Asymptotics of American Options and Implied Volatilities in Local Volatility Models

F. Bourgey[1] S. De Marco[1]

[1]Centre de Mathématiques Appliquées (CMAP)
Ecole Polytechnique

12th European Summer School in Financial Mathematics
2-6 September 2019

Motivation

- Important volumes of American options traded in exchange market.
- Typically only American (and no European) options are traded on individual stocks (as opposed to stock indexes).
- Typically short maturities $(T \leq 2Y)$.
- Calibration of a model to American options (industry practice) :
  - Compute implied vol from American options.
  - Input this smile as it was European.
  - Calibrate a model with standard methods (e.g local vol from Dupire's formula).
- How relevant is the approximation of American implied vol to European implied vol ?

- In the presentation :
  - Short-time expansions for the American put prices.
  - Estimates of the difference between American and European implied vol.
  - Upper / lower bounds for the American implied vol.

Motivation

- Important volumes of American options traded in exchange market.
- Typically only American (and no European) options are traded on individual stocks (as opposed to stock indexes).
- Typically short maturities $(T \leq 2Y)$.
- Calibration of a model to American options (industry practice) :
  - Compute implied vol from American options.
  - Input this smile as it was European.
  - Calibrate a model with standard methods (e.g local vol from Dupire's formula).
- How relevant is the approximation of American implied vol to European implied vol ?

- In the presentation :
  - Short-time expansions for the American put prices.
  - Estimates of the difference between American and European implied vol.
  - Upper / lower bounds for the American implied vol.

Motivation

- Important volumes of American options traded in exchange market.
- Typically only American (and no European) options are traded on individual stocks (as opposed to stock indexes).
- Typically short maturities $(T \leq 2Y)$.
- Calibration of a model to American options (industry practice) :
  - Compute implied vol from American options.
  - Input this smile as it was European.
  - Calibrate a model with standard methods (e.g local vol from Dupire's formula).
- How relevant is the approximation of American implied vol to European implied vol ?

- In the presentation :
  - Short-time expansions for the American put prices.
  - Estimates of the difference between American and European implied vol.
  - Upper / lower bounds for the American implied vol.

Motivation

- Important volumes of American options traded in exchange market.
- Typically only American (and no European) options are traded on individual stocks (as opposed to stock indexes).
- Typically short maturities $(T \leq 2Y)$.
- Calibration of a model to American options (industry practice) :
  - Compute implied vol from American options.
  - Input this smile as it was European.
  - Calibrate a model with standard methods (e.g local vol from Dupire's formula).
- How relevant is the approximation of American implied vol to European implied vol ?

- In the presentation :
  - Short-time expansions for the American put prices.
  - Estimates of the difference between American and European implied vol.
  - Upper / lower bounds for the American implied vol.

Motivation

- Important volumes of American options traded in exchange market.
- Typically only American (and no European) options are traded on individual stocks (as opposed to stock indexes).
- Typically short maturities ($T \leq 2Y$).
- Calibration of a model to American options (industry practice) :
  - Compute implied vol from American options.
  - Input this smile as it was European.
  - Calibrate a model with standard methods (e.g local vol from Dupire's formula).
- How relevant is the approximation of American implied vol to European implied vol ?

- In the presentation :
  - Short-time expansions for the American put prices.
  - Estimates of the difference between American and European implied vol.
  - Upper / lower bounds for the American implied vol.

Motivation

- Important volumes of American options traded in exchange market.
- Typically only American (and no European) options are traded on individual stocks (as opposed to stock indexes).
- Typically short maturities $(T \leq 2Y)$.
- Calibration of a model to American options (industry practice) :
  - Compute implied vol from American options.
  - Input this smile as it was European.
  - Calibrate a model with standard methods (e.g local vol from Dupire's formula).
- How relevant is the approximation of American implied vol to European implied vol ?

- In the presentation :
  - Short-time expansions for the American put prices.
  - Estimates of the difference between American and European implied vol.
  - Upper / lower bounds for the American implied vol.

Motivation

- Important volumes of American options traded in exchange market.
- Typically only American (and no European) options are traded on individual stocks (as opposed to stock indexes).
- Typically short maturities $(T \leq 2Y)$.
- Calibration of a model to American options (industry practice) :
  - Compute implied vol from American options.
  - Input this smile as it was European.
  - Calibrate a model with standard methods (e.g local vol from Dupire's formula).
- How relevant is the approximation of American implied vol to European implied vol ?

- In the presentation :
  - Short-time expansions for the American put prices.
  - Estimates of the difference between American and European implied vol.
  - Upper / lower bounds for the American implied vol.

Motivation

- Important volumes of American options traded in exchange market.
- Typically only American (and no European) options are traded on individual stocks (as opposed to stock indexes).
- Typically short maturities $(T \leq 2Y)$.
- Calibration of a model to American options (industry practice) :
  - Compute implied vol from American options.
  - Input this smile as it was European.
  - Calibrate a model with standard methods (e.g local vol from Dupire's formula).
- How relevant is the approximation of American implied vol to European implied vol ?

- In the presentation :
  - Short-time expansions for the American put prices.
  - Estimates of the difference between American and European implied vol.
  - Upper / lower bounds for the American implied vol.

Motivation

- Important volumes of American options traded in exchange market.
- Typically only American (and no European) options are traded on individual stocks (as opposed to stock indexes).
- Typically short maturities $(T \leq 2Y)$.
- Calibration of a model to American options (industry practice) :
  - Compute implied vol from American options.
  - Input this smile as it was European.
  - Calibrate a model with standard methods (e.g local vol from Dupire's formula).
- How relevant is the approximation of American implied vol to European implied vol ?

- In the presentation :
  - Short-time expansions for the American put prices.
  - Estimates of the difference between American and European implied vol.
  - Upper / lower bounds for the American implied vol.

Motivation

- Important volumes of American options traded in exchange market.
- Typically only American (and no European) options are traded on individual stocks (as opposed to stock indexes).
- Typically short maturities $(T \leq 2Y)$.
- Calibration of a model to American options (industry practice) :
  - Compute implied vol from American options.
  - Input this smile as it was European.
  - Calibrate a model with standard methods (e.g local vol from Dupire's formula).
- How relevant is the approximation of American implied vol to European implied vol ?

- In the presentation :
  - Short-time expansions for the American put prices.
  - Estimates of the difference between American and European implied vol.
  - Upper / lower bounds for the American implied vol.

Pricing of an American put

- Time-homogeneous LV model for stock price :

$$\frac{\mathrm{d}S_t}{S_t} = r\mathrm{d}t + \sigma\left(S_t\right)\mathrm{d}W_t.$$

- American put : holder gets the payoff $(K - S_u)_+$ if exercised at $u \leq T$.
- Optimal stopping problem (Bensoussan 84', Karatzas 88') :

$$A\left(t,s,K\right) = \sup_{u\in[t,T]} \mathbb{E}\left[e^{-r(u-t)}\left(K-S_u\right)_+ \middle| S_t = s\right], \quad u \text{ stopping time.}$$

- Parabolic obstacle PDE :

$$\begin{cases} \partial_t A + rs\partial_s A + \frac{1}{2}\sigma\left(s\right)^2 s^2 \partial_{ss} A - rA = 0, & t < T, \quad s > s^T\left(t\right) \\ A\left(t,s,K\right) > \left(K-s\right)_+, & t < T, \quad s > s^T\left(t\right) \\ A\left(t,s,K\right) = \left(K-s\right)_+, & t < T, \quad s \leq s^T\left(t\right) \\ A\left(T,s,K\right) = \left(K-s\right)_+, & t = T, \quad s > 0. \end{cases}$$

- $s^T(t)$ is the exercise boundary.

Pricing of an American put

- Time-homogeneous LV model for stock price :

$$\frac{\mathrm{d}S_t}{S_t} = r\mathrm{d}t + \sigma\left(S_t\right)\mathrm{d}W_t.$$

- American put : holder gets the payoff $(K - S_u)_+$ if exercised at $u \leq T$.
- Optimal stopping problem (Bensoussan 84', Karatzas 88') :

$$A\left(t,s,K\right) = \sup_{u\in[t,T]} \mathbb{E}\left[e^{-r(u-t)}\left(K-S_u\right)_+ \middle| S_t = s\right], \quad u \text{ stopping time.}$$

- Parabolic obstacle PDE :

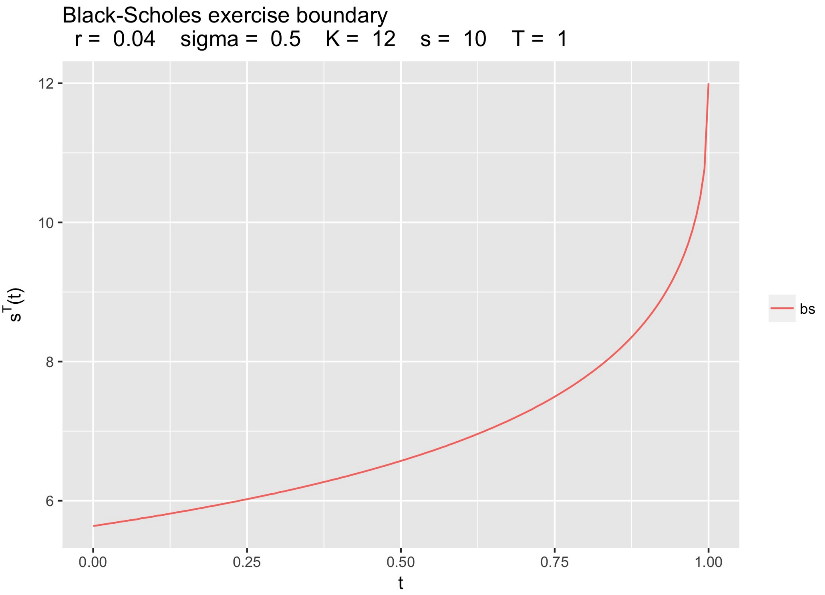
$$\begin{cases} \partial_t A + rs\partial_s A + \frac{1}{2}\sigma\left(s\right)^2 s^2 \partial_{ss} A - rA = 0, & t < T, \quad s > s^T\left(t\right) \\ A\left(t,s,K\right) > \left(K-s\right)_+, & t < T, \quad s > s^T\left(t\right) \\ A\left(t,s,K\right) = \left(K-s\right)_+, & t < T, \quad s \leq s^T\left(t\right) \\ A\left(T,s,K\right) = \left(K-s\right)_+, & t = T, \quad s > 0. \end{cases}$$

- $s^T(t)$ is the exercise boundary.

Pricing of an American put

- Time-homogeneous LV model for stock price :
$$\frac{\mathrm{d}S_t}{S_t} = r\mathrm{d}t + \sigma\left(S_t\right)\mathrm{d}W_t.$$
- American put : holder gets the payoff $(K - S_u)_+$ if exercised at $u \leq T$.
- Optimal stopping problem (Bensoussan 84', Karatzas 88') :
$$A\left(t, s, K\right) = \sup_{u \in [t,T]} \mathbb{E}\left[e^{-r(u-t)}\left(K - S_u\right)_+ \middle| S_t = s\right], \quad u \text{ stopping time.}$$
- Parabolic obstacle PDE :
$$\begin{cases} \partial_t A + rs\partial_s A + \frac{1}{2}\sigma\left(s\right)^2 s^2 \partial_{ss} A - rA = 0, & t < T, \quad s > s^T\left(t\right) \\ A\left(t, s, K\right) > \left(K - s\right)_+, & t < T, \quad s > s^T\left(t\right) \\ A\left(t, s, K\right) = \left(K - s\right)_+, & t < T, \quad s \leq s^T\left(t\right) \\ A\left(T, s, K\right) = \left(K - s\right)_+, & t = T, \quad s > 0. \end{cases}$$
- $s^T(t)$ is the exercise boundary.

Exercise boundary



Early premium formula

- $\tau := T - t$ time to maturity.
- Homogeneous Markov process : $s^T(t) = \bar{s}(T-t) = \bar{s}(\tau)\,.$
- Decomposition of the American put (*early premium formula*) :

$$\begin{aligned} A(\tau, s, K) &= \mathbb{E}_{0,s}\left[e^{-r\tau}(K - S_\tau)_+\right] + rK\int_0^\tau e^{-ru}\mathbb{E}_{0,s}\left[1_{S_u \le \bar{s}(\tau-u)}\right]\mathrm{d}u \\ &= P(\tau, s, K) + p(\tau, s, K)\,. \end{aligned}$$

see (McKean 65', Moerbeke 76', Myneni 92').

- $P(\tau, s, K)$: European price, $p(\tau, s, K)$: premium.
- Continuation condition : $A(\tau, s(\tau), K) = (K - s(\tau))$.
  - Integral equation for $\bar{s}(\tau)$.

$$K - \bar{s}(\tau) = \mathbb{E}_{0,\bar{s}(\tau)}\left[e^{-r\tau}(K - S_\tau)_+\right] + rK\int_0^\tau e^{-ru}\mathbb{E}_{0,\bar{s}(\tau)}\left[1_{S_u \le \bar{s}(\tau-u)}\right]\mathrm{d}u.$$

Early premium formula

- $\tau := T - t$ time to maturity.
- Homogeneous Markov process : $s^T(t) = \bar{s}(T-t) = \bar{s}(\tau)$.
- Decomposition of the American put (*early premium formula*) :

$$A(\tau, s, K) = \mathbb{E}_{0,s}\left[e^{-r\tau}(K - S_\tau)_+\right] + rK\int_0^\tau e^{-ru}\mathbb{E}_{0,s}\left[1_{S_u \le \bar{s}(\tau-u)}\right]\mathrm{d}u$$
$$= P(\tau, s, K) + p(\tau, s, K).$$

see (McKean 65', Moerbeke 76', Myneni 92').

- $P(\tau, s, K)$: European price, $p(\tau, s, K)$: premium.
- Continuation condition : $A(\tau, s(\tau), K) = (K - s(\tau))$.
  - Integral equation for $\bar{s}(\tau)$.

$$K - \bar{s}(\tau) = \mathbb{E}_{0,\bar{s}(\tau)}\left[e^{-r\tau}(K - S_\tau)_+\right] + rK\int_0^\tau e^{-ru}\mathbb{E}_{0,\bar{s}(\tau)}\left[1_{S_u \le \bar{s}(\tau-u)}\right]\mathrm{d}u.$$

Early premium formula

- $\tau := T - t$ time to maturity.
- Homogeneous Markov process : $s^T(t) = \bar{s}(T-t) = \bar{s}(\tau)$.
- Decomposition of the American put (*early premium formula*) :

$$\begin{aligned} A(\tau, s, K) &= \mathbb{E}_{0,s}\left[e^{-r\tau}(K - S_\tau)_+\right] + rK\int_0^\tau e^{-ru}\mathbb{E}_{0,s}\left[1_{S_u \le \bar{s}(\tau-u)}\right]\mathrm{d}u \\ &= P(\tau, s, K) + p(\tau, s, K)\,. \end{aligned}$$

see (McKean 65', Moerbeke 76', Myneni 92').

- $P(\tau, s, K)$: European price, $p(\tau, s, K)$: premium.
- Continuation condition : $A(\tau, s(\tau), K) = (K - s(\tau))$.
  - Integral equation for $\bar{s}(\tau)$.

$$K - \bar{s}(\tau) = \mathbb{E}_{0,\bar{s}(\tau)}\left[e^{-r\tau}(K - S_\tau)_+\right] + rK\int_0^\tau e^{-ru}\mathbb{E}_{0,\bar{s}(\tau)}\left[1_{S_u \le \bar{s}(\tau-u)}\right]\mathrm{d}u.$$

Early premium formula

- $\tau := T - t$ time to maturity.
- Homogeneous Markov process : $s^T(t) = \bar{s}(T-t) = \bar{s}(\tau)$.
- Decomposition of the American put (*early premium formula*) :

$$\begin{aligned} A(\tau, s, K) &= \mathbb{E}_{0,s}\left[e^{-r\tau}(K - S_\tau)_+\right] + rK\int_0^\tau e^{-ru}\mathbb{E}_{0,s}\left[1_{S_u \le \bar{s}(\tau-u)}\right]\mathrm{d}u \\ &= P(\tau, s, K) + p(\tau, s, K). \end{aligned}$$

see (McKean 65', Moerbeke 76', Myneni 92').

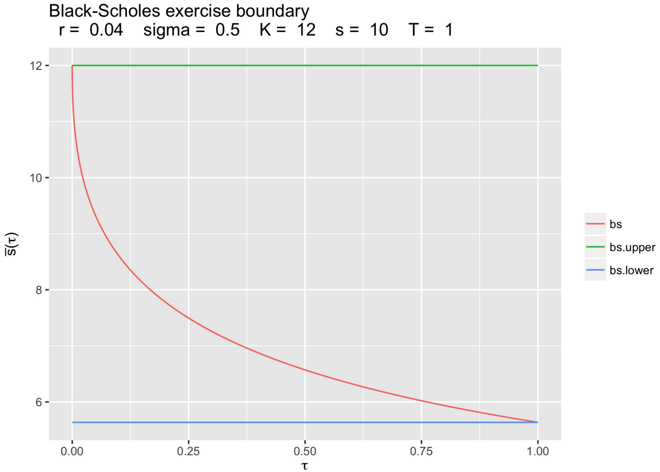
- $P(\tau, s, K)$: European price, $p(\tau, s, K)$: premium.
- Continuation condition : $A(\tau, s(\tau), K) = (K - s(\tau))$.
  - Integral equation for $\bar{s}(\tau)$.

$$K - \bar{s}(\tau) = \mathbb{E}_{0,\bar{s}(\tau)}\left[e^{-r\tau}(K - S_\tau)_+\right] + rK\int_0^\tau e^{-ru}\mathbb{E}_{0,\bar{s}(\tau)}\left[1_{S_u \le \bar{s}(\tau-u)}\right]\mathrm{d}u.$$

Exercise boundary

- $\bar{s}(0) = K$, $s(\tau)$ decreasing on $[0, T]$.
- Lower and upper bounds for the exercise boundary :

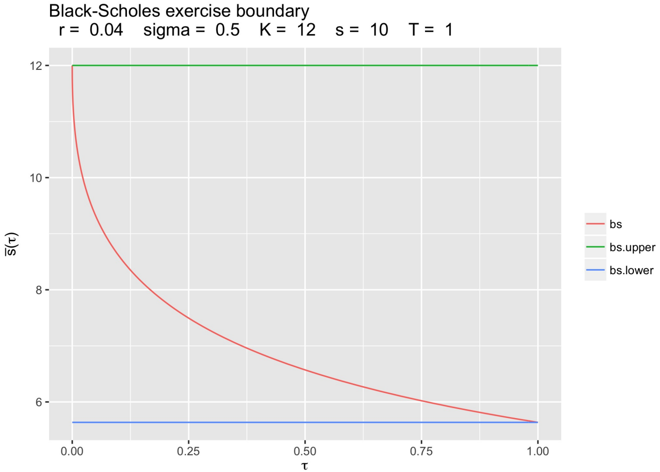
$$\forall u \in [0, \tau], \quad \bar{s}(\tau) \leq \bar{s}(\tau - u) \leq K.$$



Exercise boundary

- $\bar{s}(0) = K$, $s(\tau)$ decreasing on $[0, T]$.
- Lower and upper bounds for the exercise boundary :

$$\forall u \in [0, \tau], \quad \bar{s}(\tau) \leq \bar{s}(\tau - u) \leq K.$$



Exercise boundary close to the maturity

- Define $\bar{x}(\tau) = \ln \bar{s}(\tau)$. Geodesic distance : $\mathrm{d}(\bar{x}(\tau), \ln K) = \int_{\bar{x}(\tau)}^{\ln K} \frac{\mathrm{d}p}{\sigma(e^p)}$.

Theorem

We have as $\tau \to 0$,

$$\mathrm{d}^2\left(\bar{x}(\tau), \ln K\right) = \tau \ln\left(\frac{\gamma\left(K\right)}{\tau}\right)\left(1 + \frac{2}{\ln^2\left(\frac{\gamma(K)}{\tau}\right)} + \mathcal{O}\left(\frac{1}{\ln^3\left(\frac{1}{\tau}\right)}\right)\right),$$

Consequently :

$$\bar{x}(\tau) = \ln K - \sigma\left(K\right)\sqrt{\tau \ln\left(\frac{\gamma\left(K\right)}{\tau}\right)}\left(1 + \frac{1}{\ln^2\left(\frac{\gamma(K)}{\tau}\right)} + \mathcal{O}\left(\frac{1}{\ln^3\left(\frac{1}{\tau}\right)}\right)\right).$$

see (Lamberton 95', Chevalier 05', De Marco, Henry-Labordere 17' and many others).

- $\gamma(K) = \frac{\sigma^2(K)}{8\pi r^2}$ $\to$ homogeneous to time. It plays the role of a time-scale. In BS model $\gamma_{\mathrm{BS}} = \frac{\sigma^2}{8\pi r^2} \approx 4$ for typical values.

Short-time asymptotics for the density

- Heat-kernel expansion for the transition density of the logarithmic stock price $X_u = \ln(S_u)$ starting at $X_0 = x$:

$$f_X\left(u, y \middle| x\right) = \frac{e^{-\frac{d^2(x,y)}{2u}}}{\sqrt{2\pi u}}\left(u_0\left(x,y\right) + \mathcal{O}\left(u;x,y\right)\right), \quad (u \to 0)$$

where :

$$u_0\left(x,y\right) := \sigma\left(e^x\right)^{\frac{1}{2}}\sigma\left(e^y\right)^{-\frac{3}{2}}e^{-\frac{1}{2}(y-x)+r\int_x^y\frac{\mathrm{d}p}{\sigma^2(e^p)}},$$

$$\mathrm{d}(x,y) = \int_x^y \frac{\mathrm{d}p}{\sigma\left(e^p\right)},$$

and the following uniform estimate holds :

$$\exists C > 0, \forall\left(x,y\right) \in \mathbb{R}^2, \forall u \in (0,\tau], \quad \left|\mathcal{O}\left(u;x,y\right)\right| \leq Cu.$$

European put price expansion

- Put price is a Laplace-type integral :

$$P(\tau, s, K) = \frac{K e^{-r\tau}}{\sqrt{2\pi\tau}} \int_{-\infty}^{0} (1 - e^{z}) \left(u_0 (\ln s, \ln K + z) + \mathcal{O}(\tau) \right) e^{-\frac{d^2(\ln s, z + \ln K)}{2\tau}} \mathrm{d}z.$$

$$P(\tau, s = K) = \frac{s\sigma(s)}{\sqrt{2\pi}} \sqrt{\tau} - \frac{rs}{2}\tau + A_2(s) \frac{s}{\sqrt{2\pi}} \tau\sqrt{\tau}(1 + o(1)) \quad (K = s),$$

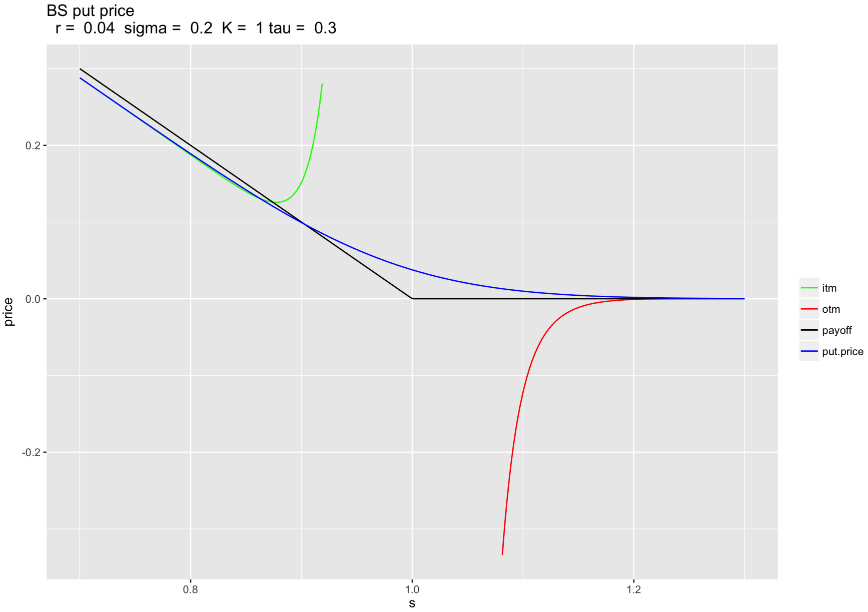
$$P(\tau, s, K) = \frac{u_0(\ln s, \ln K) K \sigma(K)^2}{\sqrt{2\pi} d^2(\ln K, \ln s)} e^{-\frac{d^2(\ln K, \ln s)}{2\tau}} \tau\sqrt{\tau}(1 + o(1)) \quad (K < s),$$

$$P(\tau, s, K) = K\left(1 - e^{-r\tau}\right) - \frac{u_0(\ln s, \ln K) K \sigma(K)^2}{\sqrt{2\pi} d^2(\ln K, \ln s)} e^{-\frac{d^2(\ln K, \ln s)}{2\tau}} \tau\sqrt{\tau}(1 + o(1)) \quad (K > s).$$

where $A_2(s)$ depends only on $\sigma(s), \sigma'(s), \sigma''(s)$.

(Ref. see e.g Henry-Labordere 08', Gatheral et al. 12').

Black-Scholes put price



European implied volatility expansion

- Define the European implied vol $\sigma_E(\tau, s, K)$ as solution to :

$$P(\tau, s, K) = P_{BS}(\tau, s, K; \sigma_E(\tau, s, K)).$$

- Using the European put price expansion gives a polynomial expansion:

$$\sigma_E\left(\tau, s, K\right) = \sigma_0\left(s, K\right) + \sigma_1\left(s, K\right)\tau + \mathcal{O}\left(\tau^2\right),$$

$$\sigma_0\left(s, K\right) = \frac{\ln\left(\frac{s}{K}\right)}{d\left(\ln K, \ln s\right)},$$

$$\sigma_1\left(s, K\right) = \frac{\sigma_0^3\left(s, K\right)}{\ln\left(\frac{s}{K}\right)^2} \ln\left(\frac{u_0\left(\ln s, \ln K\right)\sigma^2\left(K\right)}{\sigma_0\left(s, K\right)}\right) - \frac{\sigma_0^3\left(s, K\right)}{2\ln\left(\frac{s}{K}\right)} + \frac{r\sigma_0\left(s, K\right)}{\ln\left(\frac{s}{K}\right)}.$$

- Continuity of the coefficients as $K \to s$.

A time-homogeneous LV model : CEV

- CEV model : $\sigma(s) = \delta s^{\frac{\beta}{2}-1}$ with $\delta > 0$, $\beta \in \mathbb{R}$ (Schroder 89').

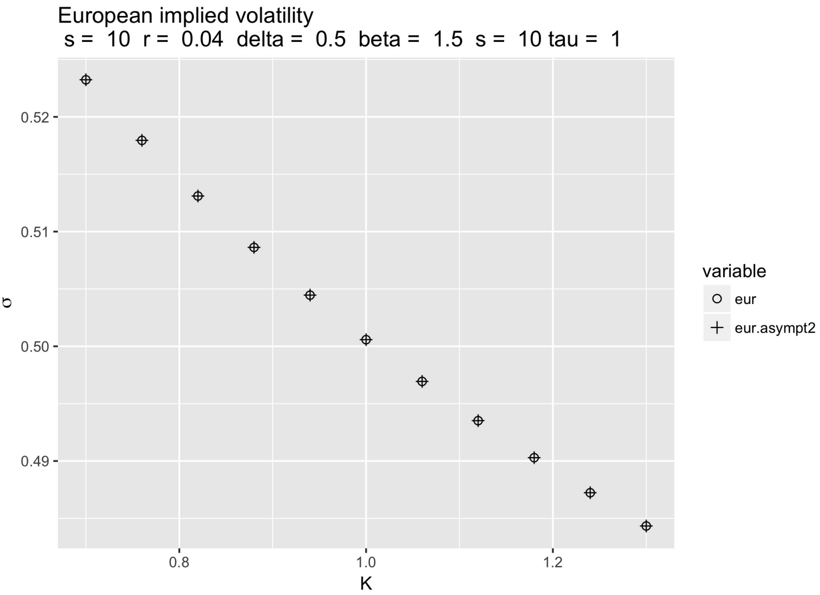
$$\mathbb{P}_{0,s}\left[S_\tau \leq K\right] = \begin{cases} \mathcal{Q}\left(2a; \frac{2}{2-\beta}, 2b\right) & \text{if } \beta < 2 \\ \mathcal{Q}\left(2b; 2 + \frac{2}{\beta-2}, 2a\right) & \text{if } \beta > 2 \end{cases},$$

  - $\mathcal{Q}(z; \nu, \varepsilon)$: complementary noncentral χ^2 c.d.f
  - ν : degrees of freedom, ε : noncentral parameter.

$$k = \frac{2r}{\delta^2 (2-\beta)\left[e^{r(2-\beta)\tau} - 1\right]},\ a = k s^{2-\beta} e^{r(2-\beta)\tau},\ b = k K^{2-\beta}.$$

- European put : closed formula.
- American put : integral equation for the CEV exercise boundary.

European implied vol asymptotics



Upper and lower bounds

- $\forall u \in [0, \tau], \quad \bar{s}(\tau) \leq \bar{s}(\tau - u) \leq K.$
- Recall the formulation for the premium :

$$p(\tau, s, K) = rK \int_0^\tau e^{-ru} \mathbb{E}_{0,s}\left[1_{S_u \leq \bar{s}(\tau-u)}\right] \mathrm{d}u$$
$$= rK \int_0^\tau e^{-ru} \int_0^{\bar{s}(\tau-u)} f_S(u, s, y) \mathrm{d}y \mathrm{d}u.$$

- Define associated upper/lower bounds for the premium :

$$\underline{p}(\tau, s, K) = rK \int_0^\tau e^{-ru} \mathbb{E}_{0,s}\left[1_{S_u \leq \bar{s}(\tau)}\right] \mathrm{d}u,$$
$$\overline{p}(\tau, s, K) = rK \int_0^\tau e^{-ru} \mathbb{E}_{0,s}\left[1_{S_u \leq K}\right] \mathrm{d}u.$$

- Corresponding bounds for the American price :

$$\underline{A}(\tau, s, K) = P(\tau, s, K) + \underline{p}(\tau, s, K),$$
$$\overline{A}(\tau, s, K) = P(\tau, s, K) + \overline{p}(\tau, s, K)$$

Upper and lower bounds

- $\forall u \in [0,\tau], \quad \bar{s}(\tau) \leq \bar{s}(\tau-u) \leq K.$
- Recall the formulation for the premium :

$$p(\tau,s,K) = rK\int_0^\tau e^{-ru}\mathbb{E}_{0,s}\left[1_{S_u \leq \bar{s}(\tau-u)}\right]\mathrm{d}u$$
$$= rK\int_0^\tau e^{-ru}\int_0^{\bar{s}(\tau-u)} f_S(u,s,y)\mathrm{d}y\mathrm{d}u.$$

- Define associated upper/lower bounds for the premium :

$$\underline{p}(\tau,s,K) = rK\int_0^\tau e^{-ru}\mathbb{E}_{0,s}\left[1_{S_u \leq \bar{s}(\tau)}\right]\mathrm{d}u,$$
$$\overline{p}(\tau,s,K) = rK\int_0^\tau e^{-ru}\mathbb{E}_{0,s}\left[1_{S_u \leq K}\right]\mathrm{d}u.$$

- Corresponding bounds for the American price :

$$\underline{A}(\tau,s,K) = P(\tau,s,K) + \underline{p}(\tau,s,K),$$
$$\overline{A}(\tau,s,K) = P(\tau,s,K) + \overline{p}(\tau,s,K)$$

Upper and lower bounds

- $\forall u \in [0, \tau], \quad \bar{s}(\tau) \leq \bar{s}(\tau - u) \leq K.$
- Recall the formulation for the premium :

$$p(\tau, s, K) = rK \int_0^\tau e^{-ru} \mathbb{E}_{0,s} \left[1_{S_u \leq \bar{s}(\tau - u)} \right] \mathrm{d}u$$
$$= rK \int_0^\tau e^{-ru} \int_0^{\bar{s}(\tau - u)} f_S(u, s, y) \mathrm{d}y \mathrm{d}u.$$

- Define associated upper/lower bounds for the premium :

$$\underline{p}(\tau, s, K) = rK \int_0^\tau e^{-ru} \mathbb{E}_{0,s} \left[1_{S_u \leq \bar{s}(\tau)} \right] \mathrm{d}u,$$
$$\overline{p}(\tau, s, K) = rK \int_0^\tau e^{-ru} \mathbb{E}_{0,s} \left[1_{S_u \leq K} \right] \mathrm{d}u.$$

- Corresponding bounds for the American price :

$$\underline{A}(\tau, s, K) = P(\tau, s, K) + \underline{p}(\tau, s, K),$$
$$\overline{A}(\tau, s, K) = P(\tau, s, K) + \overline{p}(\tau, s, K)$$

ATM premium short-time expansions

Theorem

Let $l(\tau) = \ln\left(\frac{\sigma(s)^2}{8\pi r^2 \tau}\right)$ *and* $\Sigma(\tau, s) = \frac{C_0}{l(\tau)} + C_1 \frac{\ln^2(l(\tau))}{l^2(\tau)} + C_2 \frac{\ln(l(\tau))}{l^2(\tau)} + \frac{C_3}{l^2(\tau)}$ *with* C_0, C_1, C_2, C_3 *universal constants. We have as* $\tau \to 0$ *:*

$$p(\tau, s) = \frac{rs}{2} \tau \Sigma(\tau, s)\left(1 + o(1)\right).$$

Corollary

We have as $\tau \to 0$ *:*

$$\underline{p}(\tau, s) = \frac{4r^2 s}{\sigma(s)} \frac{\tau\sqrt{\tau}}{l^{\frac{3}{2}}(\tau)}(1 + o(1)),$$

$$\overline{p}(\tau, s) = \frac{rs}{2}\tau + \frac{s}{\sqrt{2\pi}}\left(-\frac{2r^2}{3\sigma(s)} + \frac{r\sigma(s)}{3} + \frac{rs\sigma'(s)}{3}\right)\tau\sqrt{\tau}(1 + o(1)).$$

ATM premium short-time expansions

Theorem

Let $l(\tau) = \ln\left(\frac{\sigma(s)^2}{8\pi r^2 \tau}\right)$ *and* $\Sigma(\tau, s) = \frac{C_0}{l(\tau)} + C_1 \frac{\ln^2(l(\tau))}{l^2(\tau)} + C_2 \frac{\ln(l(\tau))}{l^2(\tau)} + \frac{C_3}{l^2(\tau)}$ *with* C_0, C_1, C_2, C_3 *universal constants. We have as* $\tau \to 0$ *:*

$$p(\tau, s) = \frac{rs}{2} \tau \Sigma(\tau, s) (1 + o(1)).$$

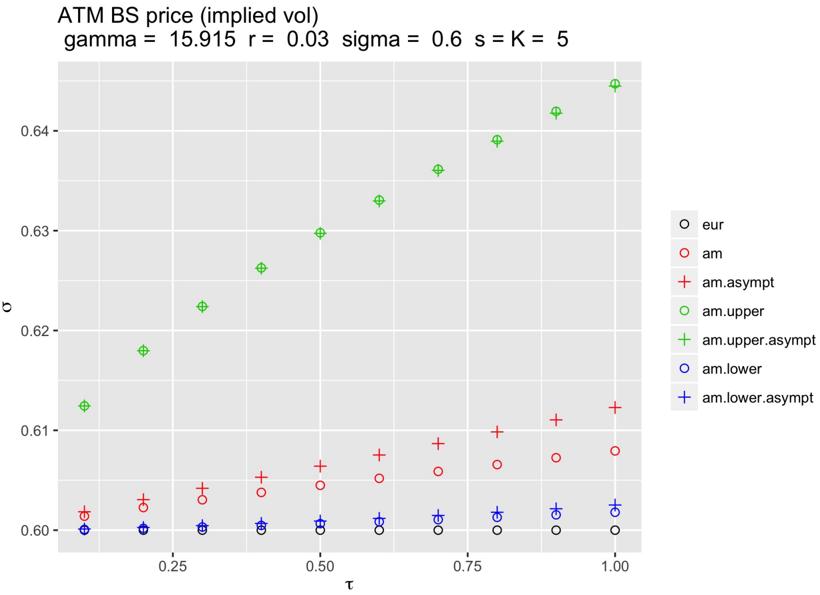
Corollary

We have as $\tau \to 0$ *:*

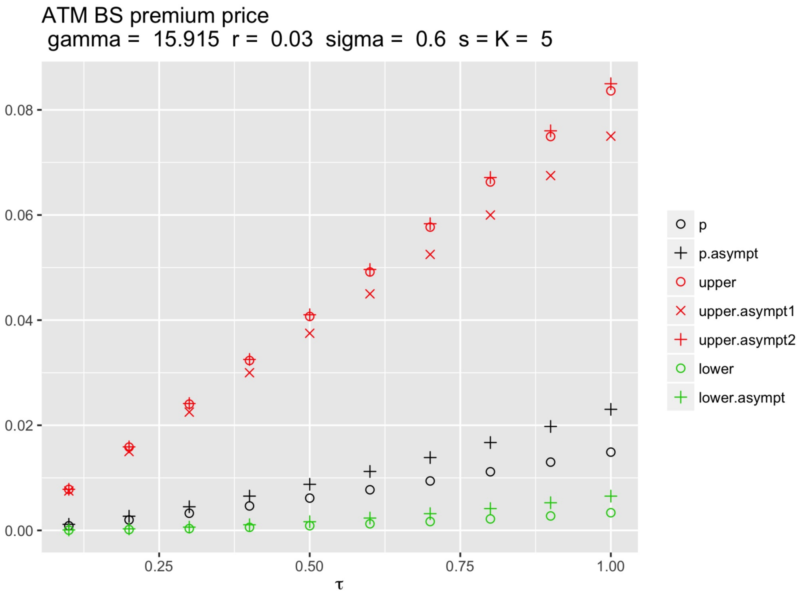
$$\underline{p}(\tau, s) = \frac{4r^2 s}{\sigma(s)} \frac{\tau\sqrt{\tau}}{l^{\frac{3}{2}}(\tau)} (1 + o(1)),$$

$$\overline{p}(\tau, s) = \frac{rs}{2}\tau + \frac{s}{\sqrt{2\pi}} \left(-\frac{2r^2}{3\sigma(s)} + \frac{r\sigma(s)}{3} + \frac{rs\sigma'(s)}{3} \right) \tau\sqrt{\tau}(1 + o(1)).$$

ATM American price approximation



ATM premium price approximation



ITM / OTM premium short-time expansions

Theorem

Suppose $K < s$, then we have as $\tau \to 0$:

$$p(\tau, s, K) = \mathcal{P}(s, K)\frac{\tau^{\frac{5}{2}}}{2l(\tau, \sigma(s))}e^{-\frac{d^2(\ln K, \ln s)}{2\tau}}(1 + o(1)),$$

$$\overline{p}(\tau, s, K) = \mathcal{P}(s, K)\tau^{\frac{5}{2}}e^{-\frac{d^2(\ln K, \ln s)}{2\tau}}(1 + o(1)),$$

$$\underline{p}(\tau, s, K) = \mathcal{P}(s, K)\tau^{\frac{5}{2}}e^{-\frac{d^2(\ln K, x)}{2\tau}}e^{-d(\ln K, x)\sqrt{\frac{\ln\left(\frac{\gamma}{\tau}\right)}{\tau}}\left(1+\mathcal{O}\left(\frac{1}{\ln^2\left(\frac{\gamma}{\gamma}\right)}\right)\right)}(1 + o(1))$$

where $\mathcal{P}(s, K) = \frac{2rKu_0(\ln s, \ln K)\sigma(K)}{\sqrt{2\pi}d^3(\ln K, \ln s)}$.

Similarly for $K > s$:

$$\underline{p}(\tau, s, K) \sim Ke^{-r\tau} - s - \underline{p}(\tau, s, K),$$

$$\overline{p}(\tau, s, K) \sim Ke^{-r\tau} - s - \overline{p}(\tau, s, K).$$

ITM / OTM premium short-time expansions

Theorem

Suppose $K < s$, then we have as $\tau \to 0$:

$$p(\tau, s, K) = \mathcal{P}(s, K)\frac{\tau^{\frac{5}{2}}}{2l(\tau, \sigma(s))}e^{-\frac{d^2(\ln K, \ln s)}{2\tau}}\left(1 + o(1)\right),$$

$$\overline{p}(\tau, s, K) = \mathcal{P}(s, K)\tau^{\frac{5}{2}}e^{-\frac{d^2(\ln K, \ln s)}{2\tau}}\left(1 + o(1)\right),$$

$$\underline{p}(\tau, s, K) = \mathcal{P}(s, K)\tau^{\frac{5}{2}}e^{-\frac{d^2(\ln K, x)}{2\tau}}e^{-d(\ln K, x)\sqrt{\frac{\ln\left(\frac{\gamma}{\tau}\right)}{\tau}}\left(1+\mathcal{O}\left(\frac{1}{\ln^2\left(\frac{\gamma}{\gamma}\right)}\right)\right)}\left(1 + o(1)\right)$$

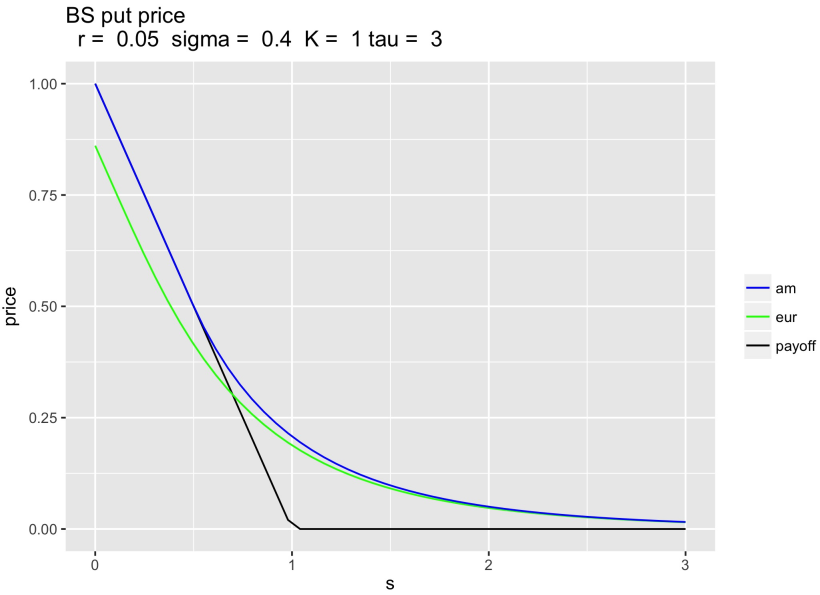
where $\mathcal{P}(s, K) = \frac{2rKu_0(\ln s, \ln K)\sigma(K)}{\sqrt{2\pi}d^3(\ln K, \ln s)}$.

Similarly for $K > s$:

$$\underline{p}(\tau, s, K) \sim Ke^{-r\tau} - s - \underline{p}(\tau, s, K),$$

$$\overline{p}(\tau, s, K) \sim Ke^{-r\tau} - s - \overline{p}(\tau, s, K).$$

American / European put price



American implied vol

Same analysis for the American implied vol ?

- Define the American implied vol $\sigma_A(\tau, s, K)$ as solution to :

$$A(\tau, s, K) = A_{BS}(\tau, s, K; \sigma_A(\tau, s, K)).$$

- Define upper/lower bounds for the American implied volatilities :

$$\underline{A}(\tau, s, K) = A_{BS}(\tau, s, K; \underline{\sigma}_A(\tau, s, K)),$$
$$\overline{A}(\tau, s, K) = A_{BS}(\tau, s, K; \overline{\sigma}_A(\tau, s, K)).$$

American implied vol

Same analysis for the American implied vol ?

- Define the American implied vol $\sigma_A(\tau, s, K)$ as solution to :

$$A(\tau, s, K) = A_{BS}(\tau, s, K; \sigma_A(\tau, s, K)).$$

- Define upper/lower bounds for the American implied volatilities :

$$\underline{A}(\tau, s, K) = A_{BS}(\tau, s, K; \underline{\sigma}_A(\tau, s, K)),$$
$$\overline{A}(\tau, s, K) = A_{BS}(\tau, s, K; \overline{\sigma}_A(\tau, s, K)).$$

ATM American implied vol

Theorem

We have as $\tau \to 0$ *:*

$$\sigma_A(\tau, s) = \sigma_E(\tau, s) + \mathcal{O}\left(\frac{\tau}{l\left(\tau\right)}\right) = \sigma_0(s) + \sigma_1(s)\tau + \mathcal{O}\left(\frac{\tau}{l\left(\tau\right)}\right),$$

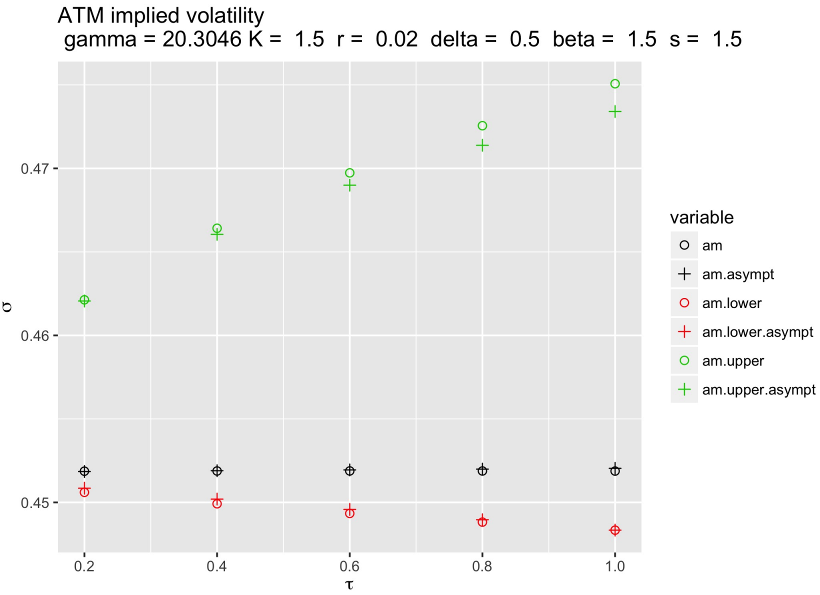
$$\overline{\sigma}_A(\tau, s) = \sigma\left(s\right) + \sqrt{\frac{\pi}{2}} r \sqrt{\tau}\left(1 + \Sigma(s, \tau)(1 + o(1))\right),$$

$$\underline{\sigma}_A(\tau, s) = \sigma\left(s\right) - \sqrt{\frac{\pi}{2}} r \sqrt{\tau} \Sigma(s, \tau)(1 + o(1)),$$

where $\Sigma(\tau, s) = \frac{C_0}{l(\tau)} + C_1 \frac{\ln^2(l(\tau))}{l^2(\tau)} + C_2 \frac{\ln(l(\tau))}{l^2(\tau)} + \frac{C_3}{l^2(\tau)}.$

- Up to the order $\frac{\tau}{l(\tau)}$, American and European implied vol match.

ATM American implied volatility

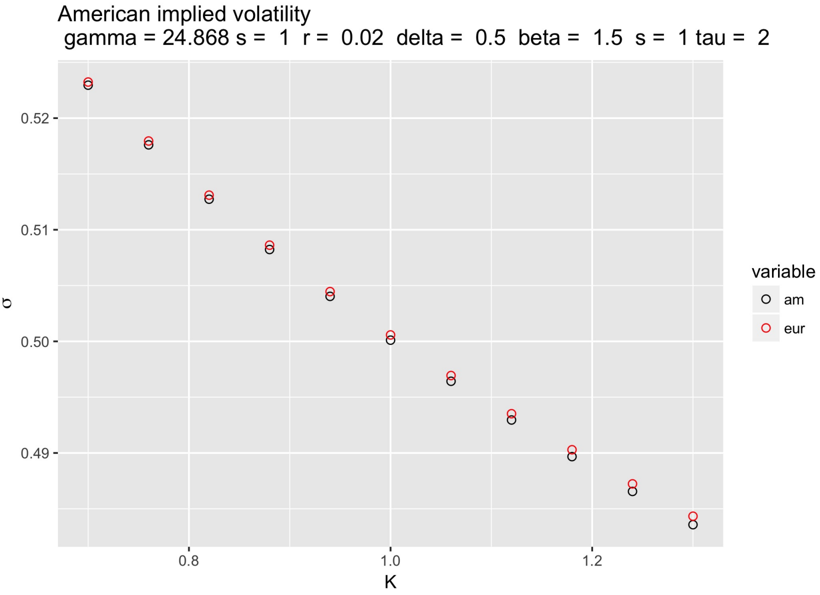


ITM / OTM American implied vol

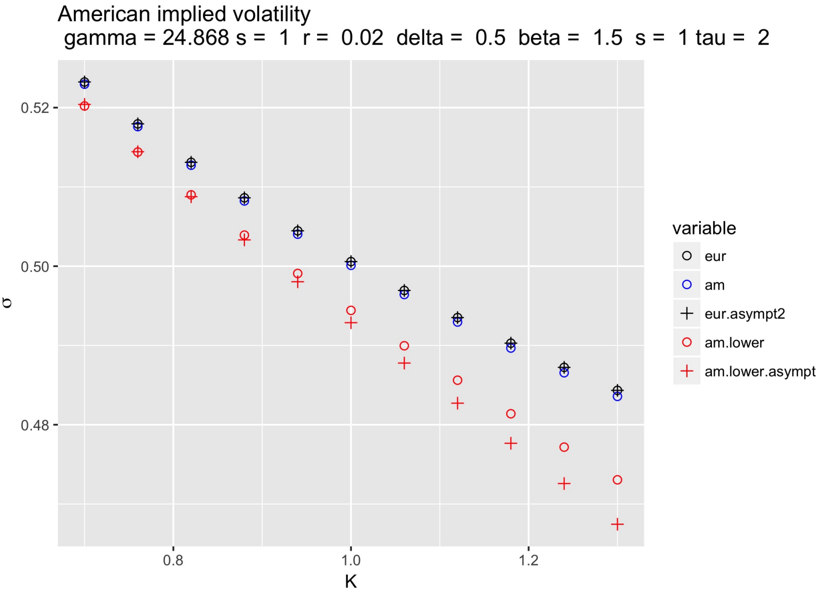
Theorem

$$\begin{aligned}\sigma_A(\tau,s,K) &= \sigma_E(\tau,s,K)+\mathcal{O}\left(\frac{\tau^2}{l(\tau)}\right)\\ &= \sigma_0(s,K)+\sigma_1(s,K)\tau+\sigma_2(s,K)\tau^2+\mathcal{O}\left(\frac{\tau^2}{l(\tau)}\right),\\ \overline{\sigma}_A(\tau,s,K) &= \sigma_0(s,K)+\sigma_1(s,K)\tau+\left(\sigma_2(s,K)-\frac{2\sigma_0(s,K)\,r}{d^3(\ln K,\ln s)\,\sigma(K)}\right)\tau^2(1+o(1)),\\ \underline{\sigma}_A(\tau,s,K) &= \sigma_0(s,K)+\sigma_1(s,K)\tau+\sigma_2(s,K)\tau^2\\ &\quad -\frac{\sqrt{2\pi}}{s}\mathcal{P}(s,K)\frac{\tau^2}{2l(\tau,\sigma(s))}e^{-\frac{d^2(\ln K,\ln s)}{2\tau}}(1+o(1)),\end{aligned}$$

- Discontinuity in the 2nd coefficient for $\overline{\sigma}_A(\tau,s,K)$.
- Up to the order $\frac{\tau^2}{l(\tau)}$, American and European implied vol match.
- Quality of the expansion depends on : $\gamma(s)=\frac{\sigma^2(s)}{8\pi r^2}$.



Error is less than $10^{-2}\%$


American implied volatility
gamma = 24.868 s = 1 r = 0.02 delta = 0.5 beta = 1.5 s = 1 tau = 2
0.52
0.50
0.48
σ
0.8
1.0
1.2
K
variable
eur
am
eur.asympt2
am.lower
am.lower.asympt

Extensions

- Derive exact coefficient for the American implied vol.
- Continuous dividend rate $q \neq 0$:

$$\frac{\mathrm{d}S_t}{S_t} = (r - q)\mathrm{d}t + \sigma(S_t)\mathrm{d}W_t$$

Different behaviours for $\bar{s}(\tau)$ (depending if $r > q$ or $r < q$) hence on $p(\tau)$.

- Inhomogeneous local volatility :

$$\frac{\mathrm{d}S_t}{S_t} = (r - q)\mathrm{d}t + \sigma(t, S_t)\mathrm{d}W_t.$$

Conclusion

- Short-time expansions for the American put prices.
- Estimates of the difference between American and European implied vol.
- Upper / lower bounds for the American implied vol.

References I

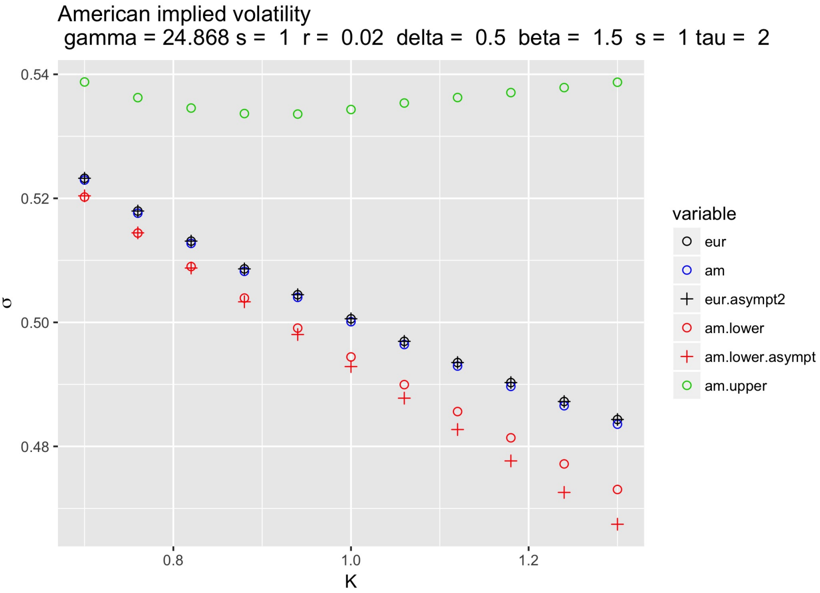
De Marco, Stefano and Henry-Labordere, Pierre.
Local volatility from american options.
Available at SSRN 2870285, 2017.

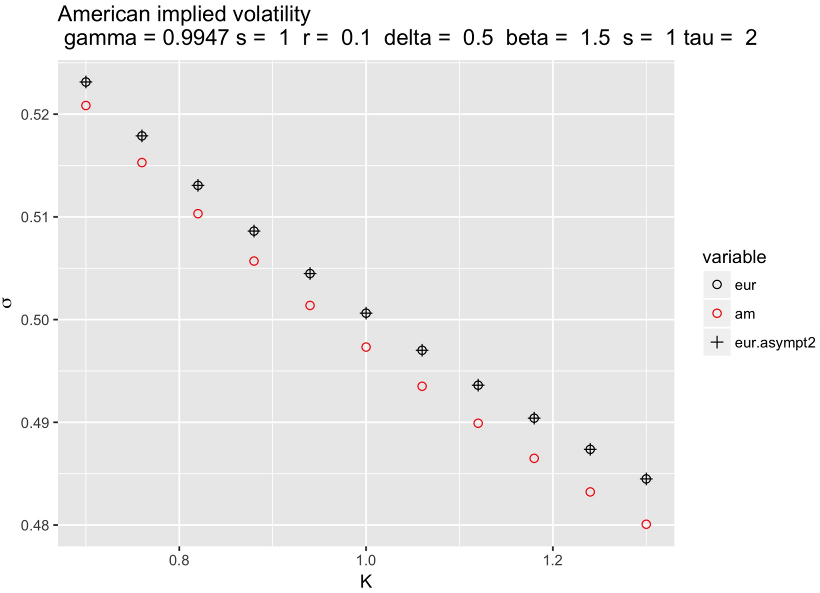
Gatheral, Jim and Hsu, Elton P and Laurence, Peter and Ouyang, Cheng and Wang, Tai-Ho.
Asymptotics of implied volatility in local volatility models.
Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics, 2012.

Chevalier, Etienne.
Critical price near maturity for an American option on a dividend-paying stock in a local volatility model.
Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics, 2005.

References II

Henry-Labordere, Pierre.
Analysis, geometry, and modeling in finance: Advanced methods in option pricing
Chapman and Hall/CRC, 2008.


American implied volatility
gamma = 24.868 s = 1 r = 0.02 delta = 0.5 beta = 1.5 s = 1 tau = 2
σ
0.54
0.52
0.50
0.48
0.8
1.0
1.2
K
variable
eur
am
eur.asympt2
am.lower
am.lower.asympt
am.upper


American implied volatility
gamma = 0.9947 s = 1 r = 0.1 delta = 0.5 beta = 1.5 s = 1 tau = 2
σ
0.52
0.51
0.50
0.49
0.48
0.8
1.0
1.2
K
variable
eur
am
eur.asympt2

Reminder : Laplace's method (see De Bruijn 1981')

- f, g sufficient smooth functions.
- g strict minimum over $[a, b]$ at an interior point c i.e :
  - $g'(c) = 0$, $g''(c) > 0$ and assume $f(c) \neq 0$.
- Leading behaviour as $\lambda \to \infty$ of the integral :

$$
\begin{aligned}
I(\lambda) = \int_a^b f(t) e^{-\lambda g(t)} \mathrm{d}t \\
&\approx e^{-\lambda g(c)} \int_{c-\epsilon}^{c+\epsilon} f(t)\, e^{-\lambda(g(t)-g(c))} \mathrm{d}t \\
&\approx e^{-\lambda g(c)} f(c) \int_{c-\epsilon}^{c+\epsilon} e^{-\frac{\lambda}{2} g''(c)(t-c)^2} \mathrm{d}t \\
&\approx e^{-\lambda g(c)} f(c) \int_{-\infty}^{\infty} e^{-\frac{\lambda}{2} g''(c)(t-c)^2} \mathrm{d}t \\
&= e^{-\lambda g(c)} f(c) \sqrt{\frac{2\pi}{\lambda g''(c)}}.
\end{aligned}
$$