Optimal default boundary in a discrete time setting

Agata Altieri and Tiziano Vargiolu

Abstract. In this paper we solve the problem of determining the default time of a firm in such a way as to maximize its total value, which includes bankruptcy costs and tax benefits, with the condition that the value of equity must be nonnegative. By applying dynamic programming in discrete time, we find results which extends those of Leland (1994) and Leland-Toft (1996).

1. Introduction

The aim of this work is to find the optimal time of default of a firm which has issued a coupon bond with a given maturity. The default time is optimal in the sense that it maximises the total value of the firm, given by its net value plus tax benefits of the coupons minus bankruptcy costs, with the constraint that equity, given by the difference between the total value of the firm minus the debt, must have a positive value at all times prior to default.

In the field of credit risk, valuation models can be divided into two categories which differ from one another in modelling default time (for a concise description of the two approaches, see [7]). In the first category (*intensity based* models), the default time τ is typically modeled as a first jump time of a Poisson process; this captures the idea that the time of a default takes the bondholders by surprise. In the second category (models based on the *value of the firm*), the default time is determined by an underlying process V describing the value of the firm; default occurs when this process hits a certain boundary (like in a barrier option), and typically represents the fact that the firm cannot repay a debt with a given maturity, so the default time is of the form $\tau = \inf\{t \mid V_t \leq f(t)\}$. In this last category, the prices of credit derivatives as corporate bonds and swaps depend on the shape of the default boundary $f(t)$. One can see different choices of f in the works [2], [5], [8], [9], [10], [11]. The choice of this default boundary in the different models seems as arbitrary as the choice of a term structure of risk-free interest rates is, and is usually exogenously suggested by personal taste. The first example of boundary built in an endogenous way is in the works [8] and [9], where a constant boundary K is determined such that the total value (including tax benefits on the debt) of the firm is maximised, with the constraint that equity must have a positive value in all the times prior to default.

Our model generalises Leland-Toft's approach by searching for a default boundary f (not of the specific kind $f(t) \equiv K$) such that the total value of the firm is maximised, with the same constraint as Leland-Toft. In order to obtain explicit results, we choose to work in discrete time. From an economic point of view, this could reflect the fact that default can be decided by the stockholders only at particular times (that could be the maturities of the coupons of the debt); from a mathematical point of view, this can be the first step for a continuous time formulation: in fact the solution of a continuous time problem of this kind can be approximated by the solutions of discrete time problems (see [6]).

We formulate this problem using the tools of stochastic control, so we write it as an optimal stopping problem with a constraint. While unconstrained optimal stopping can be treated by methods that are by now classical (see for example [14]), we found that there are no specific references for our problem, so we developed an ad-hoc method of solution that makes use of the dynamic programming principle. We find that the optimal default time is the first time at which equity falls to zero, where equity turns out to be a deterministic function of the value of the firm. In the case of a debt with no coupons, we find nice parallels between our model and Merton's: namely, under assumptions more general than the ones in [11], we find out that it is optimal for the firm to wait until the final maturity of the debt.

The paper is organised as follows. In Section 2 we describe our model and present the default-dependent quantities relevant for our work. In Section 3 we formulate and solve the optimal stopping problem. In Section 4 we apply our results to the binomial model.

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2. The model

We consider a market in which the primary assets are a riskless asset B, called the *money market account*, and a risky asset V which represents the total value of the firm. We represent the value of the firm as a stochastic process $V = (V_t)_{t\in[0,N]}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $N \in \mathbb{N}$ is a given maturity.

We assume that the market is arbitrage free and complete. This is equivalent to assume that there exists a unique martingale measure $\mathbb{Q}$ and every contingent claim can be priced via the expected value of its discounted final payoff under $\mathbb{Q}$. This situation occurs for example if V is a binomial process (i.e. the sample paths of V are pathwise constant and they can change assuming only two values) or if V is a diffusion driven by a 1-dimensional Brownian motion. We will be interested in particular in some activities related to the firm (corporate debt, equity, tax benefits, bankruptcy costs, etc.), that will be viewed as a derivative asset of the value of the firm.

We also assume that default can take place only at discrete times $n = 0, 1, \dots, N$ where $N > 0$ is a given maturity. Since we are interested only in finding which is the optimal default time among the dates $n = 0, 1, \dots, N$, we represent the dynamics of B and V as

$$B_{n+1} = B_n(1 + r) ,$$

where $r > 0$ is the deterministic risk-free interest rate, and as

$$V_{n+1} = V_n \omega_n , \quad n = 0, \dots, N ,$$

where $\{\omega_n\}_{n \in \{0,1,\dots,N\}}$ is a sequence of positive i.i.d. random variables on $(\Omega, \mathcal{F}, \mathcal{F}_n, \mathbb{P})$ and $(\mathcal{F}_n)_n$ is the filtration generated by V. Under these assumptions, V is a Markov chain with transition operator T defined by

$$Tf(v) = \mathbb{E}_{\mathbb{Q}}[f(V_{n+1}) \mid V_n = v]$$

(since the $(\omega_n)_n$ are i.i.d., the right hand side does not depend on n) for every $f : \mathbb{R}^+ \to \mathbb{R}$ measurable and bounded.

Let τ denote the time at which bankruptcy occurs. As said before, we are assuming that τ takes values in the set $\{0, \dots, N\}$. There is also the possibility that default does not take place before time N; when this happens, we will give τ the value $N + 1$. This is not a real date (because the terminal one is N) but it indicates that the firm has arrived at N without declaring bankruptcy.

Following [8] and [9], we consider three particular claims depending on the value of the firm and on default time. Since there exists a unique martingale measure $\mathbb{Q}$, the value of these contingent claims will be given by the discounted expectation under $\mathbb{Q}$ of the payments of the claim.

1. **Bankruptcy costs.** We assume that the costs connected with default are equal to αV_τ, where $\alpha \in (0, 1)$ is a fixed fraction. These costs are equivalent to a claim that does not pay coupons and is worth αV_τ in the event of bankruptcy. Then the value of bankruptcy costs at time n is given by:

$$BC(V_n, \tau) = \mathbb{E}_{\mathbb{Q}} \left[\left. \frac{\alpha V_\tau}{(1 + r)^{\tau - n}} 1_{\{n \le \tau \le N\}} \right| \mathcal{F}_n \right] .$$

2. **Debt.** We assume that debt issued by the firm pays coupon payments C_n at every date n prior to maturity or default and an amount P at the terminal date; in the event of default the bondholders receive the residual value of the firm minus bankruptcy costs. Thus the value of debt is:

$$\begin{aligned} D(V_n, \tau) \quad = \quad & \mathbb{E}_{\mathbb{Q}} \left[\sum_{i=n}^{\tau-1} \frac{C_n}{(1 + r)^{i-n}} + \frac{P}{(1 + r)^{\tau - n}} 1_{\{\tau = N+1\}} + \right. \\ & \left. + \frac{(1 - \alpha) V_\tau}{(1 + r)^{\tau - n}} 1_{\{n \le \tau \le N\}} \right| \mathcal{F}_n \Bigg] . \end{aligned}$$

3. **Tax benefits.** The tax benefits associated with debt financing are proportional to the coupon C_n via a constant $\gamma \in (0,1)$. The value of tax benefits is:
$$TB(V_n,\tau) = \mathbb{E}_{\mathbb{Q}}\left[\sum_{i=n}^{\tau-1} \frac{\gamma C_n}{(1+r)^{i-n}} \middle| \mathcal{F}_n\right] .$$

The total value of the firm is given by:
$$v(V_n,\tau) = V_n + TB(V_n,\tau) - BC(V_n,\tau) .$$

By the Modigliani-Miller theorem, the value of equity is the total value of the firm minus the value of debt:
$$E(V_n,\tau) = v(V_n,\tau) - D(V_n,\tau) .$$

3. The optimal stopping problem

We now suppose that the firm can choose the time τ at which it declares bankruptcy and we assume that the firm does it in such a way as to maximize its total value with the condition that equity must be nonnegative. Hence the problem we want to solve is
$$W_0(V_0) = \max_{\tau \in \mathcal{T}_0} v(V_0,\tau) , \tag{1}$$
where $\mathcal{T}_n$ is the set of the $\{n,\ldots,N+1\}$-valued stopping times such that $\{\tau > k\} \subseteq \{E(V_k,\tau) > 0\}$ for all $k = n,\ldots,N$. This problem is not a classical optimal stopping problem (see for instance [14]) because of the presence of a constraint for the stopping time. Therefore we present an ad-hoc method of solution that makes use of dynamic programming.

Theorem 3.1. *Given the problem (1) and defining successively the functions $h_N, h_{N-1},\ldots,h_0$, $\overline{W}_N, \overline{W}_{N-1},\ldots,\overline{W}_0$ as:*
$$h_N(V) = (V - P - (1-\gamma)C_N)^+ \ ,$$
$$\overline{W}_N(V) = \begin{cases} V(1-\alpha) & \text{if } h_N(V) = 0 , \\ V + \gamma C_N & \text{if } h_N(V) > 0 , \end{cases}$$
$$h_k(V) = \left(\frac{1}{1+r} T h_{k+1}(V) - (1-\gamma)C_k\right)^+ , \qquad k = N-1,\ldots,0 ,$$
$$\overline{W}_k(V) = \begin{cases} \gamma C_k + \dfrac{1}{1+r} T W_{k+1}(V) & \text{if } h_k(V) > 0 , \\ V(1-\alpha) & \text{if } h_k(V) = 0 , \end{cases}$$

the optimal stopping time is

$$\hat{\tau} = \begin{cases} \inf\{j \mid V_j \le v_j^*\}\ , \\ \\ N+1 \quad \textit{if the set above is empty}\ , \end{cases} \tag{2}$$

where v_j^* is the maximal solution of the equation

$$Th_{j+1}(v_j^*) = (1+r)(1-\gamma)C_j \tag{3}$$

and we have:

$$\begin{aligned} \overline{W}_0(V_0) &= W_0(V_0) = v(V_0, \hat{\tau})\ , \\ E(V_0, \hat{\tau}) &= h_0(V_0)\ . \end{aligned}$$

The proof proceeds in this way. For each time k, we assume that the optimal stopping time for the time period $\{k+1, \dots, N\}$ is given by Equation (2). The function h_k is related to the constraint $\tau \in \mathcal{T}_k$ in this way: if $h_k(V_k) = 0$, then we are forced to stop and default; conversely, if $h_k(V_k) > 0$, then we can choose either to stop or to continue. We prove that in the second situation it is always optimal to continue, so we obtain the function $\bar{W}_k$. The interested reader can find the complete proof in [1].

Corollary 3.2. *If $C_i = 0$, $i = 1, \dots, N$, and the random variables ω_n have support equal to $\mathbb{R}^+$, then the optimal stopping time is:*

$$\hat{\tau} = \begin{cases} N & \textit{if } V_N \le P\ , \\ \\ N+1 & \textit{if } V_N > P\ , \end{cases}$$

and we have:

$$\begin{aligned} W_n(V_n) &= V_n - \mathbb{E}_{\mathbb{Q}}\left[\frac{\alpha V_N}{(1+r)^{N-n}} 1_{\{V_N \le P\}} \middle| \mathcal{F}_n \right]\ , \\ D_n(V_n) &= \mathbb{E}_{\mathbb{Q}}\left[\frac{(1-\alpha) V_N}{(1+r)^{N-n}} 1_{\{V_N \le P\}} + \frac{P}{(1+r)^{N-n}} 1_{\{V_N > P\}} \middle| \mathcal{F}_n \right]\ , \\ BC_n(V_n) &= \mathbb{E}_{\mathbb{Q}}\left[\frac{\alpha V_N}{(1+r)^{N-n}} 1_{\{V_N \le P\}} \middle| \mathcal{F}_n \right]\ , \\ TB_n(V_n) &= 0\ , \\ E_n(V_n) &= V_n - \mathbb{E}_{\mathbb{Q}}\left[\frac{V_N}{(1+r)^{N-n}} 1_{\{V_N \le P\}} + \frac{P}{(1+r)^{N-n}} 1_{\{V_N > P\}} \middle| \mathcal{F}_n \right]\ . \end{aligned}$$

Proof. By Theorem 3.1 we have

$$h_N(V) = (V-P)^+\ ,$$

$$h_k(V) = \left(\frac{1}{1+r} Th_{k+1}(V) \right)^+\ , \quad k = N-1, \dots, 0\ ,$$

and the optimal stopping time is

$$\hat{\tau} = \begin{cases} \inf\{j \mid h_j(V_j) = 0\} , \\ \\ N+1 \text{ if the set above is empty} . \end{cases}$$

We prove by induction that the functions $Th_{k+1}(V)$ for $k = N-1, \ldots, 0$, are all strictly positive. In fact, if we assume that $h_{k+1} \geq 0$ and is different from zero we have:

$$Th_{k+1}(V) \quad = \quad \int_0^{+\infty} h_{k+1}(y)\; p(V, dy) > 0 \quad \forall V \in \mathbb{R} ,$$

where p is the transition function of the Markov chain $\{V_n\}_{n \in \{0,\ldots,N\}}$ under $\mathbb{Q}$. Since $h_N \geq 0$ and is different from the function zero, we deduce that

$$h_k(V) = \frac{1}{1+r} Th_{k+1}(V) > 0 \quad \forall k = N-1, \ldots, 0 ,$$

so $h_j(V_j) > 0$ for all $j = N-1, \ldots, 0$, and the only nontrivial condition to stop is $h_N(V_N) = 0$. Thus the optimal stopping time is

$$\hat{\tau} = \begin{cases} N & \text{if} \quad V_N \leq P , \\ N+1 & \text{if} \quad V_N > P . \end{cases}$$

□

This corollary draws a nice parallel between our model and the seminal one by Merton [11]. In fact it says that, if there are not intermediate coupons, then the optimal rule for the firm to default is to wait until the maturity and then to default if and only if its net value is worth less than the principal, as in Merton's model. Moreover, if we take $\omega_n = \exp((r - 1/2\sigma^2)T/N + \sigma W_N)$, where $W_n \sim N(0, T/N)$, then we obtain exactly the same quantities for D and E as Merton.

Remark 3.3. If the random variables ω_n have a support smaller than $\mathbb{R}^+$ (such as in the case when the possible values are a finite number as in the binomial model which we will discuss in the next section), then the results of Corollary 3.2 are still valid if we take the admissible default times in the set $\tilde{\mathcal{T}}_n$ of the $\{n, \ldots, N+1\}$-valued stopping times such that $\{\tau > k\} \subseteq \{E(V_k, \tau) \geq 0\}$ for all $k = n, \ldots, N$. This corresponds to allowing the equity to be exactly zero in some periods of time. This could be quite unpleasant to the intuition, but it could be justified by the fact that in our model the primary asset is not the equity but the value of the firm.

4. The binomial model

In this section we apply our results, collected in Theorem 3.1, to the case of the binomial model. The purpose is to approximate a lognormal model or a pure jump model, in order to obtain results similar to the ones in [8] and [9].

Let us suppose that the ω_n are random variables which take only the two values $1+u$ and $1+d$ with probability p and $1-p$ ($0<p<1$), respectively, under the equivalent martingale measure $\mathbb{Q}$. We want the discounted value of V to be a $\mathbb{Q}$-martingale, so

$$p=\frac{r-d}{u-d}\,, \qquad 1-p=\frac{u-r}{u-d}\,,$$

with $-1<d<r<u$. The transition operator T this time is such that

$$T\varphi(x)=p\varphi(x(1+u))+(1-p)\varphi(x(1+d))\,.$$

We now apply Theorem 3.1 to the case $C_n=0$. The functions h_n, $n=0,\dots,N$ turn out to be pathwise affine increasing functions, and it is easy to determine the optimal default boundary v_n^*, $n=0,\dots,N$.

Theorem 4.1. *If $C_n\equiv 0$, then the optimal default boundary is given by*

$$v_k^*=\frac{P}{(1+u)^{N-k}},\quad k=0,\dots,N\,,$$

and the optimal stopping time is given by Equation (2).

Proof. We have $h_N(v)=(v-P)^+$. Since $C_n=0$ and T sends nonnegative functions into nonnegative functions, then

$$h_{N-n}(v)=\frac{T^n h_N(v)}{(1+r)^n}\,.$$

Now we need to find the maximal solution v_j of the equation $Th_{j+1}(v)=0$, that becomes $T^{N-j}h_N(v)=0$. We claim that $v_j=P/(1+u)^{N-j}$ for all $j=0,\dots,N$. This is true for $j=N$, because $h_N=0$ on $(0,P]$ and $h_N>0$ on $(P,+\infty)$. We now proceed by induction, assuming that it is true for j. This means that $h_j=0$ on $(0,P/(1+u)^{N-j}]$ and $h_j>0$ on $(P/(1+u)^{N-j},+\infty)$. Then for $v\in(0,P/(1+u)^{N-j+1}]$:

$$h_{j-1}(v)=\frac{Th_j(v)}{1+r}=ph_j(v(1+u))+(1-p)h_j(v(1+d))=0\,,$$

since $v(1+u),v(1+d)\in(0,P/(1+u)^{N-j}]$. For $v\in(P/(1+u)^{N-j+1},+\infty)$,

$$h_{j-1}(v)=\frac{Th_j(v)}{1+r}=ph_j(v(1+u))+(1-p)h_j(v(1+d))>0\,,$$

since $v(1+u)\in(P/(1+u)^{N-j},+\infty)$. The result follows from Theorem 3.1. □

Now we apply Theorem 3.1 to the case $C_n\neq 0$. Also now the functions h_n, $n=0,\dots,N$, turn out to be pathwise affine increasing functions, but now it is more difficult to find explicitly the optimal default boundary v_n^*, $n=0,\dots,N$. However we find that if the condition

$$\sum_{i=0}^{N-k}\frac{(1-\gamma)C}{(1+r)^i}+\frac{P}{(1+r)^{N-k}}\leq\frac{(1-\gamma)C(r-d)}{(1+d)(1+r)},\quad k=0,\dots,N\,, \tag{4}$$

holds, then it is possible to find explicitly v_n^*, $n=0,\dots,N$.

Theorem 4.2. *If the condition in Equation (4) holds, then the optimal default boundary is given by*

$$v_k^* = \sum_{i=0}^{N-k} \frac{(1-\gamma)C}{(1+r)^i} + \frac{P}{(1+r)^{N-k}}, \quad k = 0, \dots, N \;, \tag{5}$$

and the optimal stopping time is given by Equation (2).

The proof proceeds in the same spirit as in Theorem 4.1, with some more algebra involved. Again, the interested reader can find the complete proof in [1].

The cases in the two previous theorems can be regarded as extreme cases: in fact, the optimal default boundary $(v_k^*)_k$ lies always between the two boundaries found in these cases.

Theorem 4.3. *The optimal default boundary $(v_k^*)_k$ satisfies the following bound:*

$$\frac{P}{(1+u)^{N-k}} \le v_k^* \le \sum_{i=0}^{N-k} \frac{(1-\gamma)C}{(1+r)^i} + \frac{P}{(1+r)^{N-k}}, \quad k = 0, \dots, N \;,$$

and the optimal stopping time is given by Equation (2).

Proof. We have the following bound on h_N:

$$v - P - (1-\gamma)C \le h_N(v) \le (x - P)^+ \;.$$

Since the operator T is linear and monotone, this implies:

$$v - \frac{P + (1-\gamma)C}{1+r} - (1-\gamma)C \le \frac{T h_N(v)}{1+r} - (1-\gamma)C \le \frac{T(v-P)^+}{1+r} \;.$$

By taking the positive parts of the two last quantities, this implies that

$$v - \frac{P + (1-\gamma)C}{1+r} - (1-\gamma)C \le h_{N-1}(v) \le \frac{T(v-P)^+}{1+r} \;.$$

By induction we have that

$$v - \sum_{i=0}^{n} \frac{(1-\gamma)C}{(1+r)^i} - \frac{P}{(1+r)^n} \le h_{N-n}(v) \le \frac{T^n(v-P)^+}{(1+r)^n} \;.$$

By taking positive parts we have that

$$\left(v - \sum_{i=0}^{n} \frac{(1-\gamma)C}{(1+r)^i} - \frac{P}{(1+r)^n} \right)^+ \le h_{N-n}(v) \le \frac{T^n(v-P)^+}{(1+r)^n} \;.$$

The first function is equal to zero for

$$v \le \sum_{i=0}^{n} \frac{(1-\gamma)C}{(1+r)^i} + \frac{P}{(1+r)^n} \;,$$

while by Theorem 4.1 the third is equal to zero for $v \le (1-\gamma)C/(1+u)^n$. This ends the proof. □

We notice that the default boundary v_k^* in general depends on time. In particular, it is bounded by two boundaries that evolve in an exponential way with respect to time.

Now we present two ideas of how we can use our results in order to find the optimal default boundary when the process V is a geometric Brownian motion or a geometric Poisson process.

4.1. Approximation of a geometric Brownian motion

We take a maturity T and we discretise the interval $[0, T]$ in N intervals of length T/N. If we let

$$C_n = \frac{C}{N} , \quad r_n = \frac{r}{N} , \quad u = \frac{r}{N} + \frac{\sigma}{\sqrt{N}} , \quad d = \frac{r}{N} - \frac{\sigma}{\sqrt{N}} ,$$

then the process $(V_{[t/N]})_t$ (where $[u]$ denotes the greatest integer smaller or equal to u) approximates a geometric Brownian motion

$$\tilde{V}_t = V_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right)$$

(in particular for $N \to \infty$ we have convergence in law). The condition (4) can be rewritten as

$$(1-\gamma)\frac{C}{\sqrt{N}} > \frac{P\sigma}{1 + o(N^{-1/2})} ,$$

so for $N \to \infty$ we have $P\sigma < 0$, that is never verified. This means that if we want to approximate a lognormal model with this discrete time model, for N sufficiently large the h_n become complex as n gets far from N.

4.2. Approximation of a geometric Poisson process

As before, we take a maturity T and we discretise the interval $[0, T]$ in N intervals of length T/N. If we let

$$C_n = \frac{C}{N} , \quad r_n = \frac{r}{N} , \quad u = b , \quad d = 0 ,$$

then the process $(V_{[t/N]})_t$ approximates a geometric Poisson process

$$\tilde{V}_t = V_0(1+b)^{N_t}$$

(in particular for $N \to \infty$ we have convergence in law) where N is a Poisson process having intensity r/b. This time the condition (4) can be rewritten as

$$(1-\gamma)C > Pr .$$

If this condition holds (we remind that in order not to have arbitrage we must have $C > Pr$), then the shape of the functions h_n remains stable, and we have an explicit solution. But if the condition above is not satisfied, for N sufficiently large the function h_n becomes complex as n gets far from N as in the approximation of a geometric Brownian motion.

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Department of Pure and Applied Mathematics
University of Padova
via Belzoni 7
I-35131 Padova, Italy
E-mail address: altieri@cicrp.jussieu.fr
vargiolu@math.unipd.it