'primal' utility maximisation problem into an equivalent 'dual' problem, where we minimise the so-called conjugate function of u over the set of all equivalent martingale measures: it is well known that, in our situation, an equivalent change of measure corresponds to changing the intensities of the Poisson processes. We characterise this minimiser and show that the Poisson processes still have constant intensities under this optimal martingale measure. The advantages of using the convex-duality approach instead of a direct method (Callegaro et al., 2006) are, mainly, that we explicitly obtain the optimal equivalent martingale measure $\mathbb{Q}^*$, which is very useful in many situations and that, with this formulation, $\mathbb{Q}^*$ can be clearly seen to be unique. By the use of this characterisation, we find the optimal final wealth and show that it is admissible by proving that the so-called duality gap is zero. Finally, we show that there exists a portfolio strategy which realises the optimal portfolio and, by calculating it explicitly, we find out that it corresponds to keeping fixed proportions of wealth at each time in the riskless and in the risky assets, these proportions being functions of the coefficients of the assets and of the utility function's parameters. This allows us to see a difference between our market and a diffusion market: in fact, in the latter case the optimal portfolio in the risky assets is proportional to a fixed vector of risky assets' proportions of the total wealth and this proportionality only depends on the risk-aversion coefficient γ (Karatzas et al., 1991). We show, by using a counter-example, that in our market this is no longer true.

The paper is organised as follows: in Section 2 we present the market model and state the utility maximisation problem. In Section 3 we characterise all the martingale measures of this market. In Section 4 we state and solve the dual problem. In Section 5 we characterise the optimal final wealth in terms of the optimal martingale measure and show that it is admissible for the primal problem. In Section 6 we characterise the optimal portfolio strategy. In Section 7 we analyse the complete market case and in Section 8 we obtain more explicit results for the case $N = 1$, $M = 2$. Finally, in Section 9 we show that the dependence of the optimal portfolio proportions on the risk-aversion coefficient is more representative of a simple scalar dependence.

2 The market model and the primal problem

We consider an extension of the multinomial model, in the sense that the price of the risky assets can increase or decrease due to fixed factors, but the instants of these changes are not fixed but random. The financial market considered, then, consists of a money market account with price B_t and n risky assets $S^i_t, i = 1, \ldots, n$, whose dynamics are given, under the measure $\mathbb{P}$, by the following stochastic differential equations:

$$\begin{aligned} \mathrm{d}B_t &= rB_t\mathrm{d}t \\ \mathrm{d}S^i_t &= S^i_{t^-}\left[\sum_{j=1}^{m}(e^{a_{ij}} - 1)\mathrm{d}N^j_t\right], \quad i = 1, \ldots, n, \end{aligned} \tag{1}$$

where we impose that the $n \times m$ matrix $A := (e^{a_{ij}} - 1)_{i=1,\ldots,n,j=1,\ldots,m}$ has maximum rank and $(N_t)_t = (N^1_t, \ldots, N^m_t)_t$ is an m-dimensional Poisson process, with $m \geq n$, on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_t)$, where $(\mathcal{F}_t)_t$ is the right-continuous filtration generated by N augmented by all the $\mathbb{P}$-null sets of Ω. We assume that the m components are independent and that their intensities, λ^j, $j = 1, \ldots, m$, are positive constants. Equivalently we have:

$$\begin{aligned} B_t &= B_0 e^{rt} \\ S^i_t &= S^i_0 \exp\left[\sum_{j=1}^{m} a_{ij}N^j_t\right], \quad i = 1, \ldots, n. \end{aligned}$$

Such a market is, in general, *incomplete* if $m > n$. Furthermore, if we suppose that there is *no arbitrage* possibility, this implies that there exists at least a martingale measure equivalent to $\mathbb{P}$ (possibly infinite many if the market is incomplete), which we call EMM (equivalent martingale measure) for short.

Finally, let $(\alpha, \beta) = (\alpha_t^1, \ldots, \alpha_t^n, \beta_t)_t$ be an $(n+1)$-uple of $\mathcal{F}_t$-predictable processes, representing the investment strategy at time $t \in [0, T]$, where α_t^i is the number of units of the i-th asset and β_t is the number of units of the riskless assets which are held in the portfolio at time t. The processes α_t^i and β_t have to satisfy the following integrability conditions with respect to the compensated processes $M_t^j : \forall t \geq 0$, $\forall i = 1, \ldots, n, j = 1, \ldots, m$,

$$\int_0^t |\alpha_s^i| S_s^i \lambda^j \, \mathrm{d}s < \infty, \quad \int_0^t |\beta_s| \lambda^j \, \mathrm{d}s < \infty \ \mathbb{P}\text{-a.s.} \tag{2}$$

The value at time t of a portfolio corresponding to the strategy (α, β) is, then, the $\mathcal{F}_t$-measurable random variable

$$V_t = \sum_{i=1}^n \alpha_t^i S_t^i + \beta_t B_t.$$

The portfolio is self-financing if:

$$\mathrm{d}V_t = \sum_{i=1}^n \alpha_t^i \mathrm{d}S_t^i + \beta_t \mathrm{d}B_t.$$

Notice that if the portfolio is self-financing and if we know $V_0 = v$ and $\alpha = (\alpha_t^1, \ldots, \alpha_t^n)_t$, we then, also, know V_t and β_t, $\forall t$. In this case we often indicate the portfolio as V^α.

With these elements, we can formulate the primal problem for a generic utility function: given an initial wealth v and a fixed time horizon T, maximise the expected value of the utility of the terminal value of the self-financing portfolio

$$\begin{cases} \max_\alpha \ E\{u(V_T^\alpha)\} \\ \alpha : \text{self-financing strategy such that} \\ E^{\mathbb{Q}}[B_T^{-1} V_T^\alpha] \leq v \ \forall \mathbb{Q} \ \text{EMM}. \end{cases} \tag{3}$$

When u is of the HARA class, then

$$u(x) = \begin{cases} x^\gamma / \gamma, & \gamma < 1, \gamma \neq 0 \\ \log x, & \gamma = 0. \end{cases}$$

Notice that if α is a self-financing strategy, the discounted value of the portfolio $(B_t^{-1} V_t^\alpha)_t$ is a $(\mathbb{Q}, \mathcal{F}_t)$-martingale $\forall \mathbb{Q}$ EMM. For this reason, the meaning of the constraint $E^{\mathbb{Q}}[B_t^{-1} V_t^\alpha]$ in Equation (3) is that the investor's initial wealth is less than or equal to v.

The primal problem can be solved in two steps:

Step 1: determine V_T^* which solves the problem (without α)

$$\begin{cases} \max E\{u(V_T)\} \\ V_T \in \mathcal{V}_v \end{cases} \tag{4}$$

where we define

$$\mathcal{V}_v := \left\{ V_T \text{r.v.} : E^{\mathbb{Q}}[B_T^{-1} V_T] \leq v \ \forall \mathbb{Q} \ \text{EMM} \right\}. \tag{5}$$

Step 2: determine the optimal investment strategy α^* such that

$$V_T^{\alpha^*} = V_T^* \ \mathbb{P} \, \text{a.s.}$$

3 The set of all the EMMs

Let us now consider the compensated Poisson processes under the measure $\mathbb{P}$:

$$M_t^j := N_t^j - \int_0^t \lambda^j \mathrm{d}u = N_t^j - \lambda^j t, \quad j = 1, \ldots, m \tag{6}$$

which are $(\mathbb{P}, \mathcal{F}_t)$-martingales. If we introduce the discounted prices $\tilde{S}_t^i := S_t^i / B_t$, the dynamics under the measure $\mathbb{P}$ are (Equation (1))

$$\mathrm{d}\tilde{S}_t^i = \mathrm{d}(S_t^i / B_t) = \tilde{S}_{t^-}^i \left\{ \sum_{j=1}^m (e^{a_{ij}} - 1)\, \mathrm{d}N_t^j - r\, \mathrm{d}t \right\}. \tag{7}$$

The Radon-Nikodym derivative for an absolutely continuous change of measure from $\mathbb{P}$ to $\mathbb{Q}$, that implies a change of the Poisson intensities from λ^j to $\lambda^j \psi_t^j, j = 1, \ldots, m$, is:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = Z_T(\psi^1, \ldots, \psi^m) = \exp\left\{ \int_0^T \sum_{j=1}^m (1 - \psi_t^j) \lambda^j \mathrm{d}t + \int_0^T \sum_{j=1}^m \log \psi_t^j \, \mathrm{d}N_t^j \right\} \tag{8}$$

where ψ^j, $j = 1, \ldots, m$, must be non-negative predictable processes, satisfying

$$\int_0^T \psi_s^j \lambda^j \, \mathrm{d}s < \infty \quad \mathbb{P}\text{-a.s.} \tag{9}$$

Furthermore, we want the Radon-Nikodym derivative to give a probability measure, so the processes ψ^j, $j = 1, \ldots, m$, must be such that the following condition holds true:

$$E^{\mathbb{P}}\{Z_t\} = 1 \quad \forall t \in [0, T]. \tag{10}$$

A sufficient condition for Equation (10) to be valid can be found in Brémaud (1981, Theorem VIII, T11). Defining the Poisson martingales $M^{j\mathbb{Q}}$, $j = 1, \ldots, m$, by

$$\mathrm{d}M_t^{j\mathbb{Q}} = \mathrm{d}N_t^j - \lambda^j \psi_t^j \, \mathrm{d}t$$

the dynamics of $\tilde{S}^i, i = 1, \ldots, n$, under $\mathbb{Q}$ become (Equation (7)),

$$\mathrm{d}\tilde{S}_t^i = \tilde{S}_{t^-}^i \left\{ \sum_{j=1}^m (e^{a_{ij}} - 1)\, \mathrm{d}M_t^{j\mathbb{Q}} + \sum_{j=1}^m (e^{a_{ij}} - 1)\, \lambda^j \psi_t^j \mathrm{d}t - r\mathrm{d}t \right\}, i = 1, \ldots, n. \tag{11}$$

Taking as numeraire, as usual, the money market account B_t, we immediately see that $\mathbb{Q}$ is a martingale measure, i.e., a measure under which $\tilde{S}_t$ is a martingale, if and only if the $\psi^j \geq 0, j = 1, \ldots, m$ are chosen such that

$$\sum_{j=1}^m (e^{a_{ij}} - 1) \lambda^j \psi_t^j = r, \quad \mathbb{P}\text{-a.s. for all } t \tag{12}$$

for all $i = 1, \ldots, n$. We have obtained infinitely many martingale measures characterised by the Radon-Nikodym densities and the Radon-Nikodym density processes

$$\frac{\mathrm{d}\mathbb{Q}^{\psi}}{\mathrm{d}\mathbb{P}} = Z_T^{\psi}, \qquad Z_t^{\psi} = \left.\frac{\mathrm{d}\mathbb{Q}^{\psi}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}$$

each one parameterised by the process ψ such that Equations (9), (10) and (12) hold.

The set of positive processes ψ, which parameterise the set of the EMM and make Equations (9), (10) and (12) hold, will be denoted by $\mathcal{N}_r$.

4 The dual problem and its solution

Using the theory of convex duality in Luenberger (1969), we now introduce the dual problem and we solve it in order to solve the primal problem.

First of all, define the dual functional

$$\begin{aligned} L(\psi, \lambda) : &= \sup_{V_T} \left\{ E[u(V_T)] - \lambda E^{\mathbb{Q}^{\psi}}[B_T^{-1} V_T] + \lambda v \right\} \\ &= \lambda v + \sup_{V_T} \left\{ E[u(V_T) - \lambda Z_T^{\psi} B_T^{-1} V_T] \right\}. \end{aligned}$$

Recalling the definition of *conjugate convex function* $\tilde{u}(\cdot)$ associated with $u(\cdot)$:

$$\tilde{u}(b) = \sup_{x \geq 0} \{u(x) - xb\}, \qquad b > 0$$

we have

$$L(\psi, \lambda) = \lambda v + E\left[\tilde{u}(\lambda Z_T^{\psi} B_T^{-1})\right].$$

So our dual problem is

$$\begin{cases} \min \quad L(\psi, \lambda) \\ \psi \in \mathcal{N}_r, \lambda > 0. \end{cases} \tag{13}$$

In our specific case of HARA utility functions, the conjugate convex function is

$$\tilde{u}(b) = \begin{cases} -b^{\tilde{\gamma}}/\tilde{\gamma}, & \text{if } \gamma < 1, \gamma \neq 0, \quad \text{with } \tilde{\gamma} := \dfrac{\gamma}{\gamma - 1} \\ \log\left(\dfrac{1}{b}\right) - 1, & \text{if } \gamma = 0. \end{cases}$$

In order to find the minimum of the dual problem, we first minimise L with respect to ψ, for all fixed $\lambda \in \mathbb{R}^+$ and obtain an optimum ψ^*, then we minimise with respect to λ and obtain an optimum λ^*.

4.1 The optimal ψ^*

For all fixed $\lambda > 0$, the dual problem (Equation 13) is equivalent to

$$\min_{\psi \in \mathcal{N}_r} E\left[\tilde{u}(Z_T^{\psi})\right]. \tag{14}$$

In fact, if $\gamma < 1$, $\gamma \neq 0$ we have

$$L(\psi, \lambda) = \lambda v - E\left[\frac{1}{\tilde{\gamma}}(\lambda Z_T^{\psi} B_T^{-1})^{\tilde{\gamma}}\right] = \lambda v - \left(\frac{\lambda}{B_T}\right)^{\tilde{\gamma}} E\left[\frac{1}{\tilde{\gamma}}(Z_T^{\psi})^{\tilde{\gamma}}\right] = \lambda v + \left(\frac{\lambda}{B_T}\right)^{\tilde{\gamma}} E\left[\tilde{u}(Z_T^{\psi})\right]$$

while if $\gamma = 0$ we have

$$\begin{aligned} L(\psi, \lambda) &= \lambda v - 1 - E\Big[\log(\lambda Z_T^{\psi} B_T^{-1})\Big] \\ &= \lambda v - 1 - \log\left(\frac{\lambda}{B_T}\right) - E\Big[\log(Z_T^{\psi})\Big] \\ &= \lambda v - \log\left(\frac{\lambda}{B_T}\right) + E\Big[\tilde{u}(Z_T^{\psi})\Big]. \end{aligned}$$

Theorem 4.1. The solution to Equation (14) is given by the positive process $\psi_t^* \equiv \psi^*$, where $(\psi^*, \bar{\lambda})$ is an $(m+n)$-dimensional vector which is the unique solution to the algebraic system

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \bar{\lambda}^i (e^{a_{ij}} - 1) = (\psi^j)^{\frac{1}{\gamma - 1}} - 1, \quad j = 1, \ldots, m \\ \displaystyle\sum_{j=1}^{m} (e^{a_{ij}} - 1) \lambda^j \psi^j = r, \quad i = 1, \ldots, n. \end{cases} \tag{15}$$

Remark 4.2. Note that by introducing the vectors $\bar{\lambda} = (\bar{\lambda}^1, \ldots, \bar{\lambda}^n)$ and $\Psi = ((\psi_t^1)^{\tilde{\gamma}-1} - 1, \ldots, (\psi_t^m)^{\tilde{\gamma}-1} - 1)$ and using the fact that the matrix A has maximum rank, the first m equations in Equation (15) can be rewritten in the more compact form

$$\bar{\lambda}^* = \Psi A^T (A A^T)^{-1}$$

Proof. We can see Equation (14) as a stochastic control problem of a pure-jump process of the kind

$$\Phi(z, t) = \inf_{\psi \in \mathcal{N}_r} J^{(\psi)}(z, t) = \inf_{\psi \in \mathcal{N}_r} E\Big[\tilde{u}(Z_T^{\psi}) | Z_t^{\psi} = z\Big] \tag{16}$$

where the control process is $\psi \in \mathcal{N}_r$ and the controlled jump process is Z_t^{ψ}, whose dynamics is

$$\begin{cases} \mathrm{d}Z_s^{\psi} = Z_{s^-}^{\psi} \Big\{ \sum_{j=1}^{m} (1 - \psi_s^j) \lambda^j \mathrm{d}s + \sum_{j=1}^{m} (\psi_s^j - 1) \mathrm{d}N_s^j \Big\}, \quad t < s \\ Z_t^{\psi} = z. \end{cases} \tag{17}$$

As known, the solution to Equation (16) is linked to the following HJB equation (Øksendal and Sulem, 2005, Th. 3.1)

$$\begin{aligned} \inf_{\psi \in \mathcal{N}_r} &\left\{ \left[z \sum_{j=1}^{m} (1 - \psi_t^j) \lambda^j \right] \frac{\partial \phi}{\partial z}(z, t) + \sum_{j=1}^{m} \int_{\mathbb{R}} \big[\phi(z + zy(\psi_t^j - 1), t) - \phi(z, t) \big] \lambda^j \delta_1(\mathrm{d}y) \right\} \\ &+ \frac{\partial \phi}{\partial t}(z, t) = 0 \end{aligned} \tag{18}$$

where $\lambda^j \delta_1(\mathrm{d}y) = \nu_j(\mathrm{d}y)$ is the Lévy measure of N^j, $j = 1, \ldots, n$. If we try a solution of the kind

$$\phi(z, t) = -\frac{1}{\tilde{\gamma}} k(t) z^{\tilde{\gamma}} \quad \text{for } \gamma < 1, \gamma \neq 0,$$

with k a positive $\mathcal{C}^1$ function, we get

$$\inf_{\psi\in\mathcal{N}_r}\left\{-k(t)z^{\tilde{\gamma}}\left[\sum_{j=1}^{m}(1-\psi_t^j)\lambda^j\right]-\frac{1}{\tilde{\gamma}}\,k(t)z^{\tilde{\gamma}}\left[\sum_{j=1}^{m}((\psi_t^j)^{\tilde{\gamma}}-1)\lambda^j\right]\right\}-\frac{1}{\tilde{\gamma}}\,k'(t)z^{\tilde{\gamma}}=0$$

which is equivalent to

$$k(t)z^{\tilde{\gamma}}\inf_{\psi\in\mathcal{N}_r}\left\{\sum_{j=1}^{m}(\psi_t^j-1)\lambda^j-\frac{1}{\tilde{\gamma}}((\psi_t^j)^{\tilde{\gamma}}-1)\lambda^j\right\}-\frac{1}{\tilde{\gamma}}\,k'(t)z^{\tilde{\gamma}}=0.$$

An optimal ψ^* is then a solution to the problem

$$\inf_{\psi\in\mathcal{N}_r}\sum_{j=1}^{m}\left((\psi_t^j-1)\lambda^j-\frac{1}{\tilde{\gamma}}((\psi_t^j)^{\tilde{\gamma}}-1)\lambda^j\right)$$

for all $t\in[0,T]$ and so, to find the optimal EMM we have to solve the following constrained convex optimal problem (recall Equation (12))

$$\begin{cases}\min\quad \displaystyle\sum_{j=1}^{m}\left((\psi_t^j-1)\lambda^j-\frac{1}{\tilde{\gamma}}((\psi_t^j)^{\tilde{\gamma}}-1)\lambda^j\right)\\ \psi_t^j\geq 0,\quad j=1,\ldots,m,\\ \displaystyle\sum_{j=1}^{m}(e^{a_{ij}}-1)\lambda^j\psi_t^j=r,\quad i=1,\ldots,n.\end{cases}\tag{19}$$

The solution ψ^* to Equation (19) is unique and given by the solution of Equation (15), as it will be shown in Lemma 4.3. By putting $k^*(t):=e^{r^*(t-T)}$, where

$$r^*:=\tilde{\gamma}\sum_{j=1}^{m}(\psi^*-1)\lambda^j-\sum_{j=1}^{m}((\psi^*)^{\tilde{\gamma}}-1)\lambda^j$$

we have that $\phi(z,t):=-\frac{1}{\tilde{\gamma}}k^*(t)z^{\tilde{\gamma}}$ solves the HJB Equation (18) with final condition $\phi(z,T)=-\frac{1}{\tilde{\gamma}}z^{\tilde{\gamma}}$. Then, by Øksendal and Sulem (2005), ψ^* solves our problem.

The case $\gamma=0$ is analogous (the proof in this case is even simpler recalling the definition of intensity of a Poisson process (Brémaud, 1981)).

Lemma 4.3. The solution ψ^* to the constrained convex optimisation Equation (19) is unique and it is given by the unique solution $(\psi^*,\bar{\lambda}^*)$ to Equation (15).

Proof. Our aim is to prove that, in order to determine the optimal ψ_t^*, we can consider and easily solve a problem equivalent to Equation (19), where the admissible region is compact.

The admissible region is not empty because we have assumed *absence of arbitrage* on the market and so (by the first fundamental theorem of asset pricing) there exists at least one equivalent martingale measure, i.e., the constraint in Equation (19) holds for at least one vector $(\psi_t)_t$. Now, consider one such point $\bar{\psi}_t=(\bar{\psi}_t^1,\ldots,\bar{\psi}_t^m)$ for a fixed time $t\in[0,T]$.

We now evaluate the objective function in $\bar{\psi}_t$ and define

$$m := \sum_{j=1}^{m}\left[(\bar{\psi}_t^j - 1)\lambda^j - \frac{1}{\tilde{\gamma}}((\bar{\psi}_t^j)^{\tilde{\gamma}} - 1)\lambda^j\right] =: f(\bar{\psi}_t).$$

Since we are looking for the minimum of f, this will be equal to or lower than m. We also have

$$\lim_{|\psi_t|\to+\infty} f(\psi_t) = +\infty.$$

This means that the original problem is equivalent to a problem with the same value function f and the compact admissible region

$$C_m := \left\{\psi \middle| \psi^j \geq 0, j = 1, \ldots, M; f(\psi) \leq m; \sum_{j=1}^{m}(e^{a_{ij}} - 1)\lambda^j\psi^j = r, i = 1, \ldots, n\right\}.$$

The set C_m is closed, convex and bounded. Thus, we have a continuous function defined on a compact set: it admits minimum and Equation (19) has a solution which, furthermore, is unique because of the strict convexity of f. Introducing the Lagrangian function with Lagrange multipliers $\bar{\lambda}^1, \ldots, \bar{\lambda}^n$ we can finally solve the problem using first order necessary conditions, which are the first m equations in Equation (15). By Remark 4.2, the uniqueness of $\bar{\lambda}^*$ is evident.

Remark 4.4. Since the process ψ^* is constant, Equations (9) and (10) are satisfied and, furthermore, Brémaud (1981, Theorem VIII, T11) is verified: $Z_t^{\psi^*}$ is a real Radon-Nikodym derivative and so $\mathbb{Q}^{\psi^*} = \mathbb{Q}^*$ is actually an EMM.

4.2 The optimal λ^*

In order to solve the dual problem it remains now to find the optimal λ^*. For $\gamma < 1$, $\gamma \neq 0$, then we have to solve

$$\min_{\lambda>0} L(\psi^*, \lambda) = \lambda v - \frac{1}{\tilde{\gamma}}(\lambda)^{\tilde{\gamma}}(B_T^{-1})^{\tilde{\gamma}} E\left[(Z_T^{\psi^*})^{\tilde{\gamma}}\right].$$

By differentiating $L(\psi^*, \lambda)$ with respect to λ and considering the equation

$$\frac{\partial}{\partial\lambda} L(\psi^*, \lambda) = 0$$

we have:

$$\lambda^* = \left(\frac{v(B_T)^{\tilde{\gamma}}}{E[(Z_T^{\psi^*})^{\tilde{\gamma}}]}\right)^{\frac{1}{\tilde{\gamma}-1}}. \tag{20}$$

If $\gamma = 0$ we have to minimise with respect to λ the dual functional

$$L(\psi^*, \lambda) = \lambda v - 1 - \log\ \lambda - E\left[\log\ (Z_T^{\psi^*} B_T^{-1})\right].$$

The optimal value λ^* is easily obtained, setting its first derivative with respect to λ equal to zero and so the optimal value λ^* is

$$\lambda^* = \frac{1}{v}. \tag{21}$$

Notice that this is a particular case of Equation (20), as $\gamma = 0$ implies $\tilde{\gamma} = 0$. We have finally obtained the optimal solution (ν^*, λ^*) to the dual problem.

5 The relationship between primal and dual optimal solutions: the optimum $V_T^*(\lambda^*, \psi^*)$

We now want to obtain a relationship between the optimal solution of the primal problem

$$\begin{cases} \max\limits_{V_T} E\{u(V_T)\} \\ V_T \,\text{r.v.} : E^{\mathbb{Q}}[B_T^{-1} V_T] \leq v \quad \forall \mathbb{Q}\text{EMM} \end{cases} \tag{22}$$

and the optimal solution of the dual problem (which we have just obtained in the previous section)

$$\begin{cases} \min \quad L(\psi, \lambda) \\ \psi \in \mathcal{N}_r, \;\; \lambda > 0. \end{cases} \tag{23}$$

Proposition 5.1. Let (ψ^*, λ^*) be the optimal solution of the dual problem and define

$$V_T^* = \left(\lambda^* B_T^{-1} Z_T^{\psi^*}\right)^{\frac{1}{\gamma-1}}. \tag{24}$$

Then V_T^* is admissible for the primal problem, $E^{\mathbb{Q}^{\psi^*}}[V_T^* B_T^{-1}] = E^{\mathbb{Q}^*}[V_T^* B_T^{-1}] = v$ and so it is the optimal solution of the primal Equation (22).

In order to prove the above proposition, it is important to recall the notion of 'duality-gap' and its connection with the optimal solution of an optimisation problem.

In general, when we deal with a primal and a dual optimisation problem, if both the admissible regions are non-empty, then the following result is always true: the values that the objective function of the 'max-problem' has in its admissible region are less than or equal to the analogous values of the 'min-problem'. In our case we have (Equations (22) and (23)):

$$E[u(\bar{V}_T)] \leq L(\bar{\psi}, \bar{\lambda})$$

for each $\bar{V}_T$ admissible for the primal problem and for each pair $(\bar{\psi}, \bar{\lambda})$ admissible for the dual one. It follows that

$$\sup_{V_T \in \mathcal{V}_v} E[u(V_T)] \leq \inf_{\psi \in \mathcal{N}_r, \lambda > 0} L(\psi, \lambda). \tag{25}$$

Furthermore, we define the 'duality-gap' associated to the primal admissible value $\bar{V}_T$ and the pair of dual admissible values $(\bar{\psi}, \bar{\lambda})$ as

$$E[u(\bar{V}_T)] - L(\bar{\psi}, \bar{\lambda}) \leq 0.$$

If there exist V_T^*, ψ^*, λ^* such that the duality-gap is zero, Equation (25) is satisfied as an equality relation and this implies that V_T^*, ψ^*, λ^* are the optimal solutions of our problems (Avriel, 1976; Luenberger, 1969).

Now we prove Proposition 5.1.

Proof. To prove the proposition we must show:

1 the primal admissibility of V_T^*

2 that the duality-gap is zero if in the primal and dual objective functions we substitute V_T^* and (ψ^*, λ^*), respectively.

Firstly, we observe that if V_T^* is given by Equation (24), since in this case λ^* is given by Equation (20) we have

$$\begin{aligned} E^{\mathbb{Q}^*}[B_T^{-1}V_T^*] &= E[Z_T^{\psi^*}B_T^{-1}(\lambda^* Z_T^{\psi^*}B_T^{-1})^{\tilde{\gamma}-1}] \\ &= E\left[(Z_T^{\psi^*})^{\tilde{\gamma}}(B_T^{-1})^{\tilde{\gamma}} \cdot \frac{v(B_T)^{\tilde{\gamma}}}{E[(Z_T^{\psi^*})^{\tilde{\gamma}}]}\right] = v. \end{aligned} \tag{26}$$

We start from point 2. The optimal value of the dual objective function, if we use Equations (24) and (26), is given by

$$\begin{aligned} L(\psi^*, \lambda^*) &= \lambda^* v - \frac{1}{\tilde{\gamma}} \cdot E\Big[(\lambda^*)^{\tilde{\gamma}}(B_T^{-1})^{\tilde{\gamma}}(Z_T^{\psi^*})^{\tilde{\gamma}}\Big] \\ &= \lambda^* E^{\mathbb{Q}^*}[B_T^{-1}V_T^*] - \frac{1}{\tilde{\gamma}} \cdot E\Big[(\lambda^*)^{\tilde{\gamma}}(B_T^{-1})^{\tilde{\gamma}}(Z_T^{\psi^*})^{\tilde{\gamma}}\Big] \\ &= \lambda^* E\Big[Z_T^{\psi^*}B_T^{-1}(\lambda^*)^{\tilde{\gamma}-1}(B_T^{-1})^{\tilde{\gamma}-1}(Z_T^{\psi^*})^{\tilde{\gamma}-1}\Big] \\ &\quad - \frac{1}{\tilde{\gamma}} \cdot E\Big[(\lambda^*)^{\tilde{\gamma}}(B_T^{-1})^{\tilde{\gamma}}(Z_T^{\psi^*})^{\tilde{\gamma}}\Big] \\ &= \left(1 - \frac{1}{\tilde{\gamma}}\right) E\Big[(\lambda^*)^{\tilde{\gamma}}(B_T^{-1})^{\tilde{\gamma}}(Z_T^{\psi^*})^{\tilde{\gamma}}\Big] \\ &= \frac{1}{\gamma} E\big[(V_T^*)^{\gamma}\big] = E\big[u(V_T^*)\big] \end{aligned}$$

which is, thus, the optimal value of the primal objective function, as the duality-gap is equal to zero if point 1 holds. In order to prove this, we have to show that

$$E^{\mathbb{Q}}[B_T^{-1}V_T^*] \le v \quad \forall \mathbb{Q} \quad \text{EMM}$$

that is, using Equations (24) and (20):

$$\begin{aligned} &E\Big[Z_T^{\psi}B_T^{-1}(\lambda^*)^{\tilde{\gamma}-1}(B_T^{-1})^{\tilde{\gamma}-1}(Z_T^{\psi^*})^{\tilde{\gamma}-1}\Big] \\ &\quad = E\left[\frac{Z_T^{\psi}}{B_T} \cdot \frac{v(B_T)^{\tilde{\gamma}}}{E[(Z_T^{\psi^*})^{\tilde{\gamma}}]} \cdot \frac{(Z_T^{\psi^*})^{\tilde{\gamma}-1}}{(B_T)^{\tilde{\gamma}-1}}\right] = E\left[v \frac{Z_T^{\psi}}{Z_T^{\psi^*}} \frac{(Z_T^{\psi^*})^{\tilde{\gamma}}}{E[(Z_T^{\psi^*})^{\tilde{\gamma}}]}\right] \le v \quad \forall \psi \in \mathcal{N}_r. \end{aligned}$$

If we now define a new measure $\tilde{\mathbb{Q}}$ using the Radon-Nikodym derivative $(Z_T^{\psi^*})^{\tilde{\gamma}}/E[(Z_T^{\psi^*})^{\tilde{\gamma}}]$ with respect to $\mathbb{P}$, the proof of the admissibility of V_T^* reduces to proving the following relation

$$\tilde{E}\left[\frac{Z_T^{\psi}}{Z_T^{\psi^*}}\right] \le 1 \tag{27}$$

where $\tilde{E}$ denotes the expected value under the measure $\tilde{\mathbb{Q}}$, which is proved by the following Proposition 5.2.

Proposition 5.2. Let ψ^* be the optimal solution of the dual problem. Then Equation (27) holds $\forall \psi \in \mathcal{N}_r$.

Proof. Firstly, we note that using the Radon-Nikodym derivative

$$\tilde{Z}_T := \frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}} = \frac{(Z_T^{\psi^*})^{\tilde{\gamma}}}{E[(Z_T^{\psi^*})^{\tilde{\gamma}}]},$$

for all $j = 1, \ldots, m$, the intensity of the Poisson process N_t^j changes and, under the measure $\tilde{\mathbb{Q}}$, becomes $\lambda^j(\psi^{j*})^{\tilde{\gamma}}$. In fact (Equation (9)):

$$\tilde{Z}_T = \frac{\exp\left\{\tilde{\gamma} T \sum_{j=1}^m (1-\psi^{j*})\lambda^j\right\} \cdot \prod_{j=1}^m (\psi^{j*})^{\tilde{\gamma} N_T^j}}{\exp\left\{\tilde{\gamma} T \sum_{j=1}^m (1-\psi^{j*})\lambda^j\right\} \cdot E\left\{\prod_{j=1}^m (\psi^{j*})^{\tilde{\gamma} N_T^j}\right\}}$$

and now, using the independence of the N_j and recalling that if $X \sim \mathrm{Po}(\lambda)$, then $E[c^X] = e^{\lambda(c-1)}$, we obtain

$$\tilde{Z}_T = \prod_{j=1}^m \left[(\psi^{j*})^{\tilde{\gamma}}\right]^{N_T^j} \cdot e^{\sum_{j=1}^m \lambda^j T[1-(\psi^{j*})^{\tilde{\gamma}}]}. \tag{28}$$

As $\tilde{Z}_T$ is a Radon-Nikodym derivative, it implies a change of the intensities of the Poisson processes from λ^j to $\lambda^j(\psi^{j*})^{\tilde{\gamma}}, j = 1, \ldots, m$. Using Itô's formula and recalling Equation (17) we have:

$$\mathrm{d}\left(\frac{Z_t^{\psi}}{Z_t^{\psi^*}}\right) = \frac{Z_{t^-}^{\psi}}{Z_{t^-}^{\psi^*}} \left\{ \sum_{j=1}^m \left[(1-\psi_t^j)\lambda^j + (\psi^{j*}-1)\lambda^j\right]\mathrm{d}t + \sum_{j=1}^m \left(\frac{\psi_t^j}{\psi^{j*}} - 1\right)\mathrm{d}N_t^j \right\}.$$

Since we have to calculate the expected value of $Z_T^{\nu}/Z_T^{\nu^*}$ under the measure $\tilde{\mathbb{Q}}$, it is useful to introduce the $\tilde{\mathbb{Q}}$-martingales $\tilde{M}_t^j, j = 1, \ldots, m$, with dynamics

$$\mathrm{d}\tilde{M}_t^j = \mathrm{d}N_t^j - \lambda^j(\psi^{j*})^{\tilde{\gamma}}\mathrm{d}t.$$

So we have

$$\begin{aligned}\mathrm{d}\left(\frac{Z_t^{\psi}}{Z_t^{\nu^*}}\right) &= \frac{Z_{t^-}^{\psi}}{Z_{t^-}^{\psi^*}} \sum_{j=1}^m \left(\frac{\psi_t^j}{\psi^{j*}} - 1\right)\mathrm{d}\tilde{M}_t^j \\ &\quad + \frac{Z_{t^-}^{\psi}}{Z_{t^-}^{\psi^*}} \sum_{j=1}^m \left[(1-\psi_t^j)\lambda^j + (\psi^{j*}-1)\lambda^j + \left(\frac{\psi_t^j}{\psi^{j*}} - 1\right)\lambda^j(\psi^{j*})^{\tilde{\gamma}}\right]\mathrm{d}t\end{aligned}$$

and, finally, under suitable assumptions on the coefficients of $\mathrm{d}\tilde{M}_t^j, j = 1, \ldots, m$,

$$\tilde{E}\left[\frac{Z_T^{\psi}}{Z_T^{\psi^*}}\right] = 1 + \tilde{E}\left\{\int_0^T \frac{Z_{t^-}^{\psi}}{Z_{t^-}^{\psi^*}} \left[\sum_{j=1}^m \lambda^j(\psi_t^j - \psi^{j*})((\psi^{j*})^{\tilde{\gamma}-1} - 1)\right]\mathrm{d}t\right\}. \tag{29}$$

To prove Equation (27), we will now show that the integrand random variable in Equation (29) is null, so that the expected value under the measure $\tilde{\mathbb{Q}}$ is equal to 1. The optimal $\psi^{j^*}, j = 1, \ldots, m$, as shown, are the unique solutions to the algebraic Equation (15) and, then, they satisfy both the first group of equations and the second of Equation (15), while a generic $\psi^j, j = 1, \ldots, m$ satisfy only the second group. Using conditions in Equation (15), we now show that the integrand random variable in Equation (29) is null and so the proposition is proved. In fact, we have

$$
\begin{aligned}
\tilde{E}\left[\frac{Z_T^{\psi}}{Z_T^{\psi^*}}\right] &= 1 + \tilde{E}\left\{\int_0^T \frac{Z_{t^-}^{\psi}}{Z_{t^-}^{\psi^*}}\left[\sum_{j=1}^m \lambda^j(\psi_t^j - \psi^{j^*})\sum_{i=1}^n \bar{\lambda}^i(e^{a_{ij}} - 1)\right]\mathrm{d}t\right\} \\
&= 1 + \tilde{E}\left\{\int_0^T \frac{Z_{t^-}^{\psi}}{Z_{t^-}^{\psi^*}}\left[\sum_{i=1}^n \bar{\lambda}^i \sum_{j=1}^m \lambda^j(\psi_t^j - \psi^{j^*})(e^{a_{ij}} - 1)\right]\mathrm{d}t\right\} \\
&= 1 + \tilde{E}\left\{\int_0^T \frac{Z_{t^-}^{\psi}}{Z_{t^-}^{\psi^*}}\left[\sum_{i=1}^n \bar{\lambda}^i(r - r)\right]\mathrm{d}t\right\} = 1.
\end{aligned}
$$

6 The optimal investment strategy α^*

From the previous section we know that the optimal solution of the primal problem is

$$V_T^* = (\lambda^* B_T^{-1} Z_T^{\psi^*})^{\frac{1}{\gamma-1}} = v B_T \frac{(Z_T^{\psi^*})^{\tilde{\gamma}-1}}{E[(Z_T^{\psi^*})^{\tilde{\gamma}}]}$$

where λ^* is given by Equation (20) and ψ^* is given by the solution of Equation (15).

The aim of this section is to determine the optimal strategy $\alpha^* = (\alpha_t^{1^*}, \ldots, \alpha_t^{n^*}, \beta_t^*)$, i.e., the one which satisfies

$$V_T^* = V_T^{\alpha^*} \qquad \mathbb{P}\text{-a.s.}$$

We shall solve this hedging problem using a martingale representation method: in particular we will take advantage of the property that the discounted value of a self-financing portfolio is a martingale under any martingale measure. It will also be convenient to work under the 'optimal' martingale measure $\mathbb{Q}^* := \mathbb{Q}^{\psi^*}$. We shall define a martingale corresponding to $\tilde{V}_T^*$ at time T and then we will use the dynamics of both these martingales in order to obtain relationships between α^* and the coefficients in the dynamics of the new martingale. We have

$$\tilde{V}_t = \sum_{i=1}^n \alpha_t^i \tilde{S}_t^i + \beta_t, \qquad \mathrm{d}\tilde{V}_t = \sum_{i=1}^n \alpha_t \,\mathrm{d}\tilde{S}_t^i$$

and

$$\tilde{V}_T^* = \frac{v}{E[(Z_T^{\psi^*})^{\tilde{\gamma}}]} \cdot (Z_T^{\psi^*})^{\tilde{\gamma}-1}.$$

We now introduce the $(\mathbb{Q}^*, \mathcal{F}_t)$-martingale $\tilde{M}$:

$$\tilde{M}_t := E^{\mathbb{Q}^*}[\tilde{V}_T^* | \mathcal{F}_t] = \frac{v}{E[(Z_T^{\psi^*})^{\tilde{\gamma}}]} \cdot E^{\mathbb{Q}^*}[(Z_T^{\psi^*})^{\tilde{\gamma}-1} | \mathcal{F}_t].$$

in order to compare the dynamics of the two $(\mathbb{Q}^*, \mathcal{F}_t)$-martingales $\tilde{M}_t$ and $\tilde{V}_t^*$ (note that $\tilde{M}_0 = v$ and $\tilde{M}_T = \tilde{V}_T^*$).

It is useful to recall that under the measure $\mathbb{Q}^*$ the intensities of the Poisson processes are $\lambda^j \psi^{j*}, j = 1, \ldots, m$. Recalling Equation (28) we have

$$E[(Z_T^{\psi^*})^{\tilde{\gamma}}] = \exp\left(\tilde{\gamma} T \sum_{j=1}^{m} (1 - \psi^{j*})\lambda^j + \lambda^j T \sum_{j=1}^{m} ((\psi^{j*})^{\tilde{\gamma}} - 1)\right)$$

and

$$\begin{aligned} E^{\mathbb{Q}^*}\left[(Z_T^{\psi^*})^{\tilde{\gamma}-1} | \mathcal{F}_t\right] &= E^{\mathbb{Q}^*}\left[e^{(\tilde{\gamma}-1)T\sum_{j=1}^{m}(1-\psi^{j*})\lambda^j} \prod_{j=1}^{m} (\psi^{j*})^{(\tilde{\gamma}-1)N_T^j} | \mathcal{F}_t\right] \\ &= e^{(\tilde{\gamma}-1)T\sum_{j=1}^{m}(1-\psi^{j*})\lambda^j} E^{\mathbb{Q}^*}\left[\prod_{j=1}^{m} (\psi^{j*})^{(\tilde{\gamma}-1)(N_T^j - N_t^j + N_t^j)} | \mathcal{F}_t\right] \\ &= e^{(\tilde{\gamma}-1)T\sum_{j=1}^{m}(1-\psi^{j*})\lambda^j} \prod_{j=1}^{m} (\psi^{j*})^{(\tilde{\gamma}-1)N_t^j} \prod_{j=1}^{m} E^{\mathbb{Q}^*}\left[(\psi^{j*})^{(\tilde{\gamma}-1)(N_T^j - N_t^j)}\right] \end{aligned}$$

where the last equality holds because for all t and for every $j = 1, \ldots, m$, the random variables $(N_T^j - N_t^j)$ are independent and each one is also independent of the σ-algebra $\mathcal{F}_t$. If we now recall that under $\mathbb{Q}^*$ $(N_T^j - N_t^j) \sim \mathrm{Po}(\lambda^j \psi^{j*}(T-t)), j = 1, \ldots, m$, we finally have:

$$\begin{aligned} E^{\mathbb{Q}^*}\left[(Z_T^{\psi^*})^{\tilde{\gamma}-1} | \mathcal{F}_t\right] &= e^{(\tilde{\gamma}-1)T\sum_{j=1}^{m}(1-\psi^{j*})\lambda^j} \prod_{j=1}^{m} (\psi^{j*})^{(\tilde{\gamma}-1)N_t^j} \\ &\quad \times e^{(T-t)\sum_{j=1}^{m} \lambda^j \psi^{j*}((\psi^{j*})^{\tilde{\gamma}-1} - 1)}. \end{aligned} \tag{30}$$

After some calculations we then have

$$\begin{aligned} \tilde{M}_t &= v \prod_{j=1}^{m} (\psi^{j*})^{(\tilde{\gamma}-1)N_t^j} e^{-t\sum_{j=1}^{m} \lambda^j \psi^{j*}((\psi^{j*})^{\tilde{\gamma}-1} - 1)} \\ &= \tilde{M}_0 \exp\left\{\sum_{j=1}^{m} \int_0^t \log (\psi^{j*})^{\tilde{\gamma}-1} \mathrm{d}N_s^j + \sum_{j=1}^{m} \int_0^t (1 - (\psi^{j*})^{\tilde{\gamma}-1})\lambda^j \psi^{j*} \mathrm{d}s\right\} \end{aligned} \tag{31}$$

(recall that $\tilde{M}_0 = v$). In order to obtain the optimal investment strategy α^* we now have to compare the differentials of the two martingales $\tilde{M}_t$ and $\tilde{V}_t^*$. The differential of $\tilde{M}_t$ is easily obtained from Equation (32) and, under the optimal martingale measure $\mathbb{Q}^*$, is given by

$$\begin{aligned} \mathrm{d}\tilde{M}_t &= \tilde{M}_{t^-}\left\{\sum_{j=1}^{m} ((\psi^{j*})^{\tilde{\gamma}-1} - 1)\mathrm{d}N_t^j + \sum_{j=1}^{m} (1 - (\psi^{j*})^{\tilde{\gamma}-1})\lambda^j \psi^{j*} \mathrm{d}t\right\} \\ &= \tilde{M}_{t^-}\left\{\sum_{j=1}^{m} ((\psi^{j*})^{\tilde{\gamma}-1} - 1)\mathrm{d}M_t^{j\mathbb{Q}^*}\right\}. \end{aligned}$$

If we compare the following two dynamics

$$\mathrm{d}\tilde{V}_t^* = \sum_{i=1}^{n} \alpha_t^{i*} \tilde{S}_{t^-}^i \sum_{j=1}^{m} (e^{a_{ij}} - 1) \mathrm{d}M_t^{j\mathbb{Q}^*}$$

$$\mathrm{d}\tilde{M}_t = \tilde{M}_{t^-} \sum_{j=1}^{m} ((\psi^{j*})^{\tilde{\gamma}-1} - 1) \mathrm{d}M_t^{j\mathbb{Q}^*}$$

we find that the optimal investment strategy has to satisfy the following system of m equations

$$\sum_{i=1}^{n} \alpha_t^{i*} \tilde{S}_{t^-}^i (e^{a_{ij}} - 1) = \tilde{M}_{t^-} ((\psi^{j*})^{\tilde{\gamma}-1} - 1), \quad j = 1, \ldots, m.$$

Proposition 6.1. The optimal investment strategy exists and it is uniquely determined by $\alpha_t^{i*} := \bar{\lambda}^i \dfrac{\tilde{M}_{t^-}}{\tilde{S}_{t^-}^i}, i = 1, \ldots, n$, where $\bar{\lambda} = (\bar{\lambda}^1, \ldots, \bar{\lambda}^n)$ is the vector found in Theorem 4.1.

Proof. By defining

$$\frac{\alpha_t^{i*} \tilde{S}_{t^-}^i}{\tilde{M}_{t^-}} := \bar{\lambda}^i, \quad i = 1, \ldots, n, \tag{32}$$

the system above is equivalent to the system given by the first m equations in Equation (15), which admits a unique solution.

Remark 6.2. Notice that, if we introduce the fractions $h_t^i, i = 1, \ldots, n$, of wealth invested at time t in $S_t^i, i = 1, \ldots, n$,

$$h_t^i := \frac{\alpha_t^i S_{t^-}}{V_{t^-}} = \frac{\alpha_t^i \tilde{S}_{t^-}}{\tilde{V}_{t^-}}, \quad i = 1, \ldots, n,$$

so that $h_t^{i*} = (\alpha_t^{i*} \tilde{S}_{t^-}^i)/\tilde{V}_{t^-}^*$, noting that $\tilde{V}_{t^-}^* = \tilde{M}_{t^-}$, from Equation (32), we find that the optimal fractions of wealth to invest in each risky asset are constant over time: $h_t^{i*} \equiv h^{i*}, i = 1, \ldots, n$ and they are equal to the $\bar{\lambda}^i, i = 1, \ldots, n$, of Theorem 4.1.

Remark 6.3. From Theorem 4.1 and from the previous remark, substituting the $\psi_t^j, j = 1, \ldots, m$, from the first m equations in Equation (15) into the remaining n and recalling that the optimal h_t^{i*} are equal to the $\bar{\lambda}^i, i = 1, \ldots, n$, we find that the h_t^{i*} must satisfy

$$\sum_{j=1}^{m} \lambda^j (e^{a_{ij}} - 1) \left[\sum_{i=1}^{n} h^{i*} (e^{a_{ij}} - 1) + 1 \right]^{\gamma - 1} = r \tag{33}$$

which is the same result obtained in Callegaro et al. (2006), solving directly the primal problem.

7 The complete market case

In the case when $m = n$ the market is complete and there exists only one EMM, which we will denote by $\mathbb{Q}$. It can be easily seen that Equation (12) becomes a system of m equations in m unknowns, with matrix A having maximum rank m and so it admits a unique solution $\psi^* = (\psi_1^*, \ldots, \psi_M^*)$.

In this case it is not necessary to consider and solve the dual problem, since the primal Equation (4) has only one constraint and becomes

$$\begin{cases} \max E\{u(V_T)\} \\ E^{\mathbb{Q}}[B_T^{-1} V_T] \leq v \end{cases}$$

and can be solved using the Lagrange multiplier technique. By introducing $Z_T := \frac{d\mathbb{Q}}{d\mathbb{P}}$ and the Lagrange multiplier λ, our problem becomes

$$\max_{V_T} \left[E\{u(V_T)\} - \lambda E^{\mathbb{Q}}\{B_T^{-1} V_T\} \right] = \max_{V_T} E\{u(V_T) - \lambda Z_T B_T^{-1} V_T\}. \tag{34}$$

We now notice that, thanks to the properties of the utility function $u(\cdot)$, the inverse function of its derivative exists and we will denote it by

$$I(\cdot) := (u'(\cdot))^{-1}.$$

On the other hand, maximising the expectation on the right hand side of Equation (34) is equivalent to maximising its argument for each $\omega \in \Omega$. A necessary condition, then, for V_T to be optimal is that it satisfies

$$u'(V_T) = \lambda Z_T B_T^{-1}, \qquad V_T = I(\lambda Z_T B_T^{-1}) \tag{35}$$

with the Lagrange multiplier λ satisfying the 'budget equation'

$$E^{\mathbb{Q}}\left[B_T^{-1} I(\lambda Z_T B_T^{-1})\right] = v = E\left[Z_T B_T^{-1} I(\lambda Z_T B_T^{-1})\right] =: V(\lambda).$$

Having defined $V(\cdot)$, whenever it is invertible, we finally find

$$\lambda = V^{-1}(v) \quad \text{and} \quad V_T^* = I(V^{-1}(v) Z_T B_T^{-1}).$$

In the specific setting of HARA utility functions, we have

$$I(y) = y^{\frac{1}{\gamma-1}} = y^{\tilde{\gamma}-1}$$

and so, from Equation (35),

$$V_T = (\lambda Z_T B_T^{-1})^{\frac{1}{\gamma-1}} = (\lambda Z_T B_T^{-1})^{\tilde{\gamma}-1}.$$

Furthermore

$$v = E\left[Z_T B_T^{-1} (\lambda Z_T B_T^{-1})^{\tilde{\gamma}-1}\right] = \lambda^{\tilde{\gamma}-1} (B_T^{-1})^{\tilde{\gamma}} E\left[(Z_T)^{\tilde{\gamma}}\right] = V(\lambda)$$

and so

$$\lambda^* = \left(\frac{v (B_T)^{\tilde{\gamma}}}{E\left[(Z_T)^{\tilde{\gamma}}\right]} \right)^{\frac{1}{\tilde{\gamma}-1}}$$

and

$$V_T^* = (\lambda^* Z_T B_T^{-1})^{\tilde{\gamma}-1} = v B_T \frac{(Z_T)^{\tilde{\gamma}-1}}{E\left[(Z_T)^{\tilde{\gamma}}\right]}$$

i.e., we have obtained, again, Equations (20) and (24), with the difference that in this setting there is only one EMM and so the optimal $\mathbb{Q}^*$ trivially coincides with the unique EMM.

To determine the optimal investment strategy, finally, we work as in Section 6, under the measure $\mathbb{Q}$ and we compare, as usual, the two dynamics of $\tilde{V}_t^*$ and $\tilde{M}_t$. The optimal investment strategy $\alpha_t^{i*}, i = 1, \dots, N$, is again

$$\alpha_t^{i*} = \bar{\lambda}_i \frac{\tilde{V}_{t^-}}{\tilde{S}_{t^-}^i}$$

where $\bar{\lambda} = (\bar{\lambda}^1, \dots, \bar{\lambda}^n)$ is given by $\bar{\lambda} = \Psi A^{-1}$, with $\Psi = ((\psi^{1*})^{\tilde{\gamma}-1} - 1, \dots, (\psi^{n*})^{\tilde{\gamma}-1} - 1)$ (by Remark 4.2).

8 The one-dimensional case

Let us now assume that we are allowed to trade in a single risky asset S. For a realistic model, we require that this asset can go both up and down, so that $m = 2$ Poisson processes are required to describe it. As $m = 2$, $n = 1$, the market is incomplete and so all the results up to Section 6 hold true, with the advantage that, in this case, the optimal EMM and the optimal investment strategy have an explicit form. For simplicity, we use the notation $(N_t)_{t \in [0,T]} = (N_t^+, N_t^-)$ and

$$\mathrm{d}S_t = S_{t^-}\left[(e^a - 1)\mathrm{d}N_t^+ + (e^{-b} - 1)\mathrm{d}N_t^-\right]$$

with $a > 0$, $b > 0$.

As the market is incomplete, we obtain an infinite number of martingale measures and Equation (12) reduces to the condition

$$(e^a - 1)\lambda^+ \psi_t^+ + (e^{-b} - 1)\lambda^- \psi_t^- = r \quad \mathbb{P}\text{-a.s.} \quad \forall t \tag{36}$$

where λ^+ and λ^- are the intensities of the Poisson processes N^+ and N^- under the original measure $\mathbb{P}$, respectively and the process $\psi = (\psi^+, \psi^-)$ stands on a half-line in the first quadrant, which can be parameterised as follows

$$\psi_t^- =: \nu_t \geq 0, \qquad \psi_t^+ = \frac{r - (e^{-b} - 1)\lambda^- \nu_t}{(e^a - 1)\lambda^+}.$$

Due to the introduction of $(\nu_t)_t$, the Radon-Nikodym densities and the Radon-Nikodym density processes will be denoted by

$$\frac{\mathrm{d}\mathbb{Q}^\nu}{\mathrm{d}\mathbb{P}} = Z_T^\nu, \quad Z_t^\nu = \left.\frac{\mathrm{d}\mathbb{Q}^\nu}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t}.$$

In this case the solution to Equation (15) can be made explicit: in fact we have

$$\begin{cases} \bar{\lambda}(e^a - 1) = (\psi_t^+)^{\frac{1}{\gamma - 1}} - 1 \\ \bar{\lambda}(e^{-b} - 1) = (\psi_t^-)^{\frac{1}{\gamma - 1}} - 1 \\ (e^a - 1)\lambda^+ \psi_t^+ + (e^{-b} - 1)\lambda^- \psi_t^- = r \end{cases}$$

and we find that the optimal ν^*, recalling the parameterisation Equation (36), is the unique solution to the following equation

$$(\nu)^{\tilde{\gamma}-1} - \frac{(e^{-b}-1)}{(e^a-1)}\left(\frac{r-(e^{-b}-1)\lambda^-\nu}{(e^a-1)\lambda^+}\right)^{\tilde{\gamma}-1} = 1 - \frac{(e^{-b}-1)}{(e^a-1)}.$$

If $\gamma = 0 = \tilde{\gamma}$ (log-utility case), to determine ν^* we have to solve the algebraic second degree equation

$$\nu^{*2}\lambda^- k(e^{-b}-1) - \nu^* rk + \nu^*(e^a-1)(e^{-b}-1)(\lambda^+ + \lambda^-) - r(e^a-1) = 0$$

where

$$k := \left[(e^{-b}-1) - (e^a-1)\right] < 0.$$

Since $\lambda^- k(e^{-b}-1) \times r(e^a-1) > 0$, there is only one positive solution, given by

$$\psi_t^{-*} = \nu_t^* \equiv \frac{rk - (e^a-1)(e^{-b}-1)(\lambda^+ + \lambda^-) + \sqrt{\Delta}}{2\lambda^- k(e^{-b}-1)} \quad \forall t \in [0, T],$$

where

$$\Delta = \left[rk - (e^a-1)(e^{-b}-1)(\lambda^+ + \lambda^-)\right]^2 + 4rk\lambda^-(e^a-1)(e^{-b}-1) \; > 0.$$

The optimal λ^* is, thus, obtained as in Section 4.2 and the relationship between primal and dual optimal solutions is, obviously, that in Section 5.

As concerns the determination of the optimal investment strategy, by using Proposition 6.1, it is sufficient to find the optimal $\bar{\lambda}$: from Equation (15), we have that

$$\bar{\lambda} = \frac{(\psi^{+*})^{\tilde{\gamma}-1} - 1}{e^a - 1} = \frac{(\psi^{-*})^{\tilde{\gamma}-1} - 1}{e^{-b} - 1} = \frac{(\nu^*)^{\tilde{\gamma}-1} - 1}{e^{-b} - 1},$$

so that the optimal investment strategy is

$$\alpha_t^* = \frac{\tilde{M}_{t^-}}{\tilde{S}_{t^-}} \cdot \frac{(\psi^{+*})^{\tilde{\gamma}-1} - 1}{e^a - 1} = \frac{\tilde{M}_{t^-}}{\tilde{S}_{t^-}} \cdot \frac{(\psi^{-*})^{\tilde{\gamma}-1} - 1}{e^{-b} - 1} = \frac{\tilde{M}_{t^-}}{\tilde{S}_{t^-}} \cdot \frac{(\nu^*)^{\tilde{\gamma}-1} - 1}{e^{-b} - 1}.$$

9 Fixed portfolio proportions

A peculiar result of HARA utility functions in continuous time in both complete (Merton, 1969) and incomplete markets (Karatzas et al., 1991) is that the optimal portfolio in the risky assets is proportional to a fixed vector of risky assets' proportions of the total wealth and this proportionality only depends on the risk-aversion coefficient γ. In our context, this would mean that the optimal portfolio proportions $(h^{i*})_i$ in Remark 6.3 would be proportional to a fixed vector.

A first consequence of our results is that, in the simple case when only one risky asset is available in the market, this proportionality trivially holds.

Theorem 9.1. If $n = 1$, then the optimal proportion $(h_t^*)_t$ is constant over time and depends on γ.

Proof. The result in the theorem is a direct consequence of Remark 6.3: in fact, by putting $n = 1$ we obtain that the optimal h^* must satisfy the algebraic nonlinear equation

$$\sum_{j=1}^{m} \lambda^j (e^{a_{1j}} - 1)\big(h^*(e^{a_{1j}} - 1) + 1\big)^{\gamma-1} = r.$$

From the results of Section 6, we know that there exists a unique solution h^* to this equation, which is the optimal portfolio fraction invested in the risky asset and it is constant over time.

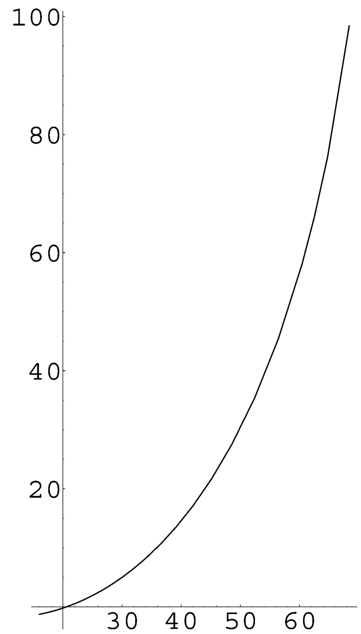
The non-trivial situation is, of course, the case when $n > 1$. In this case, we find out that, even in the simplest complete market case $m = n = 2$, this phenomenon does not always hold, thus making a difference between markets where assets follow pure diffusion processes and markets where assets can jump. The following is a counter-example, where it is shown that the optimal proportions h^* have a dependence on γ which can not be brought back to a proportionality of a fixed vector.

Example 9.2. Take $m = n = 2$, $\lambda_1 = 1$, $\lambda_2 = 0.8$, $r = 0.05$, and the multiplicative jumps

$$(e^{a_{ij}} - 1)_{i,j} = \begin{pmatrix} 1.1 & 0.9 \\ 1.2 & 0.8 \end{pmatrix}.$$

By using the results in Section 7 and solving, numerically, Equation (33) and the following for $\gamma \in (-0.3; 0.3)$, we obtain that the parametric curve obtained by seeing the optimal proportions $h = (h^{1^*}, h^{2^*})$ as a function of γ is the one in Figure 1.

Figure 1 Graph of the function $\gamma \to (h^{1^*}(\gamma), h^{2^*}(\gamma))$, with h^{1^*} in the horizontal axis and h^{2^*} in the vertical axis.



This being the situation when the market is complete, we argue that a simple scalar dependence of h^* on γ is not met also in incomplete markets, unless possibly in a few specific cases.

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