Optimal Order Placement with Random Measures

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ESSFM, Padova, 2019

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Oxford Mathematics

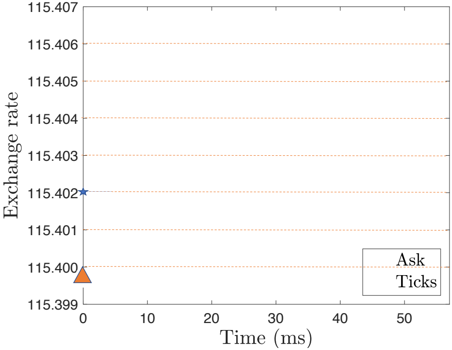
Latency

- **Latency** is the time delay between an exchange streaming market data to a trader, the trader processing information and deciding to trade, and the exchange receiving the order from the trader.

- Some facts:
  - No market participant has zero latency!
  - Latency is random.

- Main consequences for **liquidity**
  - **takers**: marketable orders are not always executed/filled at the observed price,
  - **makers**: stale limit orders are picked off.

Quote stream



Exchange streams quotes to client

Figure: Misses and latency.

Process market information

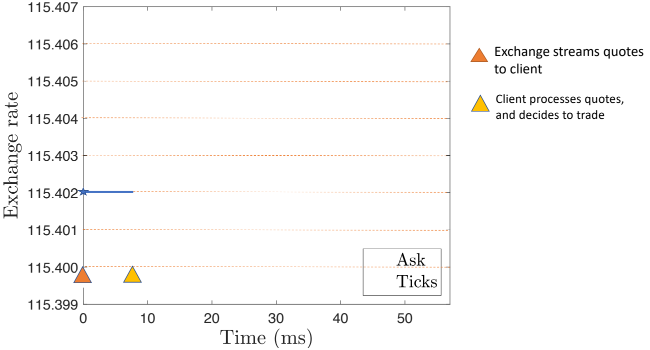


Figure: Misses and latency.

Send FoK (or IoC) order to buy at **115.402**

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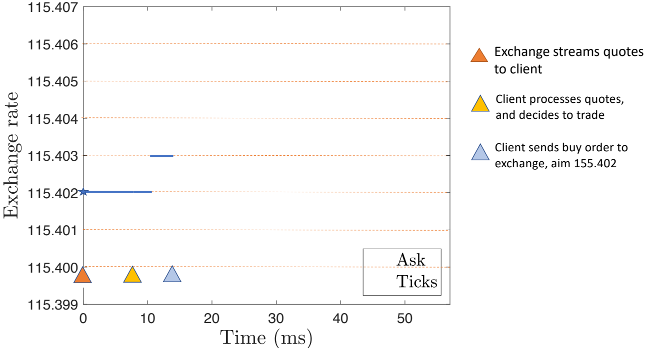

Figure: Misses and latency.

Current best ask is too high

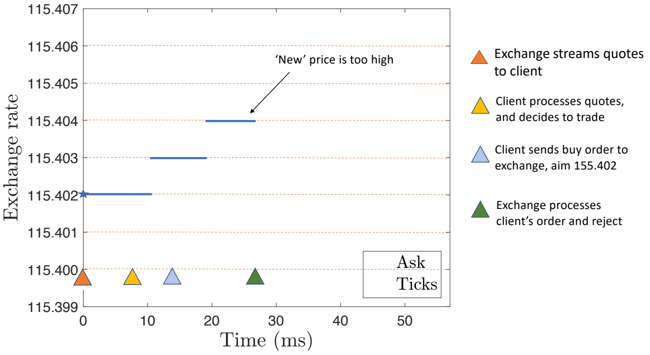

Figure: Misses and latency.

Exchange rejects order and notifies client

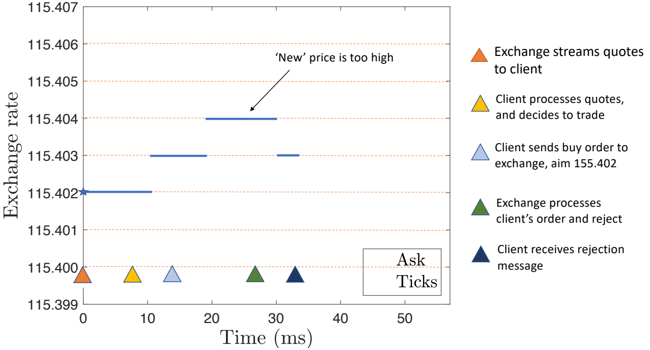

Figure: Misses and latency.

Optimal discretion: ‘Observed best ask’ plus δ

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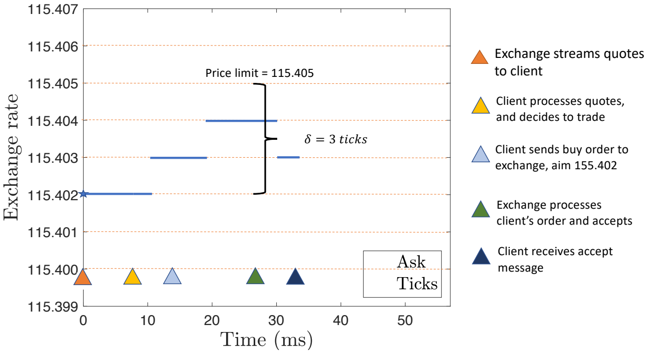

Figure: Misses, latency, and discretion.

Problem

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Is there an optimal trade-off between costs and misses?

Outline:

- The model (fairly general)
- FBSDE arising from the vanishing Gâteaux derivative
- Global minimum
- Future work

The model

Let $\mathcal{N} = \{(T_n, Z_n)\}_{(n\geq 1)}$ be a marked point process.

- (T_n) are the times at which the client trades
- (Z_n) are the changes in price due to latency

We associate the random measure p to the marked point process $\mathcal{N}$

$$p(\omega\,,\,[0,t],\,A) = \sum_n 1_{\{T_n(\omega)\leq t\}} 1_{\{Z_n(\omega)\in A\}}\,.$$

We assume the compensator is

$$\tilde{p}(\mathrm{d}z,\mathrm{d}t) = \phi_t(\mathrm{d}z)\,\lambda_t \mathrm{d}t\,.$$

For a given change in prices z, and discretion δ

- The cost is

$$z\,F(\delta - z) = \begin{cases} z, & \text{if } z \le \delta \qquad \text{trade filled} \\ 0, & \text{if } z > \delta \qquad \text{trade missed} \end{cases}$$

  where, $F(x) := \mathbb{1}_{x \ge 0}$.

- A miss trade is

$$D(z) = G(\delta - z) = \begin{cases} 0, & \text{if } z \le \delta \qquad \text{trade filled} \\ 1, & \text{if } z > \delta \qquad \text{trade missed} \end{cases}$$

  which is $1 - F(\delta - z)$.

The model

- The running **cost** of strategy δ is

$$C_t^\delta = \int_0^t \int_{\mathbb{R}} z\, F(\delta_s - z)\, p(\mathrm{d}z, \mathrm{d}s)\,.$$

- The running number of **missed trades** is

$$D_t^\delta = \int_0^t \int_{\mathbb{R}} (1 - F\,(\delta_s - z))\; p(\mathrm{d}z, \mathrm{d}s)\,.$$

The model

We study the functional $J : \mathcal{A} \mapsto \mathbb{R}$

$$J(\delta) = \mathbb{E}\left[C_T^\delta + \boldsymbol{\alpha}\, D_T^\delta + \boldsymbol{\gamma} \left(D_T^\delta \right)^2 \right],$$

where $\boldsymbol{\alpha}, \boldsymbol{\gamma} \geq 0$ and set of admissible strategies

$$\mathcal{A} = \left\{ \delta = (\delta_t)_{\{0 \leq t \leq T\}} \mid \delta \text{ is } \mathcal{F} - \text{predictable and } \mathbb{E}\left[\sup_{0 \leq t \leq T} (\delta_t)^2 \right] < \infty \right\}.$$

- **J is not a quasi-convex functional**

The model

The value function is not quasiconvex

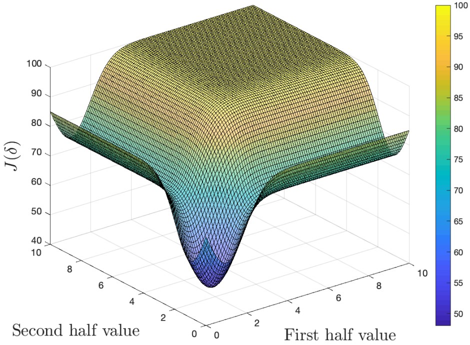

Figure: Value function for “half-constant” controls. Parameters: $T = 1$, $\alpha = 0.5$, $\gamma = 0.01$, $\lambda = 100$, $\mu = 1$, and $\sigma = 1$.

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Next steps:

- Gâteaux derivative
- Existence and uniqueness
- Global minimum

Let $w, \delta \in \mathcal{A}$. The Gâteaux derivative of $J(\delta)$ in the direction of w, is defined as

$$\langle \mathcal{D}\, J(\delta), w \rangle = \lim_{\epsilon \to 0} \frac{J(\delta + \epsilon\, w) - J(\delta)}{\epsilon} \,.$$

Theorem

The functional J is everywhere Gâteaux differentiable and

$$\langle \mathcal{D}\, J(\delta), w \rangle = \mathbb{E}\left[\int_0^T w_t\, \phi_t(\delta_t) \left[\delta_t - 2\,\gamma\, \mathbb{E}_{t^-} \left[D_T^{\delta} \right] - (\gamma + \alpha) \right] \mathrm{d}A_t \right] .$$

Theorem

The Gâteaux derivative vanishes in all directions if and only if there is $\delta^* \in \mathcal{A}$ that satisfies the FBSDE

$$\delta_t^* = 2\,\gamma\,\mathbb{E}_{t^-}\left[D_T^{\delta^*}\right] + \gamma + \alpha\,,$$

almost everywhere in $[0, T] \times \Omega$.

Next steps:

- Existence and Uniqueness of
  - BSDE
  - SDE
  - FBSDE

The FBSDE

$$\delta_t = 2\gamma\,\mathbb{E}_{t^-}\left[D_T^{\delta}\right] + \gamma + \alpha\,, \qquad (1)$$

$$D_t^{\delta} = \int_0^t \int_{\mathbb{R}} G(\delta_s - z)\,p(\mathrm{d}z, \mathrm{d}s)\,, \qquad D_0^{\delta} = 0\,,$$

with $\delta \in \mathcal{A}$ and $D \in \mathcal{C}$, is equivalent to the FBSDE

$$\tilde{\delta}_t = 2\gamma\,\mathbb{E}_t\left[\int_t^T \int_{\mathbb{R}} G(\tilde{\delta}_{s^-} + 2\gamma\,D_{s^-}^{\tilde{\delta}} - z)\,p(\mathrm{d}z, \mathrm{d}s)\right] + \gamma + \alpha\,, \qquad (2)$$

$$D_t^{\tilde{\delta}} = \int_0^t \int_{\mathbb{R}} G(\tilde{\delta}_{s^-} + 2\gamma\,D_{s^-}^{\tilde{\delta}} - z)\,p(\mathrm{d}z, \mathrm{d}s)\,, \qquad D_0^{\tilde{\delta}} = 0\,,$$

with $\tilde{\delta}$, $D^{\tilde{\delta}} \in \mathcal{C}$

Theorem

Fix $V \in \mathcal{A}$. Define Φ_t as

$$\Phi_t(x) = \int_{-\infty}^{x} \phi_t(y)\mathrm{d}y\,,$$

and take Φ_t to be uniformly Lipschitz (in $[0,T] \times \Omega$) with constant k, and let $\bar{\lambda}$ be an upper bound of the stochastic intensity λ. The functional $\Psi : \mathcal{A} \to \mathcal{A}$ has a unique fixed point:

$$\Psi(\boldsymbol{U})_t = 2\gamma\,\mathbb{E}_t\left[\int_t^T \int_{\mathbb{R}} G(\boldsymbol{U}_{s^-} + 2\gamma\, V_{s^-} - z)\, p(\mathrm{d}z, \mathrm{d}s)\right] + \gamma + \boldsymbol{\alpha}\,, \quad V \in \mathcal{A}.$$

Theorem

Fix $\boldsymbol{U} \in \mathcal{A}$. *Let the distribution function* Φ_t *be uniformly Lipschitz with constant* k *and let* $\bar{\lambda}$ *be an upper bound of the stochastic intensity. The functional* $\Theta : \mathcal{C} \to \mathcal{C}$ *has a unique fixed point:*

$$\Theta(\boldsymbol{V})_t = \int_0^t \int_{\mathbb{R}} G(\boldsymbol{U}_{s^-} + 2\gamma \boldsymbol{V}_{s^-} - z)\, p(\mathrm{d}z, \mathrm{d}s), \qquad \boldsymbol{U} \in \mathcal{C}.$$

Theorem

Let the distribution function Φ_t be uniformly Lipschitz with constant k, such that $k\,T\,\bar{\lambda}\,(\max\{1,2\gamma\})^2 < 1$. There exits unique solution to the system of FBSDEs, i.e. the functional $\Upsilon : \mathcal{C}\otimes\mathcal{C} \to \mathcal{C}\otimes\mathcal{C}$ defined by

$$\Upsilon(U,V)_t = \begin{pmatrix} H(U,V)_t \\ I(U,V)_t \end{pmatrix} = \begin{pmatrix} 2\gamma\,\mathbb{E}_t\left[\int_t^T \int_{\mathbb{R}} G(U_{s^-} + 2\gamma\,V_{s^-} - z)\,p(\mathrm{d}z,\mathrm{d}s)\right] + \gamma \\ \int_0^t \int_{\mathbb{R}} G(U_{s^-} + 2\gamma\,V_{s^-} - z)\,p(\mathrm{d}z,\mathrm{d}s) \end{pmatrix},$$

with

$$\|\Upsilon(U,V)\|_{\mathcal{C}\otimes\mathcal{C}} = \|H(U,V)\|_{\mathcal{C}} + \|I(U,V)\|_{\mathcal{C}}\ ,$$

has a unique fixed point.

Theorem

J is continuous.

Lemma

If J has a global minimum $\hat{\delta} \in \mathcal{A}$, then

$$\langle \mathcal{D} J(\hat{\delta}), w \rangle \geq 0 \qquad \forall w \in \mathcal{A}\,.$$

Theorem

If J has a global minimum $\hat{\delta} \in \mathcal{A}$, then

$$\hat{\delta} = \delta^* \qquad a.e.\ \ in \qquad [0, T] \times \Omega\,.$$

Theorem

Let $\tilde{\delta}_t = h(t, D^{\tilde{\delta}}_t, \lambda_t)$. The function h (hence $\tilde{\delta}$ and δ^*) satisfies the PIDE

$$
\begin{aligned}
0 &= \partial_t h(t,D,\lambda) + \mathcal{L}^{\lambda}_t h(t,D,\lambda) \\
&\quad + \left(\int_{h(t,D,\lambda)}^{\infty} \lambda\, \phi_t(z)\, \mathrm{d}z \right) \left(h(t,D+1,\lambda) - h(t,D,\lambda) \right) ,
\end{aligned}
\tag{3}
$$

with boundary and terminal conditions

$$
\lim_{D\to\infty} h(t,D,\lambda) = \infty \qquad \text{and} \qquad h(T,D,\lambda) = 2\,\gamma\, D + \gamma + \alpha\,.
$$

Here, $\mathcal{L}^{\lambda}_t h(t,D,\lambda)$ is the infinitesimal generator of the arrival intensity process λ acting on the function h.

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Numerical experiments:

- Take $\lambda = 100$ for all $t \in [0, T]$, so $T_n - T_{n-1} \sim \exp(100)$. And $Z_n \sim \mathcal{N}(0.2, 1)$.
- Model parameters are $\alpha = 0$, $\gamma = 0.07$ (when not specified).

Results

Optimal control

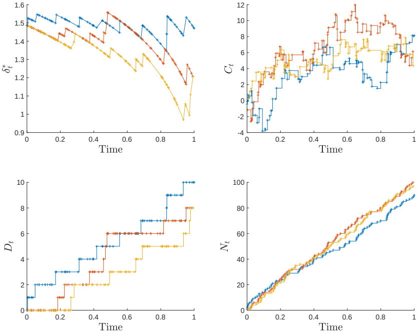

Figure: Sample paths for the optimal discretion δ^* (top left panel), number of missed trades D (lower left panel), cost of strategy C (top right panel), and number of trade attempts N (lower right panel) for three simulations of the MPP.

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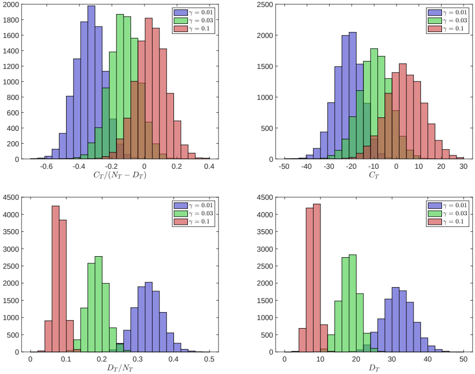

Figure: Top left panel: Histogram of the extra cost per filled trade ($C_T/(N_T - D_T)$) for various values of γ. Top right panel: Histogram of the cost (C_T) of the strategy for various values of γ. Bottom left panel: Histogram of the number of misses (D_T) for various values of γ. Bottom right panel: Histogram of percentage of misses (D_T/N_T) for various values of γ.

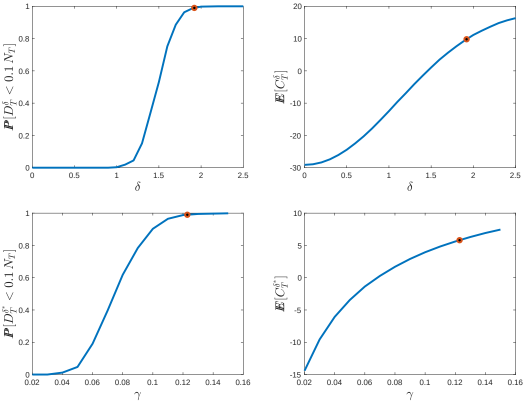

Figure: The left-hand side corresponds to fixed discretion strategies and the right-hand corresponds to the strategy δ^* when changing γ. The top panels plot $\mathbb{P}\left(D_T < 0.1\,N_T\right)$ and the bottom panels the $\mathbb{E}\left[C_T\right]$. The marker is placed where the smallest value of $\mathbb{E}\left[C_T\right]$ is when $\mathbb{P}\left(D_T < 0.1\,N_T\right) \geq 0.99$. The model parameters are: $\lambda = 100$, $\alpha = 0$, and $Z_n \sim \mathcal{N}(0.2, 1)$ for every n.

Conclusions

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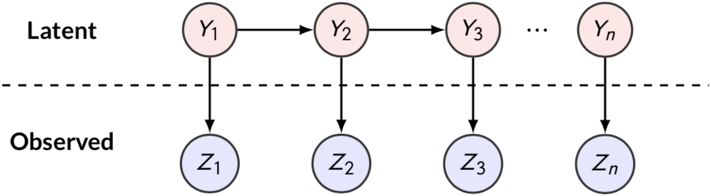
Summary:

- We found a global minimum for the tradeoff between costs and misses
- This framework can be applied to other problems (e.g. “last look”)
- We proved existence and uniqueness of a new FBSDE

- Compute the “shadow price of latency” in a similar way to Cartea, Á, and Sánchez-Betancourt, L. (2018)
- Include a latent processes modulating the shocks.

Future work

Preliminary results



The framework developed in the previous model can still be applied for HMM modulating the changes in price. Empirical experiments using FX data over April 2019 for EUR/USD show three distinguishable regimes.

Table: Probability distribution of Z_n under each regime (3 regimes in total).

Y/Z	-5	-4	-3	-2	-1	0	1	2	3	4	5
1	3.5%	3.0%	5.6%	10.3%	15.4%	26.3%	14.5%	9.5%	5.5%	3.0%	3.5%
2	0.0%	0.1%	0.6%	3.3%	12.5%	66.4%	12.7%	3.3%	0.8%	0.2%	0.0%
3	0.0%	0.0%	0.1%	0.3%	1.4%	96.2%	1.5%	0.3%	0.1%	0.0%	0.0%

Future work

Similarly, the transition matrix for the latent process Y is given by

Table: Transition probability matrix for Y.

Y/Y	1	2	3
1	77.4%	22.6%	0.0%
2	5.6%	86.8%	7.6%
3	3.1%	3.5%	93.4%

Future work

Preliminary results

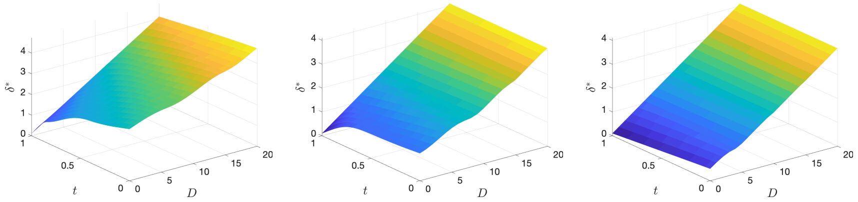


Figure: Optimal strategy under each regime. Left panel is regime one, middle panel is for regime two, and right panel is for regime three.

Thank you for your time!

Details can be found in SSRN:

- Cartea, Jaimungal, S-B. (2019) Latency and Liquidity Risk
- Cartea, S-B. (2018) The Shadow Price of Latency: Improving Intraday Fill Ratios in Foreign Exchange Markets