Nonzero-sum stochastic differential games between an impulse controller and a stopper

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Outline

Controller vs Stopper Games

Controller vs Stopper games were first studied in a discrete time zero-sum framework by Maitra and Sudderth (1996).
Zero-sum Stochastic Differential Games:

- Karatzas and Sudderth (2001);
- Karatzas and Zamfirescu (2006)-(2008);
- Bayraktar and Huang (2013);
- Nutz and Zhang (2015);
- Hernandez et al. (2015).

Nonzero-sum Stochastic Differential Games:

- Karatzas and Sudderth (2006), Karatzas and Li (2012).

Why Impulse Controls?

- if fixed and proportional costs apply intervening continuously over time is not feasible;
- more realistic financial models (e.g. fixed transaction costs and liquidity risk):
  (i) Execution delay, Bruder and Pham (2009);
  (ii) Foreign exchange, Cadenillas and Zapatero (1999) ;
  (iii) Liquiditation, Chevalier et al. (2016);
  (iv) Portfolio selection, Ly Vath et al. (2007);
- among others.

Impulse Controls in Stochastic Differential Games

- nonzero-sum impulse games:
  (i) Aïd et al. (2016) developed a general model and verification theorem;
  (ii) Ferrari-Koch (2017) studied a strategic model of pollution control;
  (iii) Basei et al. (2019) generalised Aïd et al.'s model to the N-player and Mean Field cases;
- zero-sum impulse games:
  (i) Cosso (2013) proved existence of an equilibrium in the viscosity sense;
  (ii) Azimzadeh (2017) analysed an asymmetric setting: classic controller vs impulse controller with precommitment;
  (iii) among others.

Two-Player Nonzero-sum Stochastic Differential Game

- $(\Omega, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ complete and right continuous;
- Uncontrolled state variable $X \equiv X^x$:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_{0^-} = x,$$

  existence of a strong unique solution is granted;
- P1's strategy is $u = (\tau_n, \delta_n)_{n \geq 0}$:
  - τ_n strictly increasing sequence of stopping times;
  - δ_n such that $X_{\tau_n} = X_{\tau_n^-} + \delta_n$.
- P2's strategy is a stopping time η with values in $[0, \infty]$.

Payoffs Specification

Both players want to maximize their respective objectives

$$J_1(x; u, \eta) = \mathbb{E}\left[\int_0^\eta e^{-r_1 t} f(X_t^{x,u})dt - \sum_{n:\tau_n \le \eta} e^{-r_1 \tau_n} \phi(X_{\tau_n -}^{x,u}, \delta_n) + e^{-r_1 \eta} h(X_\eta^{x,u}) \mathbf{1}_{(\eta < \infty)}\right]$$

$$J_2(x; u, \eta) = \mathbb{E}\left[\int_0^\eta e^{-r_2 t} g(X_t^{x,u})dt + \sum_{n:\tau_n \le \eta} e^{-r_2 \tau_n} \psi(X_{\tau_n -}^{x,u}, \delta_n) + e^{-r_2 \eta} k(X_\eta^{x,u}) \mathbf{1}_{(\eta < \infty)}\right]$$

- $\phi(X_{\tau_n -}, \delta_n)$ is P1's intervention cost, $\psi(X_{\tau_n -}, \delta_n)$ is P2's gain any time P1 intervenes;
- $X_t^{x;u}$ is the controlled process

$$X_t^{x;u} := x + \int_0^t b(X_s^{x;u})ds + \int_0^t \sigma(X_s^{x;u})dW_s + \sum_{n:\tau_n \le t} \delta_n$$

Nash Equilibrium

Given $x \in \mathbb{R}^d$, we say that $(u^*, \eta^*) \in \mathcal{A}_x$ is a Nash equilibrium of the game if

$$J_1(x; u^*, \eta^*) \geq J_1(x; u, \eta^*) \; \forall \; u \; s.t. \; (u, \eta^*) \in \mathcal{A}_x$$
$$J_2(y; u^*, \eta^*) \geq J_2(x; u^*, \eta) \; \forall \; \eta \; s.t. \; (u^*, \eta) \in \mathcal{A}_x$$

where $\mathcal{A}_x$ is the set of admissible pairs (u, η).
Finally, the equilibrium payoffs of the game are defined as

$$V_i(x) := J_i(x; u^*, \eta^*)$$

The QVIs' Operators

We aim at identifying a good system of QVIs for the computation of Nash equilibria.

The following operators will play a crucial role:

- $\{\delta(x)\} = \text{argmax}_{\delta \in Z}\{V_1(x+\delta) - \phi(x,\delta)\}$;
- $\mathcal{M}V_1(x) = V_1(x+\delta(x)) - \phi(x,\delta(x))$;
- $\mathcal{H}V_2(x) = V_2(x+\delta(x)) + \psi(x,\delta(x))$;
- $\mathcal{A}V(x) = b \cdot \nabla V(x) + \frac{1}{2}\text{tr}(\sigma\sigma^t D^2 V)(x)$.

The Quasi-Variational Inequalities System

We are interested in the following quasi-variational inequalities for V_1, V_2:

$\mathcal{M}V_1 - V_1 \leq 0$	everywhere
$V_2 - k \geq 0$	everywhere
$\mathcal{H}V_2 - V_2 = 0$	in $\{\mathcal{M}V_1 - V_1 = 0\}$
$V_1 = h$	in $\{V_2 = k\}$
$\max\{\mathcal{A}V_1 - r_1 V_1 + f, \mathcal{M}V_1 - V_1\} = 0$	in $\{V_2 > k\}$
$\max\{\mathcal{A}V_2 - r_2 V_2 + g, k - V_2\} = 0$	in $\{\mathcal{M}V_1 - V_1 < 0\}$

The Verification Result

Theorem

Let $V_1, V_2 : \mathbb{R}^d \to \mathbb{R}$*. Set*

$$\mathcal{C}_1 := \{\mathcal{M}V_1 - V_1 < 0\}, \quad \mathcal{C}_2 := \{V_2 - k > 0\}.$$

Moreover, assume that:

- V_1 *and* V_2 *solve the system of QVIs;*
- $V_i \in C^2(\mathcal{C}_j \setminus \partial\mathcal{C}_i) \cap C^1(\mathcal{C}_j) \cap C(\mathbb{R}^d)$*, both with polynomial growth;*
- $\partial\mathcal{C}_i$ *is Lipschitz, and* V_i*'s 2nd order derivatives are loc. bdd near* $\partial\mathcal{C}_i$*.*

Finally, let $x \in \mathbb{R}^d$ *and* $(u^*, \eta^*) \in \mathcal{A}_x$*, with* $u^* = (\tau_n, \delta_n)_{n \geq 1}$ *such that*

$$\tau_n = \inf\{t > \tau_{n-1}; X_t \in \mathcal{C}_1^c\}$$
$$\{\delta_n\} = \text{argmax}_{\delta \in Z}\{V_1(X_{\tau_n^-} + \delta) - \phi(X_{\tau_n^-}, \delta)\}$$

and $\eta^* = \inf\{t \geq 0 : V_2(X_t) = k(X_t)\}$*.*
Then, (u^*, η^*) *is a Nash Equilibrium and* $V_i = J_i(x; u^*, \eta^*)$ *for* $i \in \{1, 2\}$*.*

Example Setting

The players want to maximize the respective payoff functions:

$$J_1(x;u,\eta) = \mathbb{E}_x\left[\int_0^\eta e^{-rt}(X_t - s)dt - \sum_{n:\tau_n\le\eta} e^{-r\tau_n}(c+\lambda|\delta_n|) + ae^{-r\eta}X_\eta \mathbb{1}_{\{\eta<\infty\}}\right]$$

$$J_2(x;u,\eta) = \mathbb{E}_x\left[\int_0^\eta e^{-rt}(q - X_t)dt + \sum_{n:\tau_n\le\eta} e^{-r\tau_n}(d+\gamma|\delta_n|) - be^{-r\eta}X_\eta \mathbb{1}_{\{\eta<\infty\}}\right]$$

with:

$$X_t = X_t^{x;u} = x + \sigma W_t + \sum_{n:\tau_n\le t}\delta_n, \qquad t\ge 0$$

where s, c, λ, a, q, d, γ, b are constants in $\mathbb{R}_+$ satisfying some additional conditions:

$$a<\lambda, \quad b<\gamma, \quad 1-\lambda r>0, \quad 1-br>0.$$

The QVI system suggests the following representation for W_1 and W_2:

$$W_1(x) = \begin{cases} ax & \text{if } W_2(x) = -bx \\ \varphi_1(x) & \text{if } W_2(x) > -bx \text{ and } (\mathcal{M}W_1 - W_1)(x) < 0 \\ \mathcal{M}W_1(x) & \text{if } W_2(x) > -bx \text{ and } (\mathcal{M}W_1 - W_1)(x) = 0 \end{cases}$$

$$W_2(x) = \begin{cases} -bx & \text{if } W_2(x) + bx = 0 \\ \varphi_2(x) & \text{if } W_2(x) > -bx \text{ and } (\mathcal{M}W_1 - W_1)(x) < 0 \\ \mathcal{H}W_2(x) & \text{if } W_2(x) > -bx \text{ and } (\mathcal{M}W_1 - W_1)(x) = 0 \end{cases}$$

where φ_1 and φ_2 are

$$\varphi_1(x) = C_{11}e^{\theta x} + C_{12}e^{-\theta x} + \frac{x - s}{r}$$

$$\varphi_2(x) = C_{21}e^{\theta x} + C_{22}e^{-\theta x} + \frac{q - x}{r}$$

with $C_{11},\ C_{12},\ C_{21},\ C_{22}$ real parameters and $\theta = \sqrt{\frac{2r}{\sigma^2}}$.

Heuristics - No simultaneous Interventions

Ansatz: P1 intervenes when X is too low, whereas P2 does so when it is too high

$$W_1(x) = \begin{cases} ax & \text{in } [\bar{x}_2, +\infty) \\ \varphi_1(x) & \text{in } (\bar{x}_1, \bar{x}_2) \\ \varphi_1(x_1^*) - c - \lambda(x_1^* - x) & \text{in } (-\infty, \bar{x}_1] \end{cases}$$

$$W_2(x) = \begin{cases} -bx & \text{in } [\bar{x}_2, +\infty) \\ \varphi_2(x) & \text{in } (\bar{x}_1, \bar{x}_2) \\ \varphi_2(x_1^*) + d + \gamma(x_1^* - x) & \text{in } (-\infty, \bar{x}_1] \end{cases}$$

with $\bar{x}_1 < x_1^* < \bar{x}_2$.

Semi-Explicit Solution

The parameters involved in W_1 and W_2 must be chosen so as to satisfy the smooth pasting conditions in the verification theorem

$$
\begin{cases}
\varphi_1'(x_1^*) = \lambda \quad \text{and} \quad \varphi''(x_1^*) \le 0 & \text{(optimality of } x_1^*) \\
\varphi_1'(\bar{x}_1) = \lambda & (C^1\text{-pasting in } \bar{x}_1) \\
\varphi_2'(\bar{x}_2) = -b & (C^1\text{-pasting in } \bar{x}_2) \\
\varphi_1(\bar{x}_1) = \varphi(x_1^*) - c - \lambda(x_1^* - \bar{x}_1) & (C^0\text{-pasting in } \bar{x}_1) \\
\varphi_1(\bar{x}_2) = a\bar{x}_2 & (C^0\text{-pasting in } \bar{x}_2) \\
\varphi_2(\bar{x}_1) = \varphi_2(x_1^*) + d + \gamma(x_1^* - \bar{x}_1) & (C^0\text{-pasting in } \bar{x}_1) \\
\varphi_2(\bar{x}_2) = -b\bar{x}_2 & (C^0\text{-pasting in } \bar{x}_2)
\end{cases}
\Longrightarrow
\begin{cases}
C_{11} = -\dfrac{1-\lambda r}{r\theta}\dfrac{1}{e^{\theta x_1^*} + e^{\theta \bar{x}_1}} \\
C_{12} = \dfrac{1-\lambda r}{r\theta}\dfrac{e^{\theta(x_1^* + \bar{x}_1)}}{e^{\theta x_1^*} + e^{\theta \bar{x}_1}} \\
C_{21} = \dfrac{e^{-\theta \bar{x}_2}}{2r}\left[(1-br)\left(\bar{x}_2 + \dfrac{1}{\theta}\right) - q\right] \\
C_{22} = \dfrac{e^{\theta \bar{x}_2}}{2r}\left[(1-br)\left(\bar{x}_2 - \dfrac{1}{\theta}\right) - q\right] \\
2\left(\dfrac{1-z}{1+z}\right) + \ln z - \dfrac{cr\theta}{1-\lambda r} = 0 \\
\bar{x}_2 = \left(\dfrac{1-\lambda r}{\theta w}\dfrac{w^2 - z}{z+1} + s\right)\dfrac{1}{1-ar} \\
Aw^4 + Bw^3 + Cw^2 + Dw + E = 0
\end{cases}
$$

with $z = e^{\theta(x_1^* - \bar{x}_1)}$ and $w = e^{\theta(\bar{x}_2 - \bar{x}_1)}$.

Existence of a Nash Equilibrium

Proposition

Assume $\exists\,(\tilde{z},\ \tilde{w})$, solution to the system of QVI, satisfying some conditions, then, a Nash equilibrium exists and is given by the strategies (u^*, η^*) defined by

$$\tau_n = \inf\left\{t > \tau_{n-1}; X_t \in (-\infty, \bar{x}_1]\right\}, \quad \delta_n = (x_1^* - x)\mathbf{1}_{(-\infty,\bar{x}_1]}(x),$$
$$\eta^* = \inf\{t \geq 0 : X_t \in [\bar{x}_2, +\infty)\}$$

with:

$$x_1^* = \bar{x}_2 + \frac{\ln\tilde{z} - \ln\tilde{w}}{\theta} \qquad \bar{x}_1 = \bar{x}_2 - \tfrac{\ln\tilde{w}}{\theta} \qquad \bar{x}_2 = \left(\frac{1-\lambda r}{\theta\tilde{w}}\frac{\tilde{w}^2 - \tilde{z}}{\tilde{z}+1} + s\right)\frac{1}{1-ar}$$

Moreover, the functions $W_1,\ W_2$ coincide with the equilibrium payoff functions $V_1,\ V_2$:

$$V_1 \equiv W_1 \quad \textit{and } V_2 \equiv W_2$$

Heuristics - P1 forces P2 stop

Ansatz: As before, but this time P1 forces P2 to stop, $x_1^* \equiv \bar{x}_2$.

$$W_1(x) = \begin{cases} ax & \text{in } [\bar{x}_2, +\infty) \\ \varphi_1(x) & \text{in } (\bar{x}_1, \bar{x}_2) \\ a\bar{x}_2 - c - \lambda(\bar{x}_2 - x) & \text{in } (-\infty, \bar{x}_1] \end{cases}$$

$$W_2(x) = \begin{cases} -bx & \text{in } [\bar{x}_2, +\infty) \\ \varphi_2(x) & \text{in } (\bar{x}_1, \bar{x}_2) \\ -b\bar{x}_2 + d + \gamma(\bar{x}_2 - x) & \text{in } (-\infty, \bar{x}_1] \end{cases}$$

with $\bar{x}_1 < \bar{x}_2$.

Semi-Explicit Solution

Again, imposing the smooth pasting conditions to W_1 and W_2

$$
\begin{cases}
\varphi_1'(\bar{x}_1) = \lambda & (C^1\text{-pasting in } \bar{x}_1) \\
\varphi_1(\bar{x}_2) = a\bar{x}_2 & (C^0\text{-pasting in } \bar{x}_2) \\
\varphi_1(\bar{x}_1) = a\bar{x}_2 - c - \lambda(\bar{x}_2 - \bar{x}_1) & (C^0\text{-pasting in } \bar{x}_1) \\
\varphi_2'(\bar{x}_2) = -b & (C^1\text{-pasting in } \bar{x}_2) \\
\varphi_2(\bar{x}_2) = -b\bar{x}_2 & (C^0\text{-pasting in } \bar{x}_2) \\
\varphi_2(\bar{x}_1) = -b\bar{x}_2 + d + \gamma(\bar{x}_2 - \bar{x}_1) & (C^0\text{-pasting in } \bar{x}_1)
\end{cases}
\Rightarrow
\begin{cases}
C_{11} = \dfrac{e^{-\theta\bar{x}_1}}{2}\left[(a-\lambda)\bar{x}_2 - \left(\bar{x}_1 + \dfrac{1}{\theta}\right)\dfrac{1-\lambda r}{r} - c + \dfrac{s}{r}\right], \\
C_{12} = \dfrac{e^{\theta\bar{x}_1}}{2}\left[(a-\lambda)\bar{x}_2 - \left(\bar{x}_1 - \dfrac{1}{\theta}\right)\dfrac{1-\lambda r}{r} - c + \dfrac{s}{r}\right], \\
C_{21} = \dfrac{e^{-\theta\bar{x}_2}}{2r}\left[(1-br)\left(\bar{x}_2 + \dfrac{1}{\theta}\right) - q\right], \\
C_{22} = \dfrac{e^{\theta\bar{x}_2}}{2r}\left[(1-br)\left(\bar{x}_2 - \dfrac{1}{\theta}\right) - q\right], \\
\bar{x}_2 = \dfrac{q}{1-br} + \dfrac{w+1}{\theta(w-1)} + \dfrac{2(\theta rd - (1-\gamma r)\ln w)w}{\theta(1-br)(w-1)^2}, \\
\bar{x}_2 = \dfrac{(1-\lambda r)((\ln w - 1)w^2 + \ln w + 1) - cr\theta(w^2+1)}{\theta(1-ar)(w-1)^2} + \dfrac{s}{1-ar}
\end{cases}
$$

with $w = e^{\theta(\bar{x}_2 - \bar{x}_1)}$.

Existence of a Nash Equilibrium

Proposition

Assume $\exists\, \hat{w}$, solution to the system of QVI, satisfying some other conditions, then, a Nash equilibrium exists and is given by the strategies (u^*, η^*) defined by

$$\tau_n = \inf\{t > \tau_{n-1}; X_t \in (-\infty, \bar{x}_1]\}, \quad \delta_n = (\bar{x}_2 - x)\,\mathbf{1}_{(-\infty,\bar{x}_1]}(x)$$
$$\eta^* = \inf\{t \geq 0 : X_t \in [\bar{x}_2, +\infty)\}$$

with:

$$\bar{x}_1 = \bar{x}_2 - \frac{\ln \hat{w}}{\theta}, \quad \bar{x}_2 = \frac{q}{1 - br} + \frac{\hat{w} + 1}{\theta(\hat{w} - 1)} + \frac{2(\theta r d - (1 - \gamma r)\ln \hat{w})\hat{w}}{\theta(1 - br)(\hat{w} - 1)^2}.$$

Moreover, the functions $W_1,\ W_2$ coincide with the equilibrium payoff functions $V_1,\ V_2$:

$$V_1 \equiv W_1 \quad \text{and } V_2 \equiv W_2$$

Summary

Our main contributions:

- We formulate a general nonzero-sum impulse controller and stopper game.
- We identify a new system of QVIs and prove a verification theorem for NE.
- We study an example with linear payoffs and *multiple NE* of threshold type.

Open Questions:

- Can the two types of NE coexist?
- Applications to energy economics, finance, real options ...