Existence, uniqueness and smoothness for the Black-Scholes-Barenblatt equation

Tiziano Vargiolu *
Dipartimento di Matematica Pura ed Applicata
Università di Padova
Via Belzoni 7 - I-35131 Padova (Italy)
email: vargiolu@math.unipd.it

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Abstract

In this paper we analyse existence, uniqueness and smoothness of solutions of the BSB equation, whose main application in Mathematical Finance is the superreplication of derivative securities.

1 Introduction

The main purpose of this paper is state and prove results on the existence, uniqueness and smoothness of the solution of the Black-Scholes-Barenblatt (BSB) equation, whose main application in Financial Mathematics is the superreplication approach in stochastic volatility models for European multiasset derivatives. We show that if the final payoff h is locally Lipschitz, then the BSB equation has a unique viscosity solution with first derivative defined almost everywhere. Besides, if the BSB equation is uniformly parabolic, then the solution has also Hölder second derivatives.

We give first an outline of the practical problem where the BSB equation is commonly met. We have a riskless asset, whose value we suppose constant through time, and n risky assets whose prices $S = (S^1, \dots, S^n)$ follow the dynamics

$$dS_t = \bar{S}_t \sigma_t \, dW_t,$$

where W is a d-dimensional Brownian motion under a so-called forward-neutral measure $\mathbb{Q}$, the matrix process σ is adapted and takes values in a closed bounded set $\Sigma \subset M(n, d, \mathbb{R})$, and $\bar{S}_t = \text{diag}\,(S_t)$. We then consider an agent who wants to hedge a European contingent claim whose payoff is a deterministic function $h(\cdot)$ which is globally Lipschitz continuous, calculated in S_T. Since the market could be incomplete because of the stochastic volatility σ and the agent is not able to hedge the volatility, he chooses to hedge the option by using the superhedging approach. Following [1], [22] and [25], he fixes an initial capital at time t as $V_t = V(t, S_t)$ and builds a self-financing portfolio consisting of a quantity $\Delta_t^i = \frac{\partial V}{\partial S_i}(t, S_t)$ (a so-called Markov superhedging strategy) of the asset $S^i, i = 1, \dots, n$,

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where $V(t,s)$ is the solution of a nonlinear PDE, similar to the Black-Scholes equation, called Black-Scholes-Barenblatt (BSB) equation in analogy with [1] and [25]. This equation is a Hamilton-Jacobi-Bellman equation and it is linked to a stochastic control problem that has a nice financial interpretation as a "game against the market". Thus the problems of finding the superreplication capital and a Markov superhedging strategy is completely solved up to find a smooth solution of Equation (3). While this is not possible in general, it turns out that in most situations it is possible to prove that the solution of the BSB equation is $C_p^{1,2}$: this is the subject of this paper.

The BSB equation (3) is a fully nonlinear parabolic equation and belongs to the more general class of the so-called Hamilton-Jacobi-Bellman equations (HJB). By simplifying the subject, we can say that there are two main approaches to study this kind of equations. The first one is to look for solutions with the derivatives taken in the classical or in the Sobolev sense in spaces $C^{1,2}$ or in spaces $W^{2,p}$; the second one is to look for viscosity solutions in the sense introduced by Crandall and Lions (see [4]).

Looking at the classical solution approach we recall first the work of Krylov (see e.g. Krylov's book [17]) where classical solution of HJB equations are given by assuming that the initial datum $h \in C_b^3(\mathbb{R}^n)$ (see more precisely [11] p. 169, [17] p. 301). In Krylov's approach the solution inherits its regularity from the initial datum h. This is not good for our problem since the typical h characterizing a European asset is C^0 and with linear growth. Krylov's results could be somehow used in our problem when the datum h is semiconvex or semiconcave since in this case one could get semiconvexity or semiconcavity of the solutions. However such a result is not known at the moment for our equation.

The other dominant approach in dealing with nonlinear PDEs is the theory of viscosity solutions. This theory, developed by Crandall and Lions in a series of papers in the early '80s, gives general existence and uniqueness theorems. Moreover, there are interesting links between optimal control problems and viscosity solutions of the corresponding Hamilton-Jacobi-Bellman (HJB) equations (see Section 3): in particular, under general assumptions that are satisfied in our case, the value function is the unique viscosity solution of the HJB equation. The interested reader can see [5] for a detailed introduction to the concept of viscosity solutions and its applications, and [11] for the links between optimal stochastic control and viscosity solutions of HJB equations. The work is not yet over, because viscosity solutions are in general only continuous, and one has still to prove enough regularity (tipically $C^{1,2}$) to apply Itô's formula. In order to do this, we use interior regularity estimates for parabolic equations, a method introduced by Caffarelli and then studied by many others (see e.g. [6, 28] and the references quoted therein). This methods gives interior regularity of the solution of the parabolic equation without assuming regularity of the initial datum and is therefore good for our problem. This method, however, applies in its best performance only in the uniformly parabolic case. In this case (see Section 6) we can get $C^{1,2}$ solutions and apply Theorem 2.

Now we explain how we proceed in order to get existence, uniqueness and regularity results for our BSB equation. We consider a general HJB equation, that include as a particular case the BSB equation. For this general HJB equation we show that there exists a unique viscosity solution with space derivative defined almost everywhere and belonging to L^p_{loc}, provided the final datum is polynomially growing and locally Lipschitz. In the case when uniform parabolicity holds we show that the viscosity solution is in fact more regular, i.e. $C^{1,2}$ with Hölder continuous derivatives.

We point out that, while existence, uniqueness and smoothness results for viscosity solutions in the uniformly parabolic case can be found in the literature (see [5, 6, 11, 28]), to our knowledge these results can not be immediately used for the BSB equation, because of some particular features of the final condition h. For this reason we here give complete proofs of the results. These proofs use in fact standard ideas from the theory of viscosity solution (that can be found e.g. in [11, 14, 21]) and one could say that the results we prove were clearly expected. However, due to the practical applications of the BSB equation, we think it is useful to give a detailed statement and proof of our existence, uniqueness and regularity results. To keep the lenght of this paper shorter we will use the notations and follow the approach of [11, Chapter V], which is a well known reference on the topic, by emphasizing the points that do not fit in their assumptions and giving a complete proof of them. We also observe that the set of assumptions we use is not always sharp, but is enough for our needs.

The paper is organized as follows: in Section 2 we present the typical application of the BSB equation in Mathematical Finance, and for the reader's convenience we recall Theorem 2 that states that a smooth solution the BSB equation gives the superreplication capital and a Markov superstrategy. In Section 3 we present the links between viscosity solutions of HJB equations and stochastic optimal control. In Section 4 we present existence and uniqueness results for a general HJB equation, whose the BSB equation is a particular case. In Sections 5 and 6 we present regularity results (respectively, for the first and second derivatives) for the BSB equation.

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2 The Black-Scholes-Barenblatt equation

In this section we briefly present the usual framework in which the BSB equation is used, without giving all the details. For a more detailed presentation, the interested reader can see e.g. [12, 13, 25].

We consider a market model with a riskless asset M and n risky assets S^i, $i = 1, \ldots, n$. We make the usual assumptions that there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a complete and right-continuous filtration $(\mathcal{F}_t)_{t\in[0,T]}$, where $\mathcal{F}_t$ represents the information available up to time t and that S^i are stochastic processes adapted to $(\mathcal{F}_t)_t$. Moreover, we assume that there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, which is called **forward-neutral probability** [8], such that the value of the riskless asset M remains constantly equal to 1 through time, and the dynamics of the prices of the assets S^i under $\mathbb{Q}$ are the following:

$$dS^i_t = S^i_t \langle \sigma^i_t, dW_t \rangle$$

where $(W_t)_t$ is a d-dimensional $\mathbb{Q}$-Brownian motion adapted to $(\mathcal{F}_t)_t$, $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in $\mathbb{R}^d$ and, for all $i = 1, \ldots, n$, σ^i is a d-dimensional process such that $\sigma = (\sigma^i)_i \in \mathcal{A}(\Sigma)$, where Σ is a closed bounded set in the space of $n \times d$ real matrices $M(n, d, \mathbb{R})$ and $\mathcal{A}(\Sigma)$ (which we call set of **admissible volatilities**) is the set of Σ-valued processes progressively measurable with respect to $(\mathcal{F}_t)_t$.

We can write the dynamics of the risky assets in a more compact vectorial notation in this way:

$$dS_t = \bar{S}_t \sigma_t \, dW_t$$

where

$$\bar{S} = \text{diag}\,(S) = \begin{pmatrix} S^1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & S^n \end{pmatrix}$$

We will also write S^σ when we want to emphasize the dependence of S on the particular volatility process σ.

Now we consider an agent in the market who does not know the true volatility σ and needs to hedge a European derivative asset with payoff $h(S_T) = h(S_T^1, \dots, S_T^n)$, where h is a locally Lipschitz continuous function having polynomial growth. To this purpose, the agent builds a self-financing portfolio Π, holding η_t units of the money market account M and Δ_t^i units of the asset S^i, $i = 1, \dots, N$, at time t, where η is adapted and the Δ^i, $i = 1, \dots, N$, are progressively measurable with respect to $(\mathcal{F}_t)_t$. We indicate by Π_t the value of the portfolio at time t, defined by

$$\Pi_t = \eta_t + \langle \Delta_t, S_t \rangle \tag{1}$$

where $\Delta_t = (\Delta_t^1, \dots, \Delta_t^n)$. In order to prevent arbitrage opportunities, we assume that the portfolio is admissible and self-financing. We say that the portfolio is **admissible** if Π is a supermartingale: if this happens, then $\Pi_0 = 0$ implies $\mathbb{E}[\Pi_T] \leq 0$. We say that the portfolio is **self-financing** if Π follows the dynamics

$$d\Pi_t = \langle \Delta_t, dS_t \rangle \tag{2}$$

If the portfolio is self-financing, the initial value Π_0 and the process Δ are sufficient to characterize it, as $\Pi_t = \Pi_0 + \int_0^t \langle \Delta_t, dS_t \rangle$ and $\eta_t = \Pi_t - \langle \Delta_t, S_t \rangle$. For this reason, if the portfolio is self-financing, we will also say that Δ is an admissible strategy if the resulting portfolio is a supermartingale.

The market can be not complete because of the unknown (possibly stochastic) volatility σ, so the agent cannot expect to replicate exactly any general contingent claim, and he has to choose some criterion to hedge the claim h. Assume that he chooses to use the superhedging approach, which we present in this situation. We define the **superreplication capital** for the contingent claim h at time 0 as

$$V^+(0, S_0) = \inf \left\{ v \;\middle|\; \forall \sigma \in \mathcal{A}(\Sigma)\ \exists \Delta \text{ admissible s.t. } v + \int_0^T \langle \Delta_t, dS_t^\sigma \rangle \geq h(S_t^\sigma)\ \mathbb{P}\text{-a.s.} \right\}$$

and define **superhedging strategy**, or more briefly **superstrategy**, any such process Δ (which possibly depends on σ).

Remark 1 Notice that if $\gamma \neq \sigma$, then the laws of the price processes S^γ and S^σ can be mutually singular (see [24] for a detailed analysis in dimension 1). This is not against no-arbitrage: in fact, the absence of arbitrage is "morally" equivalent to the fact that S^σ have equivalent laws under $\mathbb{P}$ and $\mathbb{Q}$, not to the fact that S^σ has laws equivalent to that of S^γ (which are two different processes) under $\mathbb{P}$ or $\mathbb{Q}$. Since the law of S^σ under $\mathbb{P}$ is equivalent to law of S^σ under $\mathbb{Q}$, we can see that the definition of super- and subreplication capital is invariant with respect to the choice of the probability measure, as long as it is equivalent to $\mathbb{P}$ (which in general does not happen with other criteria, such as utility evaluation or minimization of risk measures). For this reason, we can work without problems under the forward neutral measure $\mathbb{Q}$.

In the above context, superstrategies depend on the particular volatility process σ, which is not known to the agent. For this reason, the agent will be interested in finding those which we call **Markov superstrategies**, i.e. superstrategies Δ of the kind $\Delta_t = \Delta(t, S_t^\sigma)$, where $(t, s) \to \Delta(t, s)$ is a deterministic function. In this way, the strategy Δ will be calculated starting from quantities that are directly observable by the agent.

As already outlined in [1], [9], [22] and [25], natural candidates to be respectively superreplication capital and a Markov superstrategy in this context are

$$V^+(0, S_0) = V(0, S_0), \qquad \Delta_t = D_s V(t, S_t), \quad t \in [0, T]$$

where V is the solution of the following PDE:

$$\begin{cases} \dfrac{\partial V}{\partial t}(t,s) + \dfrac{1}{2} \max\limits_{\gamma \in \Sigma} \mathrm{tr}\, (D_s^2 V(t,s)(\bar{s}\gamma)(\bar{s}\gamma)^*) = 0, & t \in [0,T), s \in \mathbb{R}_+^n \\ V(T,s) = h(s), & s \in \mathbb{R}_+^n \end{cases} \tag{3}$$

where $D_s V(t,s) = (\frac{\partial V}{\partial s_1}(t,s), \ldots, \frac{\partial V}{\partial s_n}(t,s))$, and $D_s^2 V(t,s) = (\frac{\partial^2 V}{\partial s_i \partial s_j}(t,s))_{ij}$. Equation (3), as in [1], will be called from now on the **Black-Scholes-Barenblatt (BSB) equation** for h. This choice has also a stochastic control interpretation of a game against the nature (which in this case is represented by the market). We send the interested reader again to [12, 13, 25].

We now present the main known result on this subject. We indicate with $C_p^{1,2}([0,T) \times \mathbb{R}_+^n)$ the space of functions V that are continuous and with polynomial growth on $[0,T) \times \mathbb{R}_+^n$ together with their first derivative in t and first and second derivatives in s. Theorem 2 below is stated (in a slightly different way) and proved in [25] (see also [1], [22] for previous similar results).

Theorem 2 *If $V \in C_p^{1,2}\left([0,T) \times \mathbb{R}_+^n\right)$ is the solution of Equation (3) and $\Delta_t = D_s V(t, S_t)$, then $V(0, S_0)$ is the superreplication capital at time 0 and Δ is a Markov superstrategy.*

This is a characterization theorem for the superstrategies. In fact, provided the solution of the BSB equation (3) $V \in C_p^{1,2}$, the theorem gives the minimal superstrategy, in the sense that there does not exist a superstrategy starting with an initial capital less than $V(0, S_0)$. Thus the problems of finding the superreplication capital and a Markov superstrategy is completely solved up to find a $C_p^{1,2}$ solution of Equation (3). This is not possible in general: in fact, there are situations where the solution of the BSB equation is not $C_p^{1,2}$, but indeed it is possible to obtain a Markov superstrategy from it [12]. However, in most situations it is possible to prove that the solution of the BSB equation is $C_p^{1,2}$: this is the subject of the rest of this paper.

3 Viscosity solutions and stochastic optimal control

The BSB equation belongs to a wider class of nonlinear second order parabolic equations, namely the Hamilton-Jacobi-Bellman (HJB) equations, that tipically arise in stochastic optimal control problems. In the rest of the paper, we will follow this strategy: we will prove general results that hold for a wide class of HJB equations, and we will apply the general results to the particular case of the BSB equations. Here we start by recalling the notion of viscosity solution of a general class of equations including HJB equations, and presenting the links between viscosity solutions and stochastic optimal control problems.

We consider an equation of the kind

$$-\frac{\partial}{\partial t}V(t,x) + F(t,x,D_xV(t,x),D_x^2V(t,x),V(t,x)) = 0 \tag{4}$$

where $(t,x) \in Q = [0,T) \times O$, O is an open subset of $\mathbb{R}^n$, and F is a continuous function such that

$$F(t,x,p,A+B,V) \le F(t,x,p,A,V)$$

for all $(t,x) \in Q$, $p \in \mathbb{R}^n$, $V \in \mathbb{R}$ and $A, B \in \mathcal{S}^n$ with $B \ge 0$, where $\mathcal{S}^n$ is the set of all $n \times n$ symmetric matrices.

We say that a function $V \in C([0,T] \times O)$

- is a **viscosity subsolution** of Equation (4) if for each $v \in C^{1,2}(Q)$,

  $$-\frac{\partial}{\partial t}v(\bar{t},\bar{x}) + F(\bar{t},\bar{x},D_xv(\bar{t},\bar{x}),D_x^2v(\bar{t},\bar{x}),v(\bar{t},\bar{x})) \le 0$$

  at every $(\bar{t},\bar{x}) \in Q$ which is a local maximum of $V - v$ on Q, with $V(\bar{t},\bar{x}) = v(\bar{t},\bar{x})$.
- is a **viscosity supersolution** of Equation (4) if for each $v \in C^{1,2}(Q)$,

  $$-\frac{\partial}{\partial t}v(\bar{t},\bar{x}) + F(\bar{t},\bar{x},D_xv(\bar{t},\bar{x}),D_x^2v(\bar{t},\bar{x}),v(\bar{t},\bar{x})) \ge 0$$

  at every $(\bar{t},\bar{x}) \in Q$ which is a local minimum of $V - v$ on Q, with $V(\bar{t},\bar{x}) = v(\bar{t},\bar{x})$.
- V is a **viscosity solution** of Equation (4) if it is both a viscosity subsolution and a viscosity supersolution of Equation (4) in Q.

A viscosity solution is a solution in a weaker sense than the classical one, in the sense that if $V \in C^{1,2}(Q)$, then V is a viscosity solution of Equation (4) in Q if and only if it is a classical solution of Equation (4) in Q (see [11] pag. 69). Indeed, it turns out that is is the "right" definition in the area of stochastic optimal control theory. On the other hand, stochastic optimal control provide a nice tool for the existence theorem of a nonlinear second order equation of the kind of Equation (4) in the case when F is convex.

We consider the following optimal control problem. We call **reference probability system** ν a 4-uple $\nu = (\Omega, (\mathcal{F}_s)_{s\in[t,T]}, \mathbb{P}, W)$, where $(\Omega, \mathcal{F}_T, \mathbb{P})$ is a probability space, and W is a Brownian motion adapted to the filtration $(\mathcal{F}_s)_s$. We denote with

$$\mathcal{A}_{t,\nu} = \{u \mid u \text{ progressively measurable } U\text{-valued process defined on } \nu\}$$

where $U \subset \mathbb{R}^m$ is a compact set. We then suppose that X is a $\mathbb{R}^n$-valued process governed by the stochastic differential equation

$$dX_t = f(t,X_t,u_t)\,dt + \sigma(t,X_t,u_t)\,dW_t, \quad t \in [0,T] \tag{5}$$

where $u \in \mathcal{A}_{t,\nu}$. Moreover, we fix $Q_0 = [0,T) \times \mathbb{R}^n$, $\bar{Q}_0$ the closure of Q_0, and we assume that

$$\begin{array}{ll} \text{(a)} & f,\ \sigma \text{ are continuous on } \bar{Q}_0 \times U, \text{ and } f(\cdot,\cdot,v),\ \sigma(\cdot,\cdot,v) \\ & \text{are of class } C^1(\bar{Q}_0) \text{ for each } v \in U \\ \text{(c)} & \text{for suitable } C_1, C_2 > 0 \\ & |f_t| + |D_xf| \le C_1, |\sigma_t| + |D_x\sigma| \le C_1 \\ & |f(t,0,v)| + |\sigma(t,0,v)| \le C_2 \text{ for each } v \in U \end{array} \tag{6}$$

We want to choose the control u to minimize

$$J(t,x;u) = \mathbb{E}_{t,x}\left[\int_t^T L(s,X_s,u_s)\;ds + \psi(X_T)\right]$$

where $\mathbb{E}_{t,x}$ is the expectation made with respect to the measure $\mathbb{P}_{t,x}(B) = \mathbb{P}(B|X_t = x)$, and L and ψ are continuous functions such that

$$\begin{array}{ll}\text{(a)} & |L(t,x,v)| \le C_3(1+|x|^k) \\ \text{(b)} & |\psi(x)| \le C_3(1+|x|^k)\end{array} \tag{7}$$

for suitable constants $C_3 > 0$, $k \ge 0$. We thus consider the infimum of J among all $u \in \mathcal{A}_{t,\nu}$:

$$V_\nu(t,x) = \inf_{u\in\mathcal{A}_{t,\nu}} J(t,x;u)$$

and we call **value function** the function $V(t,x)$ defined as

$$V(t,x) = \inf_\nu V_\nu(t,x) \tag{8}$$

This problem is linked to a PDE of the kind of Equation (4), called **dynamic programming equation** or **Hamilton-Jacobi-Bellman** (HJB) equation, obtained by imposing

$$F(t,x,p,A) = \sup_{v\in U}\left(-\langle f(t,x,v),p\rangle - \frac{1}{2}\text{tr}\;Aa(t,x,v) - L(t,x,v)\right) \tag{9}$$

where $a = \sigma\sigma^*$, with boundary condition

$$V(T,x) = \psi(x) \quad \forall x\in\mathbb{R}^n$$

We notice that the BSB equation is a HJB equation with $U = \Sigma$, $a(t,x,v) = (\bar{x}v)(\bar{x}v)^*$, $f = L = 0$. Moreover we can extend the function h defined on $\mathbb{R}^n_+$ to a function $\psi \in C_p(\mathbb{R}^n)$ in such a way that $h = \psi$ on $\mathbb{R}^n_+$.

One important tool in proving that V is a solution of Equation (4) is the following. We say that a function W has **property (DP)** (dynamic programming) if for every ν, every control $u \in \mathcal{A}_{t,\nu}$ and every stopping time θ taking values in $[t,T]$ we have

$$W(t,x) \le \mathbb{E}_{t,x}\left[\int_t^\theta L(s,X_s,u_s)\;ds + W(\theta,X_\theta)\right]$$

and for every $\delta > 0$, there exists a ν and a control $u \in \mathcal{A}_{t,\nu}$ such that

$$W(t,x) + \delta \ge \mathbb{E}_{t,x}\left[\int_t^\theta L(s,X_s,u_s)\;ds + W(\theta,X_\theta)\right]$$

for every stopping time θ taking values in $[t,T]$.

Usually the property (DP) is stated for the value function V. Here we are stating it for a generic function, because in the proof of existence of a viscosity solution of Equation (4) we will build value functions of auxiliary control problems for which property (DP) will hold.

Theorem 3 *If property (DP) holds for V (the assumptions above are satisfied) and $V \in C_p(Q_0)$, then V is a viscosity solution of Equation (4) in Q_0.*

Proof. See [11] pag. 69. □

4 Existence and uniqueness

The standard results concerning existence and uniqueness of viscosity solutions require that the final payoff is bounded (some works like [20] and [29] deal with the case of unbounded final condition but their assumptions does not include our BSB equation). For many financial applications this is not sufficient. Now we prove an existence and uniqueness theorem for h unbounded, which extends results already existing in [11].

Theorem 4 *Let Σ be compact and $h \in C_p(\mathbb{R}^n_+)$. Then the value function V^+ defined by Equation (8) is a viscosity solution of Equation (3) in $[0,T] \times \mathbb{R}^n_+$. Moreover, V^+ is the unique viscosity solution having polynomial growth that satisfies the boundary condition $V(T,s) = h(s)$ for all $s \in \mathbb{R}^n_+$.*

In order to prove this theorem, we state and prove some intermediate results about generic HJB equations.

We say that ψ and its derivatives until the k-th one have **polynomial growth**, and we indicate it with $\psi \in C^m_p(\mathbb{R}^n)$, if for all $i = 0, \cdots, m$ there exist C, k such that

$$|\psi^{(i)}(x)| \leq C(1 + |x|^k) \quad \forall x \in \mathbb{R}^n$$

Theorem 5 (Existence) *If assumptions (6) and (7) hold, and $\psi \in C_p(\mathbb{R}^n)$, then V is a viscosity solution of Equation (4) in Q_0.*

In the proof we use the following estimate (see [11] pagg. 398–399 and [16] pag. 85): for all $u \in \mathcal{A}_{t,\nu}$, $m > 0$, we have

$$\mathbb{E}_{t,x}[\|X.\|^m_\infty] \leq B_m(1 + |x|^m), \quad \mathbb{P}\{\|X.\|_\infty \geq M\} \leq \frac{B_1}{M}(1 + |x|) \tag{10}$$

where the norm $\|\cdot\|_\infty$ is the sup norm in $C([t,T])$, and the constants B_m depend only on T and the constant C_1, C_2 in assumptions (6). From this estimate, it follows that if L and ψ have polynomial growth, then by definition of V we have

$$|V(t,x)| \leq M(T-t)(1 + |x|^k)$$

where the constant M depends only on C_1, C_2 and C_3 in assumptions (6) and (7).

Theorem 6 *If assumptions (6) and (7) hold and $\psi \in C^2_p(\mathbb{R}^n)$, then V is continuous on $\bar{Q}_0$, and property (DP) holds. Moreover $V = V_\nu$ for every reference probability system ν.*

Proof. If $\psi = 0$, the proof follows from [11] pages 185–188. If $\psi \in C^2_p(\mathbb{R}^n)$, then by Dynkin's formula (see [11] p. 178) we have

$$\mathbb{E}_{t,x}[\psi(X_T)] = \psi(x) + \mathbb{E}_{t,x}\left[\int_t^T G^u\psi(X_s)\ ds\right]$$

A consequence is that

$$J(t,x;u) = \psi(x) + \mathbb{E}_{t,x}\left[\int_t^T \tilde{L}(s, X_s, u_s)\ ds\right]$$

where $\tilde{L} = L + G^u\psi$. Since $\psi(x)$ is not affected by the control u, the problem of maximizing J is equivalent to maximize the last term in the right hand side. We also notice that the function $\tilde{L}$ still satisfies assumption (7), so we can build another control problem with $\tilde{L}$ instead of L and final condition $\psi = 0$, so we are in the previous situation and the theorem holds. $\square$

Lemma 7 *If $\psi \in C_p(\mathbb{R}^n)$, then there exists a sequence $(\psi_m)_m$ in $C_p^2(\mathbb{R}^n)$ such that $\psi_m \to \psi$ uniformly on compact sets. Moreover, there exist C, k such that $|\psi_m(x)| \le V(1+|x|^k)$ uniformly with respect to m.*

Proof. For $m = 1, 2, \ldots$, let $\rho_m \in C^\infty(\mathbb{R}^n)$ be such that $\rho_m \ge 0$, $\int_{\mathbb{R}^n} \rho_m(x)\, dx = 1$ and $\rho_m(x) = 0$ if $|x| \ge 1/m$. We let

$$\psi_m(x) = (\psi * \rho_m)(x) = \int_{\mathbb{R}^n} \psi(y)\rho_m(x-y)\, dy$$

Then $\psi_m \in C^\infty(\mathbb{R}^n)$. Besides, since ψ is uniformly continuous on each compact set K, for all $\varepsilon > 0$ there exists δ (which depends on ε and K) such that

$$|\psi(x-y) - \psi(x)| < \varepsilon \quad \forall x \in K, |y| \le \delta$$

Then for all $x \in K$, $m > 1/\delta$:

$$\begin{aligned}
|\psi_m(x) - \psi(x)| &= \left| \int_{\mathbb{R}^n} (\psi(x-y) - \psi(x))\rho_m(y)\, dy \right| \le \\
&\le \int_{|y|\le 1/m} |\psi(x-y) - \psi(x)|\rho_m(y)\, dy \le \varepsilon \int_{|y|\le 1/m} \rho_m(y)\, dy = \varepsilon
\end{aligned}$$

so $\psi_m \to \psi$ uniformly on compact sets. Besides:

$$\begin{aligned}
|\psi_m(x)| &\le \int_{\mathbb{R}^n} \psi(y)\rho_m(x-y)\, dy \le \int_{|y|\le 1/m} |\psi(x-y)|\rho_m(y)\, dy \le \\
&\le \int_{|y|\le 1/m} C\left(1 + \left(|x| + \frac{1}{m}\right)^k\right)\rho_m(y)\, dy = \\
&= C\left(1 + \left(|x| + \frac{1}{m}\right)^k\right)\int_{|y|\le 1/m} \rho_m(y)\, dy \le C\left(1 + (|x|+1)^k\right)
\end{aligned}$$

so ψ_m has polynomial growth and the uniform estimate holds. $\square$

Now we prove Theorem 5.

Proof. If $\psi \in C_p^2(\mathbb{R}^n)$, then we get the result by Theorem 6 and Theorem 3. If $\psi \in C_p(\mathbb{R}^n)$, then by the previous lemma there exists a sequence $(\psi_m)_m$ in $C_p^2(\mathbb{R}^n)$ such that $\psi_m \to \psi$ uniformly on compact sets. Let $V_{m,\nu}$, V_m be the corresponding value functions. By definition of V and V_m, for each $\delta > 0$ there exists ν and $u \in \mathcal{A}_{t,\nu}$ such that

$$\begin{aligned}
&V_m(x) - V(x) + \delta \\
&\le E_{t,x}\left[\int L(s, X_s, u_s)\, ds + \psi_m(X_T)\right] - V(x) \\
&\le \mathbb{E}_{t,x}\left[\int L(s, X_s, u_s)\, ds + \psi_m(X_T)\right] - \mathbb{E}_{t,x}\left[\int L(s, X_s, u_s)\, ds + \psi(X_T)\right] = \\
&= \mathbb{E}_{t,x}\left[\psi_m(X_T) - \psi(X_T)\right] = \\
&= \mathbb{E}_{t,x}\left[(\mathbf{1}_{|X_T|\le M} + \mathbf{1}_{|X_T|>M})(\psi_m(X_T) - \psi(X_T))\right] = \\
&\le \|\psi_m - \psi\|_{B(0,M)} + \mathbb{E}_{t,x}\left[\mathbf{1}_{|X_T|>M}(\psi_m(X_T) - \psi(X_T))\right] = I_1 + I_2
\end{aligned}$$

where $\|\cdot\|_{B(0,M)}$ denotes the sup norm in $B(0,M)$. An analogous estimation holds for $V(x)-V_m(x)+\delta$. Since ψ_m and ψ have polynomial growth, we have

$$\begin{aligned}
I_2 &\leq \mathbb{E}_{t,x}\left[2V(1+|X_T|^k)\mathbf{1}_{|X_T|>M}\right] \leq \\
&\leq \left(\mathbb{P}\{|X_T|>M\}\mathbb{E}_{t,x}\left[4C^2(1+|X_T|^k)^2\right]\right)^{1/2} \leq \\
&\leq \left(\mathbb{P}\{\|X.\|_\infty>M\}C_1\left(1+\mathbb{E}_{t,x}[\|X.\|_\infty^{2k}]\right)\right)^{1/2} \leq \\
&\leq \frac{B_1}{M}(1+|x|)C_1(1+|x|^{2k})^{1/2} \leq \frac{C_2}{M}(1+|x|^{k+1})
\end{aligned}$$

So I_2 can be made arbitrarily small by choosing a suitable M and x in a given compact set. Besides I_1 converges to 0 as $m\to\infty$, so we have that $V_m\to V$ uniformly on compact sets, so V is continuous. Now let's prove that property (DP) holds for V. In order to prove the second inequality, for an arbitrary stopping time θ we calculate

$$\begin{aligned}
&\left|E_{t,x}\left[\int_t^\theta L(s,X_s,u_s)\ ds+V(\theta,X_\theta)-\int_t^\theta L(s,X_s,u_s)\ ds+V_m(\theta,X_\theta)\right]\right| \\
&= \left|\mathbb{E}_{t,x}\left[V(\theta,X_\theta)-V_m(\theta,X_\theta)\right]\right| \\
&\leq \mathbb{E}_{t,x}\left[(\mathbf{1}_{|X_\theta|\leq M}+\mathbf{1}_{|X_\theta|>M})|V(X_\theta)-V_m(X_\theta)|\right] \\
&\leq \|V-V_m\|_{B(0,M)}+\mathbb{E}_{t,x}\left[\mathbf{1}_{|X_\theta|>M}|V(X_\theta)-V_m(X_\theta)|\right] = I_3+I_4
\end{aligned}$$

By Lemma 7 we have

$$\begin{aligned}
I_4 &\leq \mathbb{E}_{t,x}\left[\mathbf{1}_{|X_\theta|>M}2C(1+|X_\theta|^k)\right] \leq \\
&\leq \left(\mathbb{P}\{|X_\theta|>M\}\mathbb{E}_{t,x}\left[4C^2(1+|X_\theta|^k)^2\right]\right)^{1/2} \leq \\
&\leq \left(\mathbb{P}\{\|X.\|_\infty>M\}C_1\left(1+\mathbb{E}_{t,x}[\|X.\|_\infty^{2k}]\right)\right)^{1/2} \leq \frac{C_2}{M}(1+|x|^{k+1})
\end{aligned}$$

Also here I_4 can be made arbitrarily small by choosing a suitable M. In particular, for each $\delta>0$ there exists M such that $I_4<\delta/3$. Moreover, there exists m such that $I_3<\delta/3$. At last, by Theorem 3, property (DP) holds for V_m, so for each $\delta>0$, there exists a ν and a control $u\in\mathcal{A}_{t,\nu}$ such that

$$V_m(t,x)+\frac{\delta}{3}\geq \mathbb{E}_{t,x}\left[\int_t^\theta L(s,X_s,u_s)\ ds+V_m(\theta,X_\theta)\right]$$

for every stopping time θ taking values in $[t,T]$. By putting together these three inequalities, we obtain property (DP) for V, so the theorem follows from Theorems 3 and 6. □

Corollary 8 *If $\psi\in C_p(\mathbb{R}^n)$, then the value function V_ψ defined in Equation (8) is a viscosity solution of Equation (3) in $[0,T]\times\mathbb{R}^n_+$. Besides, if $\psi,\psi'\in C_p(\mathbb{R}^n)$ are such that $\psi=\psi'$ in $\mathbb{R}^n_+$, then $V_\psi=V_{\psi'}$ in $[0,T]\times\mathbb{R}^n_+$.*

Proof. The controlled process S satisfies assumptions (6) and (7), so by Theorem 5 the value function V_ψ is a viscosity solution of Equation (3) in the whole space $[0,T]\times\mathbb{R}^n$, so it is a viscosity solution also in the subset $[0,T]\times\mathbb{R}^n_+$. Since the initial condition $s^i>0$

implies $S_T^i > 0$ $\mathbb{Q}$-a.s., then the second part of the theorem follows by the definition of V_ψ. $\square$

Now we cite a comparison result by Lions [21] that will imply a uniqueness theorem for Equation (4). In order to obtain the comparison result, we will assume, besides (6) and (7), that

$$\begin{array}{ll} \text{(a)} & \sup_{u\in U, t\in[0,T]} \sum_{i,j=1}^n \|\sigma_{ij}(t,\cdot,u)\|_{W^{2,\infty}(\bar B_R)} + \|f_i(t,\cdot,u)\|_{W^{1,\infty}(\bar B_R)} < +\infty \quad \forall R < +\infty \\ \text{(b)} & L \text{ is bounded continuous in } \bar B_R \text{ uniformly in } u \in U,\ t\in[0,T]\ \forall R<+\infty \\ \text{(c)} & \sigma_{ij}, f_i, L \text{ are continuous in } u \text{ for all } (t,x)\in[0,T]\times\mathbb{R}^n \\ \text{(d)} & L \text{ is continuous in } t \text{ uniformly in } u\in U,\ x\in\mathbb{R}^n \end{array} \tag{11}$$

Theorem 9 *If assumptions (6), (7) and (11) hold and $W, V \in C_p(\bar Q_0)$ are viscosity solutions of Equation (4) in $\bar Q_0$ such that $\sup_{\partial^* Q}(W-V) \geq 0$, then*

$$\sup_{(t,x)\in\bar Q_0} e^{-\lambda t}(W(t,x)-V(t,x)) = \sup_{x\in\mathbb{R}^n}(W(T,x)-V(T,x))$$

Proof. See Theorem II.3 in [21]. $\square$

Corollary 10 *Under the assumptions of Theorem 9, there is at most one viscosity solution V of Equation (4) satisfying the boundary condition*

$$V(T,x) = \psi(x) \quad \forall x \in \mathbb{R}^n$$

We notice that we could obtain the same result without assumptions (11) by a procedure similar to the one in Chapter V of [11] (see also Remark V.8.2 there) that also uses more modern and general tools. However this would have required to modify the proof, so we did not do so for sake of brevity and because the BSB equation satisfies also the stronger assumptions (11).

Now we can prove Theorem 4.

Proof. First we extend h from $\mathbb{R}^n_+$ to the whole $\mathbb{R}^n$ to a function $\psi \in C_p(\mathbb{R}^n)$ such that $\psi|_{\mathbb{R}^n_+} = h$. Then, by Theorem 5 and corollary 10, the value function V is the unique viscosity solution of Equation 3 that satisfies the boundary condition $V(T,\cdot) = \psi$ on $\mathbb{R}^n$, and its polynomial growth follows easily by its definition and by the assumptions (6.3). Therefore, V satisfies the boundary condition $V(T,\cdot) = h$ on $\mathbb{R}^n_+$. By Corollary 8, $V|_{\mathbb{R}^n_+}$ does not depend on the particular choice of ψ as long as $\psi|_{\mathbb{R}^n_+} = h$, so we have the result. $\square$

5 Smoothness: first derivative

From now on, since $L = 0$ in the BSB equation, we will always consider the case when $L = 0$.

The main result of this section is the following one.

Theorem 11 *Let Σ be compact and h be locally Lipschitz continuous. Moreover assume that for some $m \in \mathbb{N}$ we have*[1]

$$|h(s)| \leq M(1+|s|^m) \qquad |D_s h(s)| \leq M(1+|s|^m) \quad \text{for a.e. } s \in \mathbb{R}^n_+$$

Then, for every $p \in [1,+\infty]$ we have $D_s V \in L^p_{\text{loc}}([0,T] \times \mathbb{R}^n_+)$ and for suitable $M > 0$,

$$|D_s V(t,s)| \leq M(1+|s|^m)$$

for almost every $(t,s) \in [0,T] \times \mathbb{R}^n_+$. Moreover, if h is globally Lipschitz, and $\|h\|_{\text{Lip}}$ is its Lipschitz constant, then also V is globally Lipschitz and, for a suitable $M > 0$,

$$|D_s V(t,s)| \leq M\|h\|_{\text{Lip}} \tag{12}$$

The proof uses the following general result on HJB equations. We assume that ψ is locally Lipschitz and that

$$|\psi_x| \leq C_4(1+|x|^k) \tag{13}$$

for suitable constants C_4, $k \geq 0$.

Theorem 12 *If assumptions (6) and (13) holds, then $D_x V$ exists and is in $L^p_{\text{loc}}(Q_0)$ for every $p \in [1,+\infty]$ and*

$$|D_x V(t,x)| \leq M_1(1+|x|^k)$$

for almost every $(t,x) \in Q_0$, where M_1 depends on C_1, C_2, C_3, C_4, k and T.

This result is similar to that of [11, Chapter V] but there only the case of C^2 final datum is considered, while we take it only locally Lipschitz and polynomially growing.

For the proof, we introduce difference quotient approximations to derivatives, that are a tool commonly used to analyse smoothness of solutions of various PDE's.

Let $u : A \to \mathbb{R}$ be a locally integrable function, where A is an open set in $[0,T] \times \mathbb{R}^n$. We call **difference quotients** of u of size h the quantities

$$D^h_\xi u(t,x) = \frac{u(t,x+h\xi) - u(t,x)}{h},$$

where ξ is a direction in $\mathbb{R}^n$, that is $|\xi| = 1$. We also indicate $D^h_i u = D^h_{e_i} u$ and $D^h_x u(t,x) = (D^h_1 u(t,x), \ldots, D^h_n u(t,x))$.

In order to prove the smoothness of V, first we will obtain bounds for $D^h_x V$, and then we will use a general result that allows us to conclude that $D_x V \in L^p_{\text{loc}}$ for $p > 1$. Let $D^h_x J = D^h_x J(t,\cdot,u)$.

Lemma 13 *There exists M_1, that depends on C_1, C_2, C_4, k and T, such that for all directions ξ,*

$$|D^h_\xi J| \leq M_1(1+|x|^k)$$

for every $h \in (0,1]$.

[1] Recall that $D_s h$ is defined a.e. thanks to the Rademacher's Theorem.

Proof. Given $(t, x_0) \in Q_0$, let X_s be the solution of (5) with $X_t = x_0$, and let X_s^h be the solution of (5) with $X_t = x_0 + h\xi$. Also, let $\Delta X_s = (X_s - X_s^h)/h$. Since ψ is Lipschitz, then it is (weakly) differentiable in each direction and the fundamental theorem of calculus holds. Then

$$D_\xi J = \mathbb{E}\left[\frac{1}{h}(\psi(X_T^h) - \psi(X_T))\right] = \mathbb{E}\left[\int_0^1 \langle \psi_x(X_T^\lambda), \Delta X_s\rangle \, d\lambda\right]$$

where $X_s^\lambda = (1-\lambda)X_s + \lambda X_s^h$. By Equation (13),

$$\left|\int_0^1 \psi_x(X_T^\lambda) \, d\lambda\right| \le \int_0^1 C_4(1 + |X_T^\lambda|^k) \, d\lambda \le M(1 + |X_T|^k + |X_T^h|^k)$$

By Cauchy-Schwartz

$$\begin{aligned} |D_\xi J| &= M\mathbb{E}\left[\int_t^T (1 + |X_s|^k + |X_s^h|^k)^2 \, ds\right]^{1/2} \mathbb{E}\left[\int_t^T |\Delta X_s|^2 \, ds\right]^{1/2} + \\ &\quad + M\mathbb{E}\left[(1 + |X_T|^k + |X_T^h|^k)^2\right]^{1/2} \mathbb{E}\left[|\Delta X_T|^2\right]^{1/2} \end{aligned}$$

We bound the first and the third term on the right hand side using the estimate in Equation (10) with $m = 2k$ and $x = x_0, x_0 + h\xi$. We have also that $\mathbb{E}[|\Delta X_s|]^2 \le B$ (see [11] p. 400), where B depends on C_1 in (6). Since $|\xi| = 1$, $h \in (0, 1]$,

$$1 + |x|^{2k} + |x + h\xi|^{2k} \le c_k(1 + |x|^{2k})$$

for suitable c_k. This implies the lemma. □

Now we prove Theorem 12.

Proof. We take a generic open bounded set B; then by Lemma 13 we have $|D_x^h J(t, x, u)| \le M_1(1 + |x|^k)$ $\forall (t, x) \in A$. Then, since these bounds are the same for all controls u, we obtain that $|D_x^h V(t, x)| \le M_1(1 + |x|^k)$. Then we take $p > 1$ and an open set A such that $B \subseteq A$ and $\mathrm{dist}(B, \partial A) < 1 \wedge T$, and we have that $D_x^h V \in L^p(B)$ and

$$\|D_x^h V\|_{L^p(B)} \le M_3 \|(1 + |x|^k)\|_{L^p(B)}$$

for all $h \in (0, T \wedge 1)$, where M_3 depends on M_1 and M_2. This implies (see [10], p. 246–248) that $D_x V \in L^p(B)$ and $\|D_x u\|_{L^p(B)} \le M_3 \|(1 + |x|^k)\|_{L^p(B)}$. Moreover, $D_x u$ is also the derivative in the Sobolev sense. In fact, for each $\phi \in C_0^\infty((0, T) \times \mathbb{R}^n)$,

$$\int D_i u\phi = \int \lim_{h\to 0} D_i^h u\phi = \lim_{h\to 0} \int D_i^h u\phi = -\lim_{h\to 0} \int u D_i^h \phi = -\int u D_i \phi$$

and the conclusion follows. □

Now we are ready for the proof of Theorem 11.

Proof. We only have to prove that if h is globally Lipschitz, then V is globally Lipschitz. This follows easily from Equation (12) and from the fundamental theorem of calculus. □

6 Smoothness: second derivative

Now we present stronger regularity results involving the second derivative that comes from known results on uniformly parabolic equations. The main references on regularity for uniformly parabolic equations are [6] and [28]. In order to use these results, we have to make a change of variable to transform the BSB equation (which is never uniformly parabolic) into another one. We put $y_i = \log s_i$. Then the BSB equation becomes:

$$\begin{cases} \dfrac{\partial \tilde{V}}{\partial t}(t,y) + \dfrac{1}{2}\max_{\gamma\in\Sigma} \mathrm{tr}((D_s^2\tilde{V}(t,y) - \mathrm{diag}(D_s\tilde{V}(t,y)))\gamma\gamma^*) = 0, & t\in[0,T), y\in\mathbb{R}^n \\ \tilde{V}(T,\cdot) = \tilde{h}(\cdot) & y\in\mathbb{R}^n \end{cases} \tag{14}$$

where $\tilde{V}(t,y) = V(t,(e^{y_i})_i) = V(t,s)$. In fact we can see that V is a viscosity solution of Equation (3) if and only if $\tilde{V}$ is a viscosity solution of Equation (14) (see [2, p. 38-39] for change of variables in the theory of viscosity solutions). We notice that in the BSB equation (3) the domain is $s \geq 0$, while in this equation the domain is all $\mathbb{R}^n$. In order to use results of [6] and [28], we have to recall the following.

Definition 14 We say that Equation (14) is **uniformly parabolic** if there exists $M > m > 0$ such that for all $\gamma \in \Sigma$ and $\xi \in \mathbb{R}^n$,

$$m\|\xi\|^2 \leq \langle \gamma\gamma^*\xi, \xi\rangle\rangle \leq M\|\xi\|^2$$

The above definition can be rewritten in this way:

$$\exists M' > m' > 0 \text{ such that } \forall\gamma\in\Sigma, \xi\in\mathbb{R}^n, m'\|\xi\| \leq \|\gamma^*\xi\| \leq M'\|\xi\|$$

This implies that if there exists a $\gamma\in\Sigma$ which has not rank equal to n, then Equation (14) is not uniformly parabolic. In fact if we take $\xi \in \ker\gamma^*$, then $\|\gamma^*\xi\| = 0$. We show that also the converse is true. To this purpose, we use the notation $\Sigma^2 = \{\gamma\gamma^* | \gamma\in\Sigma\}$. Now we present a result, whose (easy) proof can be found in [25].

Lemma 15 *If Σ is closed and bounded in $M(n,n,\mathbb{R})$, then Equation (14) is uniformly parabolic if and only if $\Sigma^2 \subseteq \mathrm{GL}(n,\mathbb{R})$, where $\mathrm{GL}(n,\mathbb{R}) = \{\gamma\in\mathbb{R}^{n\times n} \mid \det\gamma \neq 0\}$.*

We notice that if $d < n$, then Equation (14) can not be uniformly parabolic, because $\gamma\gamma^*$ has at most rank d for $\gamma\in\Sigma$.

The above Lemma 15 allows us to use results of [6] and [28] to state the following theorem. For $\beta\in(0,1)$ we denote by C^β the space of all functions that are Hölder continuous of exponent β, and for $n\in\mathbb{N}$, $\alpha\in(0,1)$, we call $C_p^{n+\alpha}(\mathbb{R})$ the space of all functions that are polynomially growing and Hölder continuous of order of $\frac{n+\alpha}{2}$ in time and differentiable of order n in space with derivatives Hölder continuous of order α.

Theorem 16 *Let Σ be compact, h be locally Lipschitz continuous and $h, D_s h$ have polynomial growth. If $\Sigma^2 \subseteq \mathrm{GL}(n,\mathbb{R})$ then there exists $\alpha\in(0,1]$ such that the viscosity solution V of the BSB equation (3) belongs to $C_p^{2+\alpha}([0,T)\times\mathbb{R}^n)$.*

Proof. We do not give a real proof, but we just recall what kind of results can be used to get our claim giving references of them. Thanks to Lemma 15, Equation (14) is uniformly parabolic and it has a unique viscosity solution, which we denote by $\tilde{V}$. Since all the coefficients are uniformly Hölder continuous (i.e. uniformly bounded in C^{β} for some $\beta \in (0,1)$), one can prove that the solution is $C^{1,2}$ with Hölder continuous derivatives (see [6] and the discussion before Theorem 9.1). This follows substantially in two steps. The first one is to apply a result of [6] that states that the solutions of Equation (14) are of class $C^{1+\alpha}$ when all the coefficients are uniformly Hölder continuous. The second one is to use a theorem of Wang [28] that gives $C^{1,2}$ regularity (plus Hölder continuity of the derivatives) for equations of the form $-V_t + F(D_s^2 V) = f$, with $f \in C^{1+\alpha}$ and F convex. This theorem can be used thanks to the $C^{1+\alpha}$ regularity results of [6] since then the spatial gradient is just a Hölder continuous function that can be incorporated into the right hand side f. Then $\tilde{V} \in C^{1,2}(\mathbb{R}^n)$ with Hölder continuous derivatives [6]. This means that also $V(t, s_1, \ldots, s_n) = \tilde{V}(t, e^{y_1}, \ldots, e^{y_n})$ has the same regularity, so we get the claim. □

References

[1] M. Avellaneda, A. Levy, A. Parás, *Pricing and hedging derivative securities in markets with uncertain volatilities*, Applied Mathematical Finance **2** (1995), 73–88

[2] M. Bardi, and I. Capuzzo Dolcetta, OPTIMAL CONTROL AND VISCOSITY SOLUTIONS OF HAMILTON-JACOBI-BELLMAN EQUATIONS, Birkhauser, 1997

[3] F. Black, and M. Scholes, *The pricing of options and corporate liabilities*, Journal of Political Economy **81** (1973), 637–659

[4] M. G. Crandall and P. L. Lions, *Conditions d'unicité pour les solutions generalisés des equations d'Hamilton-Jacobi du premier ordre*, C.R. Acad. Sci. Paris Sér Math. **292**, (1981), 487-502

[5] M. G. Crandall, H. Ishii, and P. L. Lions, *A user's guide to viscosity solutions*, Bulletin AMS (New Series) **27** (1992), 1–67

[6] M. G. Crandall, M. Kocan, and A. Swiech, *L^p-Theory for Fully Nonlinear Parabolic Equations*, Preprint.

[7] J. Cvitanic, H. Pham, and N. Touzi, *Super-replication in stochastic volatility models under portfolio constraints*, Advances in Applied Probability **36** (1999), 523–545.

[8] N. El Karoui, H. Geman, and J. C. Rochet, *Changes of numeraire, changes of probability measures and option pricing*, Journal of Applied Probability **32** (1995), 443–458.

[9] N. El Karoui, M. Jeanblanc, and S. E. Shreve, *Robustness of the Black and Scholes formula*, Math. Finance **8 (2)** (1998), 93–126.

[10] L. C. Evans, PARTIAL DIFFERENTIAL EQUATIONS, Berkeley Mathematics Lecture Notes, 199

[11] W. H. Fleming, and H. M. Soner, CONTROLLED MARKOV PROCESSES AND VISCOSITY SOLUTIONS, Springer, 1993

[12] F. Gozzi, T. Vargiolu, *Superreplication of European multiasset derivatives with bounded stochastic volatility*, submitted.

[13] F. Gozzi, T. Vargiolu, *On the superreplication approach for European interest rates derivatives*, to appear in *Proceedings of the Ascona '99 Seminar on Stochastic Analysis, Random Fields and Applications, September 22–24, 1999*, Birkhauser

[14] H. Ishii, and P. L. Lions *Viscosity solutions of fully nonlinear second-order elliptic PDE's*, J. Differential Equat. **83** (1990), 26–78

[15] H. Johnson, *Options on the maximum or the minimum of several assets*, Journal of Financial and Quantitative Analysis **22** (1987), 277–283

[16] N. V. Krylov, CONTROLLED DIFFUSION PROCESSES, Springer, 1980

[17] N. V. Krylov, NONLINEAR ELLIPTIC AND PARABOLIC EQUATIONS OF THE SECOND ORDER, Reidel, 1987

[18] D. Lamberton, and B. Lapeyre, INTRODUCTION AU CALCUL STOCHASTIQUE APPLIQUÉE À LA FINANCE, Société de Mathématiques Appliquées et Industrielles, 1991

[19] P. L. Lions, *Control of diffusion processes in* $\mathbb{R}^N$, Comm. Pure Appl. Math. **34** (1981), 121–147

[20] P. L. Lions, GENERALIZED SOLUTIONS OF HAMILTON-JACOBI EQUATIONS, Pitman Advanced Publishing Program, 1982

[21] P. L. Lions, *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations Part II: viscosity solutions and uniqueness*, Comm. PDE **8** (1983), 1229–1276

[22] T. J. Lyons, *Uncertain volatility and the risk-free synthesis of derivatives*, Applied Mathematical Finance **2** (1995), 117–133

[23] W. Margrabe, *The price to exchange an asset for another*, The Journal of Finance **1** (1978), 177–186

[24] C. Martini, *On the marginal laws of one-dimensional stochastic integrals with uniformly elliptic integrand*, Annales de l'Institut Henry Poincarè **36 (1)** (2000), 35–43

[25] S. Romagnoli, and T. Vargiolu, *Robustness of the Black-Scholes formula in the case of options on several assets*, Finance and Stochastics **4 (3)** (2000), 325–341

[26] H. M. Soner, N. Touzi, *Superreplication under Gamma constraints*, SIAM J. Control Optimization **39 (1)** (2000), 73–96

[27] R. M. Stulz, *Options on the minimum or the maximum of two risky assets*, Journal of Financial Economics **10** (1982), 161–185

[28] L. Wang, *On the regularity theory of fully nonlinear parabolic equations: I, II*, Communications on Pure and Applied Mathematics, **45** (1992), 27–76 and 141–178

[29] J. Yong and X.Y. Zhou, STOCHASTIC CONTROL: HAMILTONIAN SYSTEMS AND HJB EQUATIONS, Springer, 1999.