Robustness of shortfall risk minimising strategies in the binomial model

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Abstract

In this paper we study the dependence on the loss function of the strategy which minimises the expected shortfall risk when dealing with a financial contingent claim in the particular situation of a binomial model. After having characterised the optimal strategies in the particular cases when the loss function is concave, linear or strictly convex, we analyse how optimal strategies change when we approximate a loss function with a sequence of suitable loss functions.
Key Words: shortfall risk minimization, binomial model, Dynamic Programming algorithm, robustness

1 Introduction

Given a market with a vector S of asset prices (one of which is non-risky), let H_N be a liability to be hedged at some future time N. If $V_N(\pi)$ is the value at time N of a portfolio corresponding to a self-financing investment strategy π, the *shortfall risk minimization problem* consists in determining $J_0(S_0, V_0)$, where for every $n = 0, 1, \dots, N$,

$$J_n(s, v) := \inf_{\pi} \mathrm{E}\left\{\ell\big([H_N - V_N(\pi)]^+\big) \;\big|\; S_n = s, V_n = v\right\}$$

for a suitable "loss function" ℓ, with S_n and V_n the (known) price at time n of the assets and the value at time n of the portfolio. The most applied side of the problem above consists in finding the optimal policy π^* and, if possible, to determine it explicitly.

The *loss function* ℓ is classically considered to be increasing and such that $\ell(0) = 0$. As it is customary to do in economic models, we shall furthermore suppose that ℓ is either concave or convex on $\mathbb{R}_+$, depending on the investor's inclination or aversion (respectively) to taking risks.

In this paper, we want to analyse the dependence of the optimal policy π^* on the loss function ℓ. In order to keep things simple, we restrict ourselves to the case in which the market follows the Cox-Ross-Rubinstein "binomial model", where the non-risky asset is supposed (by considering forward prices) identically equal to 1, and the stock S follows the dynamics

$$S_{n+1} = S_n \omega_{n+1}, \qquad n = 0, \dots, N-1$$

where $S_0 > 0$ is given and $(\omega_n)_{n=1,\dots,N}$ are i.i.d. random variables taking values in the set $\{d, u\}$ (with u and d known real numbers such that $0 < d < 1 < u$) with probability law

$$p := P\{\omega_n = u\} = 1 - P\{\omega_n = d\}, \qquad n = 1, \dots, N.$$

To avoid technicalities, we shall take as our underlying probability space the minimal one for our model; that is, we let $\Omega = \{u, d\}^N$, $\mathcal{F} = \mathcal{P}(\Omega)$.

An *investment strategy* is a sequence $\pi = (\pi_n)_{n=0,\dots,N-1}$, where $\pi_n = (\alpha_n, \beta_n)$ is the decision to hold, at time n, β_n units of the non-risky asset and α_n units of the stock S_n. We shall always suppose our investment strategies to be self-financing, namely that

$$\begin{cases} V_0 = \beta_0 + \alpha_0 S_0 \\ V_n := \beta_n + \alpha_n S_n = \beta_{n+1} + \alpha_{n+1} S_n \end{cases}$$

so that the dynamics of the portfolio value can be written as

$$V_{n+1}^{\alpha} = V_{n+1}^{\alpha}(V_n^{\alpha}, S_n, \alpha_n, \omega_{n+1}) := V_n^{\alpha} + \alpha_n S_n(\omega_{n+1} - 1).$$

By the last equality, then, the main concern in the hedging/replication problem is the determination of α_n, and this is why we use the notation V_n^{α} to stress the dependence of the portfolio dynamics upon the choice of α. Moreover, we shall suppose H to be a European contingent claim depending only on the final value S_N of the stock. Hence, the shortfall risk minimization problem can be written, for every $n = 0, \dots, N$, as

$$J_n(S_n, V_n) := \inf_{\alpha} \mathrm{E}\left\{ \ell\big([H(S_N) - V_N^{\alpha}]^+\big) \;\middle|\; S_n, V_n \right\}. \tag{1.1}$$

The paper is organised as follows. Section 2 contains the definition of the Dynamic Programming Algorithm (DPA) and some interesting results that hold when the DPA is applied to the binomial model. Section 3 briefly recalls the main results of [1] and [8], which will be a useful reference in the sequel. Section 4 is dedicated to a self-contained treatment of the optimization in particular data structures that we call "recombining binomial trees", and which is a key tool for the results in the next section. In Section 5, after the determination of the optimal control and optimal value for the case when ℓ is concave in $\mathbb{R}_+$, it will be surprisingly clear that the investor with "linear" loss function behaves in exactly the same way as a risk prone investor. Moreover, the same Section 5 explains also a way to approach the incomplete information case (i.e., the situation when p is unknown) for a risk prone investor. Section 6 treats the case when the loss function is convex, and the explicit determination of the optimal strategy is made using a Neyman-Pearson technique similar to [2]. In Section 7 we study the robustness of the optimal strategy with respect to the loss function ℓ; in particular, we shall study three cases, namely the case when we approximate a strictly convex loss function ℓ with a sequence $(\ell_n)_n$ of strictly convex functions, the case when the functions in the approximating sequence $(\ell_n)_n$ are strictly convex and ℓ is linear, and the case when ℓ and the ℓ_ns are concave.

2 The Dynamic Programming Algorithm

In [8], the key tool used for solving the shortfall risk minimization problem is the *Dynamic Programming Algorithm* (DPA for short). Since it will be also used in the present work, we summarize here its main features.

The DPA is computed according to the following backwards recursion:

$$
\begin{aligned}
J_N(s,v) &:= \ell\big([H(s)-v]^+\big) \\
J_n(s,v) &:= \inf_{\alpha} \mathrm{E}\{J_{n+1}(S_{n+1}, V_{n+1}^{\alpha}) \mid S_n = s, V_n^{\alpha} = v\},\ n = 0,\dots,N-1.
\end{aligned}
$$

Once solved, the DP algorithm allows us to compute the optimal value $J_0(S_0,V_0)$ for our problem; moreover, if the inf operators in the various recursive steps are realized as min, the minimizing α is the optimal strategy.

For the DPA applied to the shortfall risk minimization in the binomial model, there are some remarkable results that do not depend on the choice of the loss function ℓ. First of all, there exists a unique equivalent martingale measure P^* such that

$$
p^* = P^*\{\omega_n = u\} := \frac{1-d}{u-d}, \qquad 1-p^* = P^*\{\omega_n = d\} = \frac{u-1}{u-d}. \tag{2.1}
$$

In particular, for every $n = 0,\dots,N$, the arbitrage free price of $H(S_N)$ at time n is $V_n^*(S_n) := \mathrm{E}^*\{H(S_N) \mid S_n\}$ (Cox-Ross-Rubinstein valuation formula). Note that, in particular, $V_N^*(S_N) = H(S_N)$. It is also well known that, if $V_0 \geqslant V_0^*(S_0)$, there is a replicating strategy given by

$$
\alpha_n^* := \frac{V_{n+1}^*(S_n u) - V_{n+1}^*(S_n d)}{S_n(u-d)} \tag{2.2}
$$

As a consequence, the shortfall risk minimization problem for this binomial model is non trivial only in the case $V_0 < V_0^*(S_0)$. Henceforth, we shall therefore suppose to be dealing with this case. The problem has already been solved in the case $\ell(x) = x$ in [8], even when there is incomplete information on the underlying model (see [1] and [8] again, and [4] for a different approach).

Other results on the DPA are gathered in the following

2.1. Proposition. *In the notations of the present section and for every $n = 0,\dots,N-1$,*

1. *$J_n(s,v)$ is decreasing in v;*
2. *$J_n(s,v) = 0$ for $v \geqslant V_n^*(s)$;*
3. *there exist $\overline{\alpha} < \widetilde{\alpha}$ such that*

$$
J_n(s,v) = \inf_{\alpha\in[\overline{\alpha},\widetilde{\alpha}]} \mathrm{E}\{J_{n+1}(S_{n+1}, V_{n+1}^{\alpha}) \mid S_n = s, V_n^{\alpha} = v\};
$$

4. *if $v \leqslant V_n^*(s)$, in (3) above one can choose*

$$
\overline{\alpha} = \alpha_n^{(1)}(s,v) := \frac{V_{n+1}^*(sd) - v}{s(d-1)}, \quad \widetilde{\alpha} = \alpha_n^{(2)}(s,v) := \frac{V_{n+1}^*(su) - v}{s(u-1)}. \tag{2.3}
$$

Proof. We shall prove the result by backwards induction on n.

$(n = N-1)$. The expansion of the expected value gives

$$\begin{aligned} J_{N-1}(s,v) &= \inf_\alpha \mathrm{E}\left\{\ell\big([H(S_N) - V_N^\alpha]^+\big) \;\big|\; S_{N-1} = s, V_{N-1}^\alpha = v\right\} \\ &= \inf_\alpha \big\{p\ell\big([H(su) - v - \alpha s(u-1)]^+\big) + \\ &\qquad + (1-p)\ell\big([H(sd) - v - \alpha s(d-1)]^+\big)\big\}, \end{aligned}$$

so that (1) follows from the monotonicity of ℓ.

Write

$$\begin{aligned} j_N^u(s,v,\alpha) &:= p\ell\big([H(su) - v - \alpha s(u-1)]^+\big), \\ j_N^d(s,v,\alpha) &:= (1-p)\ell\big([H(sd) - v - \alpha s(d-1)]^+\big), \end{aligned}$$

so that $J_{N-1}(s,v) = \inf_\alpha\{j_N^u(s,v,\alpha) + j_N^d(s,v,\alpha)\}$. In this expression, note that for $\alpha > \alpha_{N-1}^{(2)}(s,v)$ (as defined in (2.3)) one gets $H(su) - v - \alpha s(u-1) < 0$, so $j_N^u(s,v,\alpha) = 0$. In the same way, $j_N^d(s,v,\alpha) = 0$ for $\alpha < \alpha_{N-1}^{(1)}(s,v)$. So, since j_N^u and j_N^d are, respectively, decreasing and increasing in α, we have that their sum is decreasing in α for $\alpha < \alpha_{N-1}^{(1)}(s,v)$ and increasing in α for $\alpha > \alpha_{N-1}^{(2)}(s,v)$. As a consequence, the inf of $j_N^u + j_N^d$ must be attained in the interval with extremal points $\alpha_{N-1}^{(1)}(s,v)$ and $\alpha_{N-1}^{(2)}(s,v)$, and this proves (3).

To compare $\alpha_{N-1}^{(1)}(s,v)$ and $\alpha_{N-1}^{(2)}(s,v)$, note that

$$\begin{aligned} \alpha_{N-1}^{(2)}(s,v) - \alpha_{N-1}^{(1)}(s,v) &= \frac{H(su) - v}{s(u-1)} - \frac{H(sd) - v}{s(d-1)} \\ &= \frac{(1-d)H(su) + (u-1)H(sd) - (u-d)v}{s(u-1)(1-d)} \\ &= \frac{u-d}{s(u-1)(1-d)}(V_{N-1}^*(s) - v), \end{aligned}$$

namely, that $\alpha_{N-1}^{(1)}(s,v) < \alpha_{N-1}^{(2)}(s,v)$ if and only if $v < V_{N-1}^*(s)$, and this proves (4). Also, if $v \geqslant V_{N-1}^*(s)$ then any value of $\alpha \in [\alpha_{N-1}^{(2)}(s,v), \alpha_{N-1}^{(1)}(s,v)]$ realizes the absolute minimum of 0 for $j_u + j_d$, which implies that we are able to obtain perfect replication of the claim H at the final date N and that the shortfall risk minimization problem trivially becomes the CRR hedging problem. This also proves (2).

$(n < N-1)$. Expanding the expected value as before, one gets

$$\begin{aligned} J_n(s,v) &= \inf_\alpha \mathrm{E}\{J_{n+1}(S_n, V_n^\alpha) \mid S_n = s, V_n^\alpha = v\} \\ &= \inf_\alpha \big\{pJ_{n+1}(su, v + \alpha s(u-1)) + \\ &\qquad + (1-p)J_{n+1}(sd, v + \alpha s(d-1))\big\}, \end{aligned}$$

and (1) follows from the (inductively assumed) monotonicity of $J_{n+1}(s,\cdot)$.

As in the first part of the proof, define $j_{n+1}^u(s,v,\alpha) := pJ_{n+1}(su, v + \alpha s(u-1))$, $j_{n+1}^d(s,v,\alpha) := (1-p)J_{n+1}(sd, v + \alpha s(d-1))$ and note that they are, respectively, decreasing and increasing in α. Since $J_{n+1}(s,v) = 0$ for $v \geqslant V_{n+1}^*(s)$ by the inductive hypothesis, we have that

$$\alpha > \alpha_n^{(2)}(s,v) \Rightarrow v + \alpha s(u-1) > V_{n+1}^*(su) \Rightarrow j_{n+1}^u(s,v,\alpha) = 0,$$

and in the same way that $j^d_{n+1}(s,v,\alpha)=0$ for $\alpha<\alpha^{(1)}_n(s,v)$. Hence it follows that $j^u_{n+1}+j^d_{n+1}$ is decreasing in α for $\alpha<\alpha^{(1)}_n(s,v)$ and increasing in α for $\alpha>\alpha^{(2)}_n(s,v)$. As before, this proves (3).

To compare $\alpha^{(1)}_n(s,v)$ and $\alpha^{(2)}_n(s,v)$, it suffices to compute

$$\begin{aligned}\alpha^{(2)}_n(s,v)-\alpha^{(1)}_n(s,v)&=\frac{V^*_{n+1}(su)-v}{s(u-1)}-\frac{V^*_{n+1}(sd)-v}{s(d-1)}\\&=\frac{(1-d)V^*_{n+1}(su)+(u-1)V^*_{n+1}(sd)-(u-d)v}{s(u-1)(1-d)}\\&=\frac{u-d}{s(u-1)(1-d)}(V^*_n(s)-v),\end{aligned}$$

where the last equality comes from the fact that, for every n,

$$\begin{aligned}\mathrm{E}^*\{V^*_{n+1}(S_{n+1})\mid S_n=s\}&=\mathrm{E}^*\{\mathrm{E}^*\{H(S_N)\mid S_{n+1}\}\mid S_n=s\}\\&=\mathrm{E}^*\{H(S_N)\mid S_n=s\}=V^*_n(s).\end{aligned}\tag{2.4}$$

Following the same argument as in the first part of the proof, then, also (4) and (2) are proved. □

The next two sections will make wide use of this result and of the structure of the functions to be minimized at each recursive step.

3 Preliminary results

This section briefly reviews the most important points of [1] and [8], which completely solve the problem in the case $\ell(x)=x$. These results are reported here for the convenience of the reader, as they permit a better understanding of the meaning of the results and remarks contained in the next section.

Here, we shall often use the expressions defined in (2.3) in correspondence of the current values $v=V_n$ and $s=S_n$ for the portfolio and the risky asset respectively. For simplicity of notation, then, we shall write $\alpha^{(i)}_n$ instead of $\alpha^{(i)}_n(S_n,V_n)$. The reader should nevertheless keep in mind the dependence of the $\alpha^{(i)}_n$s not only on the current time (via the Cox-Ross-Rubinstein valuation formula) but also on S_n and V_n.

In the case $\ell(x)=x$, $V_0<V^*_0(S_0)$, the recursive step in the dynamic programming algorithm defined in previous section reduces, at each time n, to the minimization of a function which is piecewise linear in α on the three disjoint intervals $(-\infty,\alpha^{(1)}_n)$, $(\alpha^{(1)}_n,\alpha^{(2)}_n)$ and $(\alpha^{(2)}_n,+\infty)$. As a consequence, the inf is attained as a min either for $\alpha=\alpha^{(1)}_n$ or for $\alpha=\alpha^{(2)}_n$. A straightforward calculation (see [8, Theorem 4.1]) allows one to check that the optimal control $\alpha^\dagger$ is calculated at each time n by choosing

$$\alpha^\dagger_n=\begin{cases}\alpha^{(1)}_n & \text{if } p<p^*\\ \alpha^{(2)}_n & \text{if } p>p^*\end{cases}.\tag{3.1}$$

(The case $p=p^*$ corresponds to S and V being martingales, so that any choice of the control would give the same results in the mean.) Note that the choice between the strategy $\alpha^{(1)}$ and $\alpha^{(2)}$ does not depend on n, so the optimal control is either $\alpha_n^\dagger \equiv \alpha_n^{(1)}$ for $p < p^*$ or $\alpha_n^\dagger \equiv \alpha_n^{(2)}$ for $p > p^*$. In particular, the choice of the optimal control can be made "a priori". The optimal value for the shortfall risk is then $\min\{\frac{p}{p^*}, \frac{1-p}{1-p^*}\}^N (V_0^*(S_0) - V_0)^+$.

Note also that the strategies $\alpha^{(1)}$ and $\alpha^{(2)}$ are in some sense a modification of the optimal replicating strategy of Cox, Ross and Rubinstein (see the Introduction). Strategy $\alpha^{(1)}$ (respectively, $\alpha^{(2)}$), indeed, is obtained from (2.2) by substituting u and $V_{n+1}^*(S_n u)$ (respectively, d and $V_{n+1}^*(S_n d)$) by 1 and V_n. Moreover, note that $p < p^*$ (respectively, $p > p^*$) if and only if $\mathrm{E}\{S_n \mid S_0\}$ is decreasing (respectively, increasing) in n. In some sense, then, the optimal strategy in the case $V_0 < V_0^*(S_0)$ is built by replacing in the CRR formula the "less significant" event (i.e., the one against the mean tendency of S to decrease or increase) with the present value.

In [1], some additional properties of the strategies belonging to the class

$$\Pi := \big\{(\alpha_n)_n \;\big|\; \alpha_n \in \{\alpha_n^{(1)}, \alpha_n^{(2)}\} \text{ for every } n\big\} \tag{3.2}$$

are investigated. In particular, a slight modification of the proof of Theorem 4.1 in [8] shows that for every $\alpha \in \Pi$,

$$\mathrm{E}\{H(S_N) - V_N^\alpha \mid S_0, V_0\} = \left(\frac{p}{p^*}\right)^{\lambda_N} \left(\frac{1-p}{1-p^*}\right)^{N-\lambda_N} [V_0^*(S_0) - V_0]^+, \tag{3.3}$$

where $\lambda_N := \#\{n \mid \alpha_n = \alpha_n^{(1)}\}$. Moreover, it is possible to define a one to one correspondence between controls $\alpha \in \Pi$ and "events" $\omega = (\omega_n)_{n=0,\dots,N-1}$ by defining

$$\big(\omega(\alpha)\big)_n := \begin{cases} u & \text{if } \alpha_n = \alpha_n^{(1)} \\ d & \text{if } \alpha_n = \alpha_n^{(2)} \end{cases}. \tag{3.4}$$

In other words, $\omega(\alpha)$ is the only event such that the stock S goes up every time the investor "bets" on it going down on average, and the stock goes down every time the investor bets on it going up on average. Now, it is straightforward to prove that *any strategy $\alpha \in \Pi$ gives perfect hedging in all events $\omega \neq \omega(\alpha)$*. This is quite a strong result, which has two main consequences. First of all, Π is a class of "quasi-replicating" controls, in the sense that every $\alpha \in \Pi$ gives perfect hedging with probability close to 1. More in detail, since $\mathrm{E}\{(H(S_N) - V_N^\alpha)^+\} > 0$ and $H(S_N)(\omega) - V_N^\alpha(\omega) = 0$ in all events $\omega \neq \omega(\alpha)$, one has

$$P\{H(S_N) - V_N^\alpha > 0\} = P\{\omega(\alpha)\} = p^{\lambda_N}(1-p)^{1-\lambda_N} \tag{3.5}$$

with λ_N defined as above. Moreover, from (3.3), it follows that in the only "critical" event $\omega(\alpha)$ we have

$$H(S_N(\omega(\alpha))) - V_N^\alpha(\omega(\alpha)) = \left(\frac{1}{p^*}\right)^{\lambda_N} \left(\frac{1}{1-p^*}\right)^{N-\lambda_N} [V_0^*(S_0) - V_0]^+,$$

which is generally much greater than the initial "lack" of capital.

4 Optimal paths in recombining binomial trees

This section contains some technical definitions and results that formalise a "branch-and-bound" type algorithm for the purpose of determining an optimal path in a particular directed graph, which we call a recombining binomial tree. These results will constitute a key tool for the proof of Theorem 5.2 concerning the solution of the shortfall risk minimization problem in the case when ℓ is concave.

4.1. Definition. For $n \in \mathbb{N}$, a *recombining binomial tree* (r.b.t. for short) of depth n is a directed graph $T(n)$ with $\frac{(n+1)(n+2)}{2}$ nodes τ_k^i, where $k = 0, \dots, i$, $i = 0, \dots, n$. This way, $T(n)$ features $i+1$ nodes at each "depth" i. In particular, at the depth 0 there is only one node τ_0^0, which will be called the *vertex* of the tree $T(n)$.

For $i = 0, \dots, n-1$, each node τ_k^i is supposed to be connected with the two nodes τ_k^{i+1} and τ_{k+1}^{i+1}. The nodes at depth n are thus "terminal" nodes, and will be called the *leaves* of the tree. We shall refer to moving from τ_k^i to τ_k^{i+1} (respectively, to τ_{k+1}^{i+1}) with the expression *branching left* (respectively, *branching right*).

Since the graph is directed, the maximum length paths on the graph start from the vertex τ_0^0 and reach a leaf τ_k^n for some k. Note that there is a one-to-one correspondence between maximum length paths and branching sequences $(\vartheta_i)_{i=0}^{n-1} \in \{l, r\}^n$, where $\vartheta_i = l$ (respectively, $\vartheta_i = r$) means the decision to branch left (respectively, right) when passing from depth i to depth $i+1$.

The problem we want to solve is the following. Suppose that each leaf τ_k^n of the r.b.t. $T(n)$ is associated with some value $r_k \in \mathbb{R}$. Then, we want to find a branching sequence so as to reach a leaf associated with the minimum value, that we shall call, respectively, an *optimal branching sequence* (or *strategy*) and an *optimal leaf.* (The problem of reaching the maximum value can trivially be reformulated as a minimum problem by associating the values $-r_k$ to the leaves.)

Note that, even if the correspondence between paths and branching sequences is one to one, in general the strategy leading to a chosen leaf τ_k^n is not unique. Actually, any strategy (starting from the vertex) which branches left $n-k$ times and right k times in any order will end in leaf τ_k^n, and it is clear that there are $\binom{n}{k}$ such strategies. Thus, the choice of the optimal branching strategy is not a consequence of the determination of the optimal leaf.

The algorithm we propose below solves the problem of choosing an optimal strategy by deciding the branching sequence while scanning the leaves to determine the minimum value.

4.2. Proposition. *Let $n \in \mathbb{N}$, and $T(n)$ be the recombining binomial tree of depth n, with values r_k, $k = 0, \dots, n$ associated with the leaves.*

Define

$$\vartheta_0 = \begin{cases} l & \text{if } r_0 \leqslant r_n \\ r & \text{if } r_0 > r_n \end{cases}$$

and then, recursively for every $i = 1, \dots, n$,

$$\begin{aligned} k_{\vartheta,i} &:= \#\{j < i \mid \vartheta_j = r\}, \\ \vartheta_i &= \begin{cases} l & \text{if } r_{k_{\vartheta,i}} \leqslant r_{n-i+k_{\vartheta,i}} \\ r & \text{if } r_{k_{\vartheta,i}} > r_{n-i+k_{\vartheta,i}}. \end{cases} \end{aligned}$$

Then the leaf $\tau^n_{k_{\vartheta,n}}$ is associated with the minimum value, and $\vartheta = (\vartheta_i)_{i=1}^n$ is an optimal branching sequence.

Proof. Induction on n.

($n = 1$). In this case, there are only the two leaves τ^1_0 and τ^1_1 associated with the values r_0 and r_1 respectively. The thesis translates into the obvious decision to branch left and reach τ^1_0 if r_0 is the minimum value, and to branch right and reach τ^1_1 otherwise.

($n > 1$). To determine the branching at depth 0 the following considerations can be made. Note that the two outmost leaves τ^n_0 and τ^n_n are both reached from the vertex by a unique branching strategy, namely, τ^n_0 can only be reached by branching always left and τ^n_n can only be reached by branching always right. Any other leaf can be reached with a suitable branching sequence whatever ϑ_0 is. Thus, the proposed strategy decides to branch so as to make unreachable the outmost leaf which is associated with the higher (and, thus, non-optimal) value and reach the node $\tau^1_{k_{\vartheta,1}}$.

One can now consider the"sub-tree" T_1 of depth $n-1$ with vertex $\tau^1_{k_{\vartheta,1}}$ and leaves τ^n_k, $k = k_{\vartheta,1}, \ldots, n-1+k_{\vartheta,1}$. Note that the minimum value has to be associated with one of the leaves of T_1, since the value associated with the "discarded" leaf is greater than a value associated with a leaf of T_1. The recursive step then corresponds to the first step in this sub-tree, and so the proposition follows by recursion. □

5 Risk prone investor: the concave case

Throughout this section ℓ will be supposed to be a concave function on $\mathbb{R}_+$. As in Section 3, we use the convention of writing $\alpha_n^{(i)}$ instead of $\alpha_n^{(i)}(S_n, V_n)$.

Before starting the discussion, we need an extension of Proposition 2.1.

5.1. Proposition. *If ℓ is concave then, in the notations of Section 2, the function $J_n(s,\cdot)_{|(-\infty, V_n^*(s)]}$ is concave for every $n = 0, \ldots, N$.*

Proof. We proceed again by backwards induction on n. For $n = N$ the statement follows immediately from the fact that $V_n^*(s) = H(s)$ and that ℓ is concave.

For $n < N$, choose $v' < \overline{v} < v'' \leqslant V_n^*(s)$, and let λ', λ'' be the convex combinators such that $\lambda' + \lambda'' = 1$, $\lambda' v' + \lambda'' v'' = \overline{v}$. Expanding the expected value as in the proof of Proposition 2.1 one gets (as explained below)

$$\begin{aligned}
J_n(s,\overline{v}) &= \inf_\alpha \big\{pJ_{n+1}(su, \overline{v} + \alpha s(u-1)) + \\
&\qquad + (1-p)J_{n+1}(sd, \overline{v} + \alpha s(d-1))\big\} \\
&= \inf_\alpha \big\{pJ_{n+1}\big(su, \lambda'(v' + \alpha s(u-1)) + \lambda''(v'' + \alpha s(u-1))\big) + \\
&\qquad + (1-p)J_{n+1}\big(sd, \lambda'(v' + \alpha s(d-1)) + \lambda''(v'' + \alpha s(d-1))\big)\big\} \\
&\geqslant \inf_\alpha \big\{\lambda'[pJ_{n+1}(su, v' + \alpha s(u-1)) + (1-p)J_{n+1}(sd, v' + \alpha s(d-1))] + \\
&\qquad + \lambda''[pJ_{n+1}(su, v'' + \alpha s(u-1)) + (1-p)J_{n+1}(sd, v'' + \alpha s(d-1))]\big\} \\
&\geqslant \lambda' J_n(s, v') + \lambda'' J_n(s, v'').
\end{aligned}$$

The last inequality comes from the fact that whenever a, b are positive one has $\inf_x\{af(x) + bg(x)\} \geqslant a \inf_x\{f(x)\} + b \inf_x\{g(x)\}$. The first inequality is not straightforward, because J_{n+1} is not concave on the entire real line, but can be proved as follows. By (3) of Proposition 2.1, in the computation of the inf we can restrict α to take values in the interval $[\alpha_n^{(1)}(s,\overline{v}), \alpha_n^{(2)}(s,\overline{v})]$. In this interval, both inequalities

$$\begin{aligned}\overline{v} + \alpha s(d-1) \leqslant V^*_{n+1}(sd)\\ \overline{v} + \alpha s(u-1) \leqslant V^*_{n+1}(su)\end{aligned}$$

hold, so that both j^u_{n+1} and j^d_{n+1} (in the same notations as for the proof to 2.1) are calculated in the (inductively assumed) concavity domain of J_{n+1}. This way $j^u_{n+1} + j^d_{n+1}$, being a sum of concave functions, is concave, and hence the first inequality follows. □

The fact that the concavity "propagates" along the DP algorithm allows us to find a straightforward explicit solution for our problem.

5.2. Theorem. *Let ℓ be a concave function and, for every $n = 0, \dots, N$, let J_n be the function defined in Equation (1.1). Suppose that $V_0 < V^*_0(S_0)$. Then, for every n,*

$$J_n(S_n, V_n) = \min_{k=n,\dots,N} p^{k-n}(1-p)^{N-k}\ell\left(\frac{V^*_n(S_n) - V_n}{(p^*)^{k-n}(1-p^*)^{N-k}}\right). \tag{5.1}$$

Define, for every $n = 0, \dots, N-1$,

$$\alpha_n^{(1)} := \frac{V^*_{n+1}(S_n d) - V_n}{S_n(d-1)}, \qquad \alpha_n^{(2)} := \frac{V^*_{n+1}(S_n u) - V_n}{S_n(u-1)}.$$

Then the optimal strategy is given by

$$\alpha^*_n = \begin{cases} \alpha_n^{(1)} & \text{if } p^{N-n}\ell\left(\frac{V^*_n(S_n)-V_n}{(p^*)^{N-n}}\right) \leqslant (1-p)^{N-n}\ell\left(\frac{V^*_n(S_n)-V_n}{(1-p^*)^{N-n}}\right) \\ \alpha_n^{(2)} & \text{if } p^{N-n}\ell\left(\frac{V^*_n(S_n)-V_n}{(p^*)^{N-n}}\right) > (1-p)^{N-n}\ell\left(\frac{V^*_n(S_n)-V_n}{(1-p^*)^{N-n}}\right) \end{cases}. \tag{5.2}$$

5.3. Remark. As the proof of this theorem will make clear, we can always restrict the class of admissible controls to the class Π defined in (3.2). In particular, this allows a proof by induction via a straightforward calculation that from $V_0 \leqslant V^*_0(S_0)$ follows $V_n \leqslant V^*_n(S_n)$ almost surely (i.e., for every event ω) for every n. This is the reason why in the expression for $J_n(S_n, V_n)$ there is no need to consider the positive part of the argument of ℓ.

Proof of Theorem 5.2. For every $n = 0, \dots, N-1$ define j^u_n and j^d_n as in the proof of Proposition 2.1, so that

$$J_n(S_n, V_n) = \inf_\alpha \{j^u_{n+1}(s, v, \alpha) + j^u_{n+1}(s, v, \alpha)\}.$$

By (3) and (4) of Proposition 2.1 we can restrict the computation of the inf to the interval $[\alpha_n^{(1)}, \alpha_n^{(2)}]$ where, as already seen in the proof of Proposition 5.1, $j^u_{n+1} + j^d_{n+1}$ is concave. So, since a concave function on an interval can attain its lowest value only at the extremal points, the problem reduces to comparing the values taken by the function at the extrema. Namely, on recalling that for

$\alpha = \alpha_n^{(1)}$ (respectively, $\alpha = \alpha_n^{(2)}$) one has $j_{n+1}^d = 0$ (respectively, $j_{n+1}^u = 0$), one finds that

$$\begin{aligned} J_n(S_n, V_n) = &\min\{pJ_{n+1}(S_n u, V_n + \alpha_n^{(1)} S_n(u-1)), \\ &\qquad (1-p)J_{n+1}(S_n d, V_n + \alpha_n^{(2)} S_n(d-1))\}, \\ \operatorname{argmin}[\cdots] = \alpha_n^{(1)} \iff &\ pJ_{n+1}(S_n u, V_n + \alpha_n^{(1)} S_n(u-1)) \leqslant \\ &\qquad \leqslant (1-p)J_{n+1}(S_n d, V_n + \alpha_n^{(2)} S_n(d-1)), \end{aligned}$$

and, as in [1, Section 2] (see also Section 3 here), we are allowed to restrict the class of admissible controls to the class Π defined in (3.2). Thanks to (2.4), it is straightforward to verify by backward induction on n that for every $\alpha \in \Pi$, defining

$$\lambda_n := \#\{k \geqslant n \mid \alpha_k = \alpha_k^{(1)}\}, \qquad \mu_n := \#\{k \geqslant n \mid \alpha_k = \alpha_k^{(2)}\} = N - n - \lambda_n,$$

gives

$$\mathrm{E}_{S_n,V_n}\{\ell([H(S_N) - V_N^\alpha]^+)\} = p^{\lambda_n}(1-p)^{\mu_n}\,\ell\left(\frac{V_n^*(S_n) - V_n}{(p^*)^{\lambda_n}(1-p^*)^{\mu_n}}\right). \tag{5.3}$$

Hence it follows that the optimal value is as in (5.1).

To prove that the strategy given in (5.2) is optimal, we start by observing that from (5.3) it follows that, for the purpose of determining the shortfall risk, only the number of times that $\alpha_n = \alpha_n^{(1)}$ matters, and not the particular sequence of choices. The evolution of the shortfall risk with respect to the chosen control from time n on can then be seen as a "recombining binomial tree" of depth $N-n$ (see Definition 4.1 for the definition) as follows. For every $m = n, \dots, N-1$, associate the choice $\alpha_m = \alpha_m^{(1)}$ (respectively, $\alpha_m = \alpha_m^{(2)}$) with the decision of branching left (respectively, right) at depth $m-n$, so as to set a one-to-one correspondence between strategies in the class Π and branching sequences for the tree. Formula (5.3) suggest associating with each leaf τ_k^{N-n} the value $p^k(1-p)^{N-n-k}\ell\big(\frac{V_n^*(S_n)-V_n}{(p^*)^k(1-p^*)^{N-n-k}}\big)$, i.e. the shortfall risk associated with any strategy leading to that leaf. The proof is now completed by observing that the proposed strategy is exactly the "optimal branching sequence" of Proposition 4.2. $\square$

5.4. Remark. At the end of Section 3 we noted that any strategy in the class Π is "quasi replicating", i.e., it leads to perfect hedging in all events $\omega \neq \omega(\alpha)$ as defined in (3.4). This result does not depend on the function ℓ taken into consideration, so we can conclude that the expected value (5.3) found in the proof to Theorem 5.2 corresponds to a shortfall of $\frac{V_n^*(S_n)-V_n}{(p^*)^{\lambda_n}(1-p^*)^{\mu_n}}$ in the only "critical" non-hedging event $\omega(\alpha)$, whose probability is $P(\omega(\alpha)) = p^{\lambda_n}(1-p)^{\mu_n}$ as seen in (3.5).

5.5. Remark. In the case of "complete information" (i.e., when the probability p is known by the investor), the optimal strategy can be chosen *a priori* instead of on a step-by-step basis. Actually, once the $\overline{k}$ that minimizes $p^k(1-p)^{N-k}\ell\big(\frac{V_0^*(S_0)-V_0}{(p^*)^k(1-p^*)^{N-k}}\big)$ is determined, any strategy choosing $\overline{k}$ times $\alpha_n = \alpha_n^{(1)}$ is optimal. Note also that all of these strategies have the same probability $p^{\overline{k}}(1-p)^{N-\overline{k}}$ of being non-hedging and the same shortfall $\ell\big(\frac{V_0^*(S_0)-V_0}{(p^*)^{\overline{k}}(1-p^*)^{N-\overline{k}}}\big)$

in the "critical" event, so that the choice of the "preferred" optimal strategy has to be made according to completely subjective criteria.

The expression (5.2) for the optimal control has been chosen because it can easily be adapted in the spirit of [1, Section 3] (summarized in Section 3) to elaborate optimal adaptive controls in the case of incomplete knowledge of the model.

5.6. Remark. In the case $\ell(x) = x$ examined in [1] and [8], a quite remarkable fact is that the (generally unique) optimal strategy consists of choosing either $\alpha \equiv \alpha^{(1)}$ or $\alpha \equiv \alpha^{(2)}$. In other words, in this case the optimal leaf in the tree described in the proof of Theorem 5.2 is either at the extreme "left" or at the extreme "right". This may not happen in the general case, as the following (perhaps quite artificial) example shows even in the very simple case of three possible final outcomes.

5.7. Example. Take $\ell(x) := \sqrt{x} + \sqrt[5]{x}$, $N = 2$, $p = .245$, $u = 3.17$ and $d = .8$ (so that $p^* = .0845$). Consider a contingent claim H such that $V_0^*(S_0) > 170$ and choose V_0 such that $V_0^*(S_0) - V_0 = 170$. The three values associated with the final leaves are then

$$\begin{aligned} p^2\ell\left(\frac{V_0^*(S_0) - V_0}{(p^*)^2}\right) &\sim 9.77, \\ p(1-p)\ell\left(\frac{V_0^*(S_0) - V_0}{(p^*)(1-p^*)}\right) &\sim 9.54, \\ (1-p)^2\ell\left(\frac{V_0^*(S_0) - V_0}{(1-p^*)^2}\right) &\sim 9.72. \end{aligned}$$

This way the optimal "leaf" is the central one, and two optimal controls can be built by choosing either $\alpha_0 = \alpha_0^{(1)}, \alpha_1 = \alpha_1^{(2)}$ or $\alpha_0 = \alpha_0^{(2)}, \alpha_1 = \alpha_1^{(1)}$. Note also that the policy proposed in Theorem 5.2 will choose $\alpha_1 = \alpha_1^{(2)}$.

5.8. Example. Consider the case when $\ell(x) = x^\kappa$ for some $0 \leq \kappa \leq 1$. In particular, the case $\kappa = 1$ corresponds to minimising the mean shortfall risk, and, due to the assumption that $\ell(0) = 0$, the case $\kappa = 0$ corresponds to minimising the probability of positive shortfall. These two particular cases have already been considered in [1] and [8]: when $\kappa = 1$, the optimal strategy is the $\alpha^\dagger$ defined in (3.1), and when $\kappa = 0$ the optimal strategy $\alpha^\ddagger$ is calculated by choosing

$$\alpha_n^\ddagger = \begin{cases} \alpha_n^{(1)} & p < 0.5 \\ \alpha_n^{(2)} & p > 0.5 \end{cases}$$

(when $p = 0.5$, all strategies $\alpha \in \Pi$ are optimal).

When $0 < \kappa < 1$, the solution still takes an appearance close to the cited results, namely, the optimal policies always choose either $\alpha \equiv \alpha^{(1)}$ or $\alpha \equiv \alpha^{(2)}$. Actually, in this case, for every $n = 0, \ldots, N-1$ the optimal control α^* is computed by choosing

$$\alpha_n^* = \alpha_n^{(1)} \iff \frac{p}{(p^*)^l} \leqslant \frac{1-p}{(1-p^*)^\kappa} \quad \text{i.e.,} \quad \frac{p}{1-p} \leqslant \left(\frac{p^*}{1-p^*}\right)^\kappa,$$

independent of n.

Moreover, let $\overline{\kappa} = \frac{\log(p)-\log(1-p)}{\log(p^*)-\log(1-p^*)}$ (namely, the value for which $\frac{p}{1-p} = \left(\frac{p^*}{1-p^*}\right)^{\overline{\kappa}}$). Excluding the "undecidable" cases $p = 0.5$ (all events have the same probability), $p^* = 0.5$ (division by zero in $\overline{\kappa}$) and $p = p^*$ (the stock and the portfolio are martingales under the real world probability measure), the following cases may then occur.

$p^* < p < 0.5$ $(0 < \kappa < 1)$	$\alpha^* = \begin{cases} \alpha^\ddagger = \alpha^{(1)} & \kappa < \overline{\kappa} \\ \alpha^\dagger = \alpha^{(2)} & \kappa > \overline{\kappa} \end{cases}$
$p^* < 0.5$, $p \neq (p^*, 0.5)$ $(\kappa < 0$ or $\kappa > 1)$	$\alpha^* = \alpha^\ddagger = \alpha^\dagger = \alpha^{(1)}$
$0.5 < p < p^*$ $(0 < \kappa < 1)$	$\alpha^* = \begin{cases} \alpha^\ddagger = \alpha^{(2)} & \kappa < \overline{\kappa} \\ \alpha^\dagger = \alpha^{(1)} & \kappa > \overline{\kappa} \end{cases}$
$p^* > 0.5$, $p \neq (0.5, p^*)$ $(\kappa < 0$ or $\kappa > 1)$	$\alpha^* = \alpha^\ddagger = \alpha^\dagger = \alpha^{(2)}$

Note that if $\alpha^\ddagger = \alpha^\dagger$, then the optimal control α^* for the shortfall risk minimization problem coincides with both $\alpha^\ddagger$ and $\alpha^\dagger$ for every κ. On the other hand, if $\alpha^\ddagger \neq \alpha^\dagger$, the exponent $0 < \overline{\kappa} < 1$ is "critical", in the sense that $\alpha^* = \alpha^\ddagger$ for $\kappa \in (0, \overline{\kappa})$ and $\alpha^* = \alpha^\dagger$ for $\kappa \in (\overline{\kappa}, 1)$. In other words, the study of the shortfall risk minimization problem with $\ell(x) = x^\kappa$ can be reduced to the two fundamental problems with $\kappa = 0$ and $\kappa = 1$.

6 Risk averse investor: the convex case

Throughout this section ℓ will be assumed to be a convex function on $\mathbb{R}_+$. As in the previous setting, we can give an extension of Proposition 2.1.

6.1. Proposition. *If ℓ is convex then, in the notations of Section 2, the function $J_n(s, \cdot)$ is convex for every $n = 0, \ldots, N$.*

Proof. We proceed by backwards induction on n.

$(n = N)$. Since $\ell(0) = 0$, the definition

$$\bar{\ell}(x) := \begin{cases} \ell(x) & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases}$$

extends ℓ to a function which is convex on the entire real line, and $\ell\big(x^+\big) = \bar{\ell}(x)$. We can then write

$$J_N(s, v) = \bar{\ell}(H(s) - v)$$

and the conclusion follows from the convexity of $\bar{\ell}$.

$(n < N)$. Write

$$\begin{aligned} J_n(s, v) &= \inf_\alpha \mathrm{E}\{J_{n+1}(S_{n+1}, V^\alpha_{n+1} \mid S_n = s, V^\alpha_n = v\} \qquad (6.1) \\ &= \inf_\alpha \{pJ_{n+1}(su, v + \alpha s(u-1)) + (1-p)J_{n+1}(sd, v + \alpha s(d-1))\}, \end{aligned}$$

and define, as in the proof of Proposition 2.1,

$$\begin{aligned} j^u_{n+1}(s, v, \alpha) &:= pJ_{n+1}(su, v + \alpha s(u-1)), \\ j^d_{n+1}(s, v, \alpha) &:= (1-p)J_{n+1}(sd, v + \alpha s(d-1)) \end{aligned}$$

so that $J_n(s,v) = \inf_\alpha\{j^u_{n+1}(s,v,\alpha) + j^d_{n+1}(s,v,\alpha)\}$. Since $J_{n+1}(s,\cdot)$ is convex by the inductive hypothesis, it is straightforward to prove that $j^u_{n+1}(s,\cdot,\cdot)$ and $j^d_{n+1}(s,\cdot,\cdot)$ are convex, in the sense that for every $v_1 \leqslant \overline{v} \leqslant v_2$, $\alpha_1 \leqslant \overline{\alpha} \leqslant \alpha_2$ such that $\overline{v} = \lambda_1 v_1 + \lambda_2 v_2$, $\overline{\alpha} = \lambda_1\alpha_1 + \lambda_2\alpha_2$ (with λ_1, λ_2 convex combinators) it is

$$\begin{aligned} j^u_{n+1}(s,\overline{v},\overline{\alpha}) &\leqslant \lambda_1 j^u_{n+1}(s,v_1,\alpha_1) + \lambda_2 j^u_{n+1}(s,v_2,\alpha_2) \\ j^d_{n+1}(s,\overline{v},\overline{\alpha}) &\leqslant \lambda_1 j^d_{n+1}(s,v_1,\alpha_1) + \lambda_2 j^d_{n+1}(s,v_2,\alpha_2). \end{aligned}$$

In particular, $j_{n+1}(s,v,\alpha) := j^u_{n+1}(s,v,\alpha) + j^d_{n+1}(s,v,\alpha)$ is convex in α, so the inf of (6.1) is realized as min.

Choose now $v' \leqslant \widetilde{v} \leqslant v''$, so that $\widetilde{v} = \lambda' v' + \lambda'' v''$ for some convex combinators λ',λ''. By convexity of $j_{n+1}(s,v,\cdot)$, there exist $\widetilde{\alpha}$, α', α'' such that

$$\begin{aligned} j_{n+1}(s,\widetilde{v},\widetilde{\alpha}) &= \min_\alpha j_{n+1}(s,\widetilde{v},\alpha) = J_n(s,\widetilde{v}) \\ j_{n+1}(s,v',\alpha') &= \min_\alpha j_{n+1}(s,v',\alpha) = J_n(s,v') \\ j_{n+1}(s,v'',\alpha'') &= \min_\alpha j_{n+1}(s,v'',\alpha) = J_n(s,v''), \end{aligned}$$

and by convexity of $j_{n+1}(s,\cdot,\cdot)$ and minimality of $\widetilde{\alpha}$ we have that

$$\begin{aligned} \lambda' J_n(s,v') + \lambda'' J_n(s,v'') &= \lambda' j_{n+1}(s,v',\alpha') + \lambda'' j_{n+1}(s,v'',\alpha'') \\ &\geqslant j_{n+1}(s,\widetilde{v},\lambda'\alpha' + \lambda''\alpha'') \\ &\geqslant j_{n+1}(s,\widetilde{v},\widetilde{\alpha}) = J_n(s,\widetilde{v}). \end{aligned}$$

□

This proposition has as an immediate consequence the existence of an optimal strategy for the convex case. It also follows that, if ℓ is strictly convex, then the optimal strategy is unique. Nevertheless, when trying to determine explicitly the optimal strategy in general form by using dynamic programming arguments one is led to quite complex calculations. For the sake of simplicity, then, from now on we shall shift from a DPA-based approach to another one, with techniques similar to those in [2]. Notice that, for the case when ℓ is strictly convex, we could also use techniques based on convex duality as in [4] or [6]. However, due to the absence of lower bounds on the claim $H(S_N)$ and on the portfolio X, it would not be possible to apply these techniques to the case when $\ell(x) = x$.

6.2. Definition. We define the set of the *modified contingent claims* as

$$\mathcal{X} := \{X \mid X \leqslant H(S_N) \text{ (a.s.)}, \mathrm{E}^*\{X\} \leqslant V_0\}. \tag{6.2}$$

Roughly speaking, $\mathcal{X}$ is the set of all the claims less than $H(S_N)$ which can be replicated with initial capital (less than or equal to) V_0 or, equivalently, the set of all the possible final states of adapted, self-financing strategies starting from initial capital (less than or equal to) V_0.

We can now consider the shortfall risk minimization problem from a "static" point of view:

$$\min_{X\in\mathcal{X}} \mathrm{E}_{S_0}\left\{\ell\big(H(S_N) - X\big)\right\}. \tag{6.3}$$

We now want to show that the modified contingent claim that solves (6.3) coincides with the payoff of the optimal portfolio for the shortfall risk minimization problem. We shall start from the linear case ($\ell(x) = x$) and from this case we shall derive the solution for the strictly convex case.

The following lemma, a key tool for the proof of the main Theorem 6.4, shows how this approach can be applied to the study of the "mean shortfall risk minimization problem", i.e., the shortfall risk minimization problem in the case $\ell(x) = x$. Note that, in this case, (6.3) reduces to the problem $\max_{x\in\mathcal{X}} \mathrm{E}\{X\}$.

6.3. Lemma. *If $X^* \in \mathcal{X}$ solves the problem $\max_{x\in\mathcal{X}} \mathrm{E}\{X\}$, then the hedging strategy α^* for the claim X^* also solves the mean shortfall risk minimization problem $\min_\alpha \left\{ \mathrm{E}_{S_0,V_0}\{(H(S_N) - V_N^\alpha)^+\}\right\}$. Moreover, define $c_{es} := \min_\omega \frac{\mathrm{d}P}{\mathrm{d}P^*}$, where P and P^* are, respectively, the "real world" and the martingale probability measures. Set $E := \{\omega \mid \frac{\mathrm{d}P}{\mathrm{d}P^*}(\omega) = c_{es}\}$, and*

$$X^* := H(S_N)1_{E^c} + \gamma 1_E$$

where γ is any random variable such that $\mathrm{E}^*\{X^*\} = V_0$. Then X^* solves the problem (6.3).

Proof. This proposition can be proved directly as in [9], but we give a shorter proof that uses the results of Section 3. Consider the strategy $\alpha^\dagger$ defined in (3.1). Since the corresponding portfolio value $V_N^{\alpha^\dagger}$ is the payoff of a self-financing strategy starting from the initial capital V_0 and $V_N^{\alpha^\dagger} \leqslant H(S_N)$, then $V_N^{\alpha^\dagger}$ itself is a modified contingent claim in the sense of (6.2). Moreover, if $X \in \mathcal{X}$, then one has $\mathrm{E}_{S_0,V_0}\{(H(S_N)-X)^+\} = \mathrm{E}_{S_0,V_0}\{H(S_N)\}-\mathrm{E}_{S_0,V_0}\{X\}$, so that the first part of the proposition follows.

As for the second part, if $p = p^*$ then all the modified contingent claims $X \in \mathcal{X}$ are solutions. If $p \neq p^*$, then from Section 3 we also deduce that the $V_N^{\alpha^\dagger}$ is equal to $H(S_N)$ on all events except for the "least probable" one, where the payoff is equal to

$$H(S_N) - \frac{1}{\min\{(p^*)^N, (1-p^*)^N\}}(V_0^*(S_0) - V_0).$$

In particular, it is straightforward to check that $\mathrm{E}^*\{V_N^{\alpha^\dagger}\} = V_0$, and thus that $V_N^{\alpha^\dagger}$ can be defined as in the statement above. □

To apply this approach to the convex case, we make the further assumption that ℓ is strictly convex, continuously differentiable and such that $\ell'(0) = 0$. Indeed, these properties appear to be the right ones to state the following results in a reasonable notation, but we believe that a generalization to non-$\mathcal{C}^1$ functions, in terms of the sub-differential, should be straightforward.

6.4. Theorem. *Set $I := (\ell')^{-1}$ and define the modified contingent claim*

$$X^* := H(S_N) - I\left(c^*\frac{\mathrm{d}P^*}{\mathrm{d}P}\right)$$

with $c^* > 0$ chosen in such a way that $E^*\{X^*\} = V_0$. Then X^* solves the "static" problem (6.3).

Proof. Fix an $X^* \in \mathcal{X}$ and define, for every $X \in \mathcal{X}$,

$$X_\varepsilon := (1-\varepsilon)X^* + \varepsilon X, \quad F_X(\varepsilon) := \mathrm{E}\left\{\ell\big(H(S_N) - X_\varepsilon\big)\right\}.$$

Then $X_\varepsilon \in \mathcal{X}$ for every $\varepsilon \in [0,1]$ and it is clear that X^* is optimal for the problem (6.3) if and only if

$$0 \leqslant F'_X(0^+) = \mathrm{E}\left\{(X^* - X)\ell'\big(H(S_N) - X^*\big)\right\}$$

i.e., if and only if $\mathrm{E}\left\{X^*\ell'\big(H(S_N) - X^*\big)\right\} \geqslant \mathrm{E}\left\{X\ell'\big(H(S_N) - X^*\big)\right\}$, for every $X \in \mathcal{X}$.

Define now the measure Q by

$$\frac{\mathrm{d}Q}{\mathrm{d}P} := \frac{\ell'\big(H(S_N) - X^*\big)}{\mathrm{E}\left\{\ell'\big(H(S_N) - X^*\big)\right\}},$$

so that X^* minimizes the shortfall risk if and only if it is optimal for the problem $\max_{X\in\mathcal{X}} \mathrm{E}^Q\{X\}$. According to Proposition 6.3, the latter maximization problem is solved by an X^* of the form

$$X^* := H(S_N)1_{F^c} + \gamma 1_F,$$

where $F := \{\omega \mid \frac{\mathrm{d}Q}{\mathrm{d}P^*}(\omega) = \bar{c}\}$, $\bar{c} := \min_\omega \frac{\mathrm{d}Q}{\mathrm{d}P^*}(\omega)$ and γ is such that $\mathrm{E}^*\{X^*\} = V_0$.

Note that by definition of Q,

$$\frac{\mathrm{d}Q}{\mathrm{d}P^*} = \frac{\mathrm{d}Q}{\mathrm{d}P} \cdot \frac{\mathrm{d}P}{\mathrm{d}P^*} = \frac{\ell'\big(H(S_N) - X^*\big)}{\mathrm{E}\left\{\ell'\big(H(S_N) - X^*\big)\right\}} \cdot \frac{\mathrm{d}P}{\mathrm{d}P^*}.$$

Since ℓ' is increasing and $\ell'(0) = 0$, the above equality implies that $\left\{\frac{\mathrm{d}Q}{\mathrm{d}P^*} = 0\right\} = \{H(S_N) = X^*\}$. As a consequence, if $\bar{c} = 0$, then $P\{H(S_N) = X^*\} > 0$ and one would have $X^* = H(S_N)$ on the set $\left\{\frac{\mathrm{d}Q}{\mathrm{d}P^*} > 0\right\} = \{H(S_N) > X^*\}$, a contradiction. One then has that $\bar{c} > 0$, i.e., Q and P^* are equivalent and, moreover, $H(S_N) > X^*$ almost surely. By definition of X^*, it follows then that $\frac{\mathrm{d}Q}{\mathrm{d}P^*} = \bar{c}$ almost surely, so that $X^* = \gamma 1_F$, i.e., $\gamma = X^*$. As a consequence,

$$\begin{aligned}
\frac{\mathrm{d}Q}{\mathrm{d}P^*} \equiv \bar{c} = 1 &\Rightarrow \frac{\mathrm{d}Q}{\mathrm{d}P}\frac{\mathrm{d}P}{\mathrm{d}P^*} = 1 \\
&\Rightarrow \frac{\mathrm{d}P^*}{\mathrm{d}P} = \frac{\mathrm{d}Q}{\mathrm{d}P} = \frac{\ell'\big([H(S_N) - X^*]^+\big)}{c^*} \\
&\Rightarrow \ell'\big([H(S_N) - X^*]^+\big) = c^*\frac{\mathrm{d}P^*}{\mathrm{d}P} \\
&\Rightarrow H(S_N) - X^* = I\left(c^*\frac{\mathrm{d}P^*}{\mathrm{d}P}\right) \\
&\Rightarrow X^* = H(S_N) - I\left(c^*\frac{\mathrm{d}P^*}{\mathrm{d}P}\right),
\end{aligned}$$

with $c^* := \mathrm{E}\left\{\ell'\big(H(S_N) - X^*\big)\right\}$. Since X^* minimizes the shortfall risk, one must have that $\mathrm{E}^*\{X^*\} = V_0$, i.e.,

$$\mathrm{E}^*\left\{I\left(c^*\frac{\mathrm{d}P^*}{\mathrm{d}P}\right)\right\} = \mathrm{E}^*\{H(S_N)\} - V_0. \tag{6.4}$$

Since ℓ is convex and C^1, then ℓ' is increasing and continuous and so is I. This means that the function $c \to \mathrm{E}^*\{I(c\frac{\mathrm{d}P^*}{\mathrm{d}P})\}$ is continuous and increasing, thus there exists a unique c^* such that (6.4) is satisfied. $\square$

6.5. Remark. Once the optimal modified contingent claim X^* is determined, the optimal strategy is simply the Cox-Ross-Rubinstein replicating strategy for X^* as in (2.2), that is,

$$\alpha_n^* = \frac{\mathrm{E}^*\{X^*|S_{n+1} = S_n u\} - \mathrm{E}^*\{X^*|S_{n+1} = S_n d\}}{S_n(u-d)}.$$

7 Robustness

In this section we assume that we are given a sequence $(\ell_k)_k$ of loss functions that converge pointwise to a loss function ℓ. For each function ℓ_k (resp. for ℓ), we call X_k^* (resp. X^*) the optimal modified claim which solves the problem in (6.3), and $\alpha^{*,k}$ (resp. α^*) the corresponding optimal strategy. Throughout the section, we shall also write

$$\Delta := \mathrm{E}^*\{H(S_N)\} - V_0 = V_0^*(S_0) - V_0.$$

We distinguish three cases: the case when the limit loss function ℓ is strictly convex, the case $\ell(x) = x$ and the case when the limit loss function ℓ is concave. In the first case, we use a technique similar to [3].

7.1. Theorem. *Let $(\ell_k)_{k\in\mathbb{N}}$ be a sequence of strictly convex, $\mathcal{C}^1$ functions such that $\ell_k'(0) = 0$ for every k. If $\lim_k \ell_k = \ell$ pointwise with ℓ strictly convex, $\mathcal{C}^1$ and such that $\ell'(0) = 0$, then $X_k^* \to X^*$ almost surely.*

Proof. We start by proving that $c_k^* \to c^*$. To do this, define

$$\varphi_k(c) = \mathrm{E}\left\{ I_k\left(c\frac{\mathrm{d}P^*}{\mathrm{d}P}\right)\right\}$$

and note that c^* and the c_k^*s have the property that $\Delta = \varphi_k(c_k^*) = \varphi(c^*)$ (where φ is defined in the obvious way). Since the probability space Ω is finite, it is straightforward to check that $\varphi_k(c) < +\infty$ for all $c \geq 0$ and that φ_k is continuous for all k.

Since $I_k \to I$ pointwise, we have that for every $c \in \mathbb{R}+$

$$I_k\left(c\frac{\mathrm{d}P^*}{\mathrm{d}P}(\omega)\right) \leq \sup_{k\in\mathbb{N}} I_k\left(c\frac{\mathrm{d}P^*}{\mathrm{d}P}(\omega)\right) \leq \sup_{\omega\in\Omega}\sup_{k\in\mathbb{N}} I_k\left(c\frac{\mathrm{d}P^*}{\mathrm{d}P}(\omega)\right) =: K < \infty,$$

i.e., the sequence $I_k(c\frac{\mathrm{d}P^*}{\mathrm{d}P}(\omega))$ is dominated. We know from [3] that if $\ell_k \to \ell$ pointwise then $I_k \to I$ uniformly on compact sets. This means that $\varphi_k(c) \to \varphi(c)$ for all $c \geq 0$. Since φ and the φ_k are continuous and strictly increasing, it follows that $\varphi_k^{-1} \to \varphi^{-1}$ pointwise. Moreover, these functions are continuous, so $c_k^* = \varphi_k^{-1}(\Delta) \to \varphi^{-1}(\Delta) = c^*$.

Now we are able to prove the theorem. First of all notice that by Theorem 6.4

$$X_k^* = H(S_N) - I_k\left(c_k^*\frac{\mathrm{d}P^*}{\mathrm{d}P}\right).$$

Since I_k converges uniformly on compact sets to I, the right hand side converges to $H(S_N) - I(c^*\frac{\mathrm{d}P^*}{\mathrm{d}P})$ almost surely, so that one gets $X_k^* \to X^*$ almost surely. $\square$

Now we consider the case $\ell(x) = x$. For this case, since I cannot be defined, we can no longer use techniques similar to [3], and we have to develop an ad-hoc technique. Recall from Lemma 6.3 that an optimal claim in this case is given by

$$X^* = H(S_N) - \frac{\Delta}{P^*(E)} 1_E \tag{7.5}$$

with $E = \left\{ \frac{\mathrm{d}P}{\mathrm{d}P^*} = c_{es} \right\}$.

7.2. Theorem. *Let $(\ell_k)_{k\in\mathbb{N}}$ be a sequence of strictly convex, $\mathcal{C}^1$ functions such that $\ell_k'(0) = 0$ for every k. If $\lim_k \ell_k(x) = x$ pointwise, then $X_k^* \to X^*$ almost surely, where X^* is defined as in (7.5).*

Proof. Since $\ell_k(x) \to x$ pointwise and the ℓ_k are convex, then $\ell_k'(x) \to 1$ uniformly on compact sets of $\mathbb{R}_+$ (see [3] or [7]). Since the ℓ_k' are all increasing functions, it is easy to prove that

$$\lim_{k\to\infty} I_k(x) = \begin{cases} 0 & \text{if } x < 1 \\ +\infty & \text{if } x > 1 \end{cases}$$

The convergence is uniform on compact sets of $(0, 1)$, and also the convergence to $+\infty$ is "uniform" in the sense that for every $\varepsilon > 0$ and $M > 0$ there exists $\overline{k}$ such that $I_k(x) > M$ for all $k > \overline{k}$ and $x > 1 + \varepsilon$.

The convergence above implies that $c_k^* \to c_{es}$. Indeed, for every $\varepsilon > 0$ there exists $\overline{k}$ such that $0 < I_k(x) < \varepsilon$ for every $k > \overline{k}$, $x \in (0, 1-\varepsilon)$. Then

$$\Delta = \mathrm{E}^* \left\{ I_k \left(c_k^* \frac{\mathrm{d}P^*}{\mathrm{d}P} \right) \right\} \leq \varepsilon + \mathrm{E}^* \left\{ I_k \left(c_k^* \frac{\mathrm{d}P^*}{\mathrm{d}P} \right) 1_{\left\{ c_k^* \frac{\mathrm{d}P^*}{\mathrm{d}P} \geqslant 1-\varepsilon \right\}} \right\}$$

so $\mathrm{E}^* \left\{ I_k(c_k^* \frac{\mathrm{d}P^*}{\mathrm{d}P}) 1_{\{c_k^* \frac{\mathrm{d}P^*}{\mathrm{d}P} \geq 1-\varepsilon\}} \right\} \geq \Delta - \varepsilon$; in particular, $\left\{ c_k^* \frac{\mathrm{d}P^*}{\mathrm{d}P} \geq 1 - \varepsilon \right\}$ is not empty for all $k > \bar{k}$, and this means that $c_k^* \max_\omega \frac{\mathrm{d}P^*}{\mathrm{d}P}(\omega) \geq 1 - \varepsilon$, so $c_k^* < (1-\varepsilon) \min_\omega \frac{\mathrm{d}P}{\mathrm{d}P^*}(\omega) = (1-\varepsilon)c_{es}$. Conversely, for all $\varepsilon > 0$ and $M > \Delta/P^*(E)$ there exists $\bar{k}$ such that $I_k(x) > M$ for all $k > \bar{k}$, $x > 1 + \varepsilon$. This implies that

$$\Delta = \mathrm{E}^* \left\{ I_k \left(c_k^* \frac{\mathrm{d}P^*}{\mathrm{d}P} \right) \right\} \geq \mathrm{E}^* \left\{ I_k \left(c_k^* \frac{\mathrm{d}P^*}{\mathrm{d}P} \right) 1_E \right\} = I_k \left(\frac{c_k^*}{c_{es}} \right) P^*(E)$$

Thus we must have $\frac{c_k^*}{c_{es}} \leq 1+\varepsilon$. In fact, if $\frac{c_k^*}{c_{es}} > 1+\varepsilon$, we obtain $\Delta \geq M \cdot P^*(E)$, but we took $M > \Delta/P^*(E)$, so this is absurd. In conclusion, we have that for all ε, $(1-\varepsilon)c_{es} \leq c_k^* \leq (1+\varepsilon)c_{es}$ from a certain $\bar{k}$ on, which yields $c_k^* \to c_{es}$.

Now we only have to prove that $I_k(c_k^* \frac{\mathrm{d}P^*}{\mathrm{d}P}) \to \frac{\Delta}{P^*(E)} 1_E$. Since $c_k^* \to c_{es}$, it follows that $\lim_{k\to\infty} c_k^* \frac{\mathrm{d}P^*}{\mathrm{d}P} = c_{es} \frac{\mathrm{d}P^*}{\mathrm{d}P}$, which is equal to 1 on E and less than one on E^c. Since $I_k \to 0$ uniformly on compact sets of $(0, 1)$, we have that $I_k(c_k^* \frac{\mathrm{d}P^*}{\mathrm{d}P}) \to 0$ on E^c, and the limit is uniform. Thus for all $\varepsilon > 0$ there exists $\bar{k}$ such that for all $k > \bar{k}$ we have

$$\Delta - \mathrm{E}^* \left\{ I_k \left(c_k^* \frac{\mathrm{d}P^*}{\mathrm{d}P} \right) 1_E \right\} = \mathrm{E}^* \left\{ I_k \left(c_k^* \frac{\mathrm{d}P^*}{\mathrm{d}P} \right) 1_{E^c} \right\} \in (0, \varepsilon)$$

This means that $I_k(\frac{c_k^*}{c_{es}}) P^*(E) \to \Delta$, and finally that $I_k(c_k^* \frac{\mathrm{d}P^*}{\mathrm{d}P}) \to \frac{\Delta}{P^*(E)} 1_E$. The proof is finished. $\square$

7.3. Corollary. *Under the assumptions of Theorems 7.1 or 7.2, we have that* $\alpha_n^{*,k} \to \alpha_n^*$ *almost surely for all* $n = 0, \dots, N-1$.

Proof. Recall that the optimal strategy for the shortfall risk minimization problem with loss function ℓ_k is

$$\alpha_n^{*,k} = \frac{\mathrm{E}^*\{X_k^* | S_{n-1}u\} - \mathrm{E}^*\{X_k^* | S_{n-1}d\}}{S_{n-1}(u-d)}.$$

(see Remark 6.5). Since $X_k^* \to X^*$ almost surely and the probability space Ω is finite, the conditional expectations above converge almost surely to

$$\frac{\mathrm{E}^*\{X^* | S_{n-1}u\} - \mathrm{E}^*\{X^* | S_{n-1}d\}}{S_{n-1}(u-d)},$$

i.e., $\alpha_n^{*,k} \to \alpha_n^*$ almost surely for all $n = 0, \dots, N-1$. □

In the concave case similar results hold, but the formulation is not straightforward due to the fact that the optimal solution is in general not unique. Recall that in the concave case the strategy is given by (5.2) and thus the optimal modified contingent claim is given by

$$X^* = H(S_N) - \frac{\Delta}{P\{\omega(\alpha^*)\}} 1_{\{\omega(\alpha^*)\}},$$

where $\omega(\alpha^*)$ is the "critical" event defined in (3.4).

7.4. Theorem. *Let* $(\ell_k)_{k\in\mathbb{N}}$ *be a sequence of concave functions such that* $\ell_k \to \ell$ *pointwise. Then there exists a* $(k_h)_h$ *such that* $X_{k_h}^* \to X^*$ *almost surely and* $\alpha_n^{*,k_h} \to \alpha_n^*$ *almost surely for all* $n = 0, \dots, N-1$, *where* X^* *is an optimal solution of problem (6.3) and* α^* *is the corresponding optimal strategy.*

Proof. We start by proving that, under the given hypotheses, there exist a limit strategy α^* and a subsequence $(\alpha^{*,k_h})_h$ such that $\alpha^{*,k_h} \equiv \alpha^*$ from some $\overline{h}$ on.

Define for every $k \in \mathbb{N}$,

$$\varphi_k^0(S_0, V_0) := p^N \ell_k \left(\frac{V_0^*(S_0) - V_0}{(p^*)^N} \right) - (1-p)^N \ell_k \left(\frac{V_0^*(S_0) - V_0}{(1-p^*)^N} \right),$$

so that according to (5.2) the optimal strategy at time 0 for the problem associated to ℓ_k is to choose $\alpha_0^{*,k} = \alpha_0^{(1)}$ (respectively, $\alpha_0^{*,k} = \alpha_0^{(2)}$) if $\varphi_k^0 \leqslant 0$ (respectively, $\varphi_k^0 > 0$). Note also that, since $\alpha_0^{(1)} = \frac{V_1^*(S_0 d) - V_0}{S_0(d-1)}$, $\alpha_0^{(2)} = \frac{V_1^*(S_0 u) - V_0}{S_0(u-1)}$ (see (2.3) for the definition) and $V_n^*(S_n) = \mathrm{E}\{H(S_N) \mid S_n\}$ is independent of the loss function, $\alpha_0^{(1)}$ and $\alpha_0^{(2)}$ do not depend on the loss function either. We can now distinguish two cases:

- if either $\varphi_k^0 \leqslant 0$ or $\varphi_k^0 > 0$ from some $\overline{k}$ on, then the sequence $(\alpha_0^{*,k})_k$ is constant from $\overline{k}$ on, and thus it converges to a limit α_0^*,
- if φ_k^0 converges to 0 taking both positive and negative values infinitely many times, consider the subsequence $(\varphi_{k_{h_0}}^0)_{h_0}$ formed either by the positive or with the non-positive values taken by $(\varphi_k^0)_k$ and the problem reduces to the previous case.

This way, we are dealing with a subsequence $(\ell_{k_{h_0}})_{h_0}$ such that $(\alpha_0^{*,k_{h_0}})_{h_0} \equiv \alpha_0^*$ from some $\overline{h}_0$ on. Note that, since $V_1^\alpha = V_0 + \alpha_0 S_0(\omega_0 - 1)$ only depends on the chosen strategy and not on the loss function ℓ, the optimal portfolios of the problems associated to the $\ell_{k_{h_0}}$s for $h_0 \geqslant \overline{h}_0$ all follow the same evolution in the first time interval.

The existence of the limit strategy α^* now follows by induction on n in a similar way, i.e., defining at each step $n = 1, \dots, N-1$

$$\varphi_k^n(S_n, V_n) := p^{N-n} \ell_k \left(\frac{V_n^*(S_n) - V_n}{(p^*)^{N-n}} \right) - (1-p)^{N-n} \ell_k \left(\frac{V_n^*(S_n) - V_n}{(1-p^*)^{N-n}} \right)$$

for every $k \in \mathbb{N}$, and extracting from the sequence $(\ell_{k_{h_{n-1}}})_{h_{n-1}}$ a subsequence $(\ell_{k_{h_n}})_{h_n}$ such that $(\alpha_n^{*,k_{h_n}})_{h_n} \equiv \alpha_n^*$ from some $\overline{h}_n$ on.

Now remember that, by Remark 5.4 and by the results recalled in Section 3, in the "concave" case there is a one-to-one correspondence between optimal strategies belonging to the class Π defined in (3.2) and "critical" events defined as in (3.4). Note that this is equivalent to saying that there is a one to one correspondence between strategies $\alpha \in \Pi$ and modified contingent claims of the form

$$X(\alpha) = H(S_N) - \frac{\Delta}{P\{\omega(\alpha)\}} 1_{\{\omega(\alpha)\}}.$$

In particular, since $(\alpha_n^{*,k_{h_n}})_{h_n} \equiv \alpha_n^*$ from some $\overline{h}_n$ on, this implies that the optimal modified contingent claims $X_{k_h}^*$ corresponding to α^{*,k_h} must converge to the contingent claim X^* corresponding to α^*.

It only remains to show that α^* is an optimal strategy for the problem associated to ℓ. Note that, by Equation (5.1) and the independence between the strategies and the loss functions, the optimal values of the problems associated to the ℓ_ks converge to the shortfall associated to strategy α^* under the loss function ℓ. On the other hand, since for every sequence of functions $(f_n)_n$ convergin pointwise to f one has $\liminf f_n \leqslant \inf f$, this limit value necessarily must be the optimal value for the problem associated to ℓ. □

7.5. Remark. Note that the optimal limit solution X^* is not necessarily unique, i.e., there might be different subsequences of $(X_k^*)_k$ converging to different optimal solutions to the problem (6.3). Note also that one does *not* need *strict* concavity of the ℓ_ks.

References

[1] G. Favero, *Shortfall risk minimization under model uncertainty in the binomial case: adaptive and robust approaches*, Math. Meth. Oper. Research 53, 493-503 (2001).

[2] H. Föllmer and P Leukert, *Efficient hedging: cost versus shortfall risk*, Finance and Stoch. 4, 117-146 (2000).

[3] E. Jouini and C. Napp, *Convergence of utilityu functions and convergence of optimal strategies*, Working Paper, CEREMADE, Université Paris IX-Dauphine (2001)

[4] M. Kirch, *Efficient hedging in incomplete markets under model uncertainty*, Ph.D. Thesis, Humboldt University, Berlin (2001)

[5] M. Musiela and M. Rutkowski, *Martingale Methods in Financial Modelling*, Springer-Verlag, Berlin (1997).

[6] H. Pham, *Dynamic* L^p*-hedging in discrete time under cone constraints*, SIAM J. Control and Opt. 38, 665-682 (2000).

[7] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ (1970)

[8] W. J. Runggaldier, B. Trivellato and T. Vargiolu, *A Bayesian adaptive control approach to risk management in a binomial model*. In: R. C. Dalang et al. (eds.), *Proceedings of the Ascona '99 Seminar on Stochastic Analysis, Random Fields and Applications*, Progress in Probability vol. 52, Birkhauser, pp. 243–258 (2002).

[9] M. Schulmerich and S. Trautmann, *Local expected shortfall-hedging - Part I: the discrete model*, Working Paper, Johannes Gutenberg Universität, Mainz (2001)