Nonzero-Sum Submodular Monotone-Follower Games: Existence and Approximation of Nash Equilibria

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A famous example of singular control problem...

The Monotone-Follower Problem (from [Karatzas, Shreve 1984]).
Given a standard Brownian motion W and an initial point $x_0 \in \mathbb{R}$, consider the problem of choosing an increasing, adapted, càdlàg process A with $A_0 \geq 0$ to control a state process

$$X_t = x_0 + \sigma W_t - A_t, \quad \sigma \geq 0, \quad x_0 \in \mathbb{R},$$

in order to minimize the cost functional

$$\mathcal{J}(A) := \mathbb{E}\left[\int_0^T h(t, X_t)\, dt + g(X_T) + \underbrace{\int_{[0,T]} f_t \, dA_t}_{\text{cost of intervention}} \right].$$

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- characterization of minimizers (HJB and PMP)
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The Monotone-Follower Game

Assume to be given N players, indexed by $i \in \{1, ..., N\}$, a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and on it

- a càdlàg process $L : \Omega \times [0, T] \to \mathbb{R}$
- continuous semimartingales $f^i : \Omega \times [0, T] \to [0, \infty)$
- the filtration $\bar{\mathbb{F}}_+^{f,L}$ generated by $f := (f^1, ..., f^N)$ and L
- continuous functions $h^i, g^i : \mathbb{R}^{1+N} \to [0, \infty)$.

We consider the game in which each player $i = 1, ..., N$ is allowed to choose a process A^i in the set of admissible strategies

$$\mathcal{A} := \left\{ A : \Omega \times [0, T] \to \mathbb{R} : A \text{ is } \bar{\mathbb{F}}_+^{f,L}\text{-adapted, càdlàg, increasing, with } A_0 \geq 0 \right\}$$

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$$\mathcal{J}^i(A^i, A^{-i}) := \mathbb{E}\left[\int_0^T h^i(L_t, A_t^i, A_t^{-i})\, dt + g^i(L_T, A_T^i, A_T^{-i}) + \int_{[0,T]} f_t^i \, dA_t^i \right],$$

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Definition (Open-Loop Nash Equilibrium)

Given, for $i = 1, \dots, N$, admissible strategies $A^i \in \mathcal{A}$, the profile strategy $\mathbf{A} = (A^1, \dots, A^N)$ is a Nash equilibrium (NE) of the Monotone-Follower game if, for each $i = 1, \dots, N$, we have

$$\mathcal{J}^i(A^i, A^{-i}) \leq \mathcal{J}^i(V, A^{-i}), \quad \forall\, V \in \mathcal{A}.$$

What about existence of NE?

Symmetric game:

- [Steg 2012] game with symmetric payoff
- [Ferrari, Riedel, Steg 2016] symmetric payoff, include classical controls
- [Fu, Horst 2017], [Guo, Lee 2019], [Guo, Xu 2019] mean field games with singular controls

Non symmetric game:

- [Guo, Tang, Xu 2018], [Kwon 2018] existence and analysis of Markovian equilibria for specif data

The lack of general existence results motivetes our study.

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Existence of Nash equilibria under submodularity condition

Assumption (E)

- convexity: for each $(I, a^{-i}) \in \mathbb{R} \times \mathbb{R}^{(N-1)}$, the functions $h^i(I, \cdot, a^{-i})$ and $g^i(I, \cdot, a^{-i})$ are strictly convex;
- decreasing differences: $\forall I \in \mathbb{R}$ and $a, \bar{a} \in \mathbb{R}^N$ s.t. $\bar{a} \geq a$, for $\varphi^i \in \{h^i, g^i\}$,

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  Example: $\varphi^i(I, a^1, a^2) = F^i(I)\big(a^i - \frac{1}{N-1}\sum_{j\neq i} a^j\big)^2$, with $F^i \geq 0$
- uniform coercivity condition: there exist two constants $K, \kappa > 0$ such that, for each $i = 1, ..., N$,

$$\mathcal{J}^i(A^i, A^{-i}) \geq \kappa\,\mathbb{E}[A^i_T] \quad \forall\, \mathbf{A} \in \mathcal{A}^N \quad \text{s.t.} \quad \mathbb{E}[A^i_T] \geq K;$$

- boundedness of the values there exists constant $M > 0$ such that, for each $i = 1, ..., N$,

$$\forall\, \mathbf{A} \in \mathcal{A}^N \quad \exists\, r^i(\mathbf{A}) \in \mathcal{A} \quad \text{s.t.} \quad \mathcal{J}^i(r^i(\mathbf{A}), A^{-i}) \leq M.$$

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Existence of Nash Equilibria

Theorem (Existence of NE for the Submodular Monotone-Follower Game)

Under Assumption (E), there exists a Nash equilibrium of the Monotone-Follower game.

Proof: in the spirit of [Topkis 1978], Tarski's Fixed Point theorem

Remarks

- The set of NE has a lattice structure.
- The theorem above can be proved also for $T = \infty$.
- Adding finite fuel constraints like $\mathbb{E}[A^i_T] \leq w^i$, the theorem above can be proved only with the assumptions of continuity, convexity and submodularity of the costs.
- Extension with multidimensional controls: $A^i \in \mathbb{R}^d$.
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Stochastic Differential Games with Singular Control

For $i = 1, ..., N$, player i can choose a process $\xi^i \in \mathcal{A}$ to control its state, which evolves according to the stochastic differential equation (SDE)

$$dX_t^i = \mu^i X_t^i \, dt + \sigma^i X_t^i \, dW_t^i + d\xi_t^i, \quad t \in [0, T], \quad X_{0-}^i = x_0^i,$$

in order to minimize its expected cost

$$\mathcal{J}^i(\xi^i, \xi^{-i}) := \mathbb{E}\bigg[\int_0^T h^i(L_t, X_t^i, X_t^{-i}) dt + g^i(L_T, X_T^i, X_T^{-i}) + \int_{[0,T]} f_t^i d\xi_t^i \bigg].$$

Corollary

Under Assumption (E), there exists a NE of the stochastic differential game.

Remark

Another example is given by controlled Ornstein–Uhlenbeck processes

$$dX_t^i = \theta^i(\mu^i - X_t^i) \, dt + \sigma^i \, dW_t^i + d\xi_t^i, \quad t \in [0, T], \quad X_{0-}^i = x_0^i > 0,$$

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Algorithm for Approximating the Least NE

Recall the definition of the best reply maps

$$R^i(\mathbf{A}) = \arg\min_{V \in \mathcal{A}} \mathcal{J}^i(V, A^{-i}), \quad \text{and} \quad R = (R^1, ..., R^N) : \mathcal{A}^N \to \mathcal{A}^N.$$

Consider the algorithm $\{\mathbf{B}^n\}_{n \in \mathbb{N}} \subset \mathcal{A}^N$ defined by:

- $\mathbf{B}^0 = 0 \in \mathcal{A}^N$;
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Theorem

Suppose Assumption (E) holds. Assume moreover that h^i, g^i and f^i are bounded
Then, the sequence $\{\mathbf{B}^n\}_{n \in \mathbb{N}}$ is monotone increasing in $(\mathcal{A}^N, \leq)$ and it converges to the least NE of the Monotone-Follower game.

Algorithm for Approximating the Least NE

Recall the definition of the best reply maps

$$R^i(\mathbf{A}) = \arg\min_{V \in \mathcal{A}} \mathcal{J}^i(V, A^{-i}), \quad \text{and} \quad R = (R^1, ..., R^N) : \mathcal{A}^N \to \mathcal{A}^N.$$

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Approximation of weak NE
through
NE of a sequence of *Lipschitz* games

The n-Lipschitz Game - The Approximating Game

For each $n \in \mathbb{N}$, the n-Lipschitz Game is the game in which each player $i = 1, ..., N$ is allowed to choose a process A^i in the set of admissible n-Lipschitz strategies

$$\mathcal{L}(n) := \{A \in \mathcal{A} \,|\, A \text{ is Lipschitz with } Lip(A) \leq n \;\text{ and }\; A_0 = 0\} \subset \mathcal{A},$$

in order to minimize her cost functional $\mathcal{J}^i(\cdot, A^{-i})$.

Theorem (Existence of NE for the Submodular n-Lipschitz Game)

Assume that, for each $i = 1, ..., N$,

- *regularity: h^i and g^i are continuous and strictly convex in a^i;*
- *h^i and g^i have decreasing differences.*

Then, for each $n \in \mathbb{N}$, there exists a Nash equilibrium of the n-Lipschitz game, i.e., there exist $(A^1, ..., A^N) \in \mathcal{L}(n)^N$ such that, for each $i = 1, ..., N$, we have

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Weak Formulation of the Monotone-Follower Game

IDEA = make the source of randomness $(\Omega, \mathcal{F}, \mathbb{P}, f, L)$ part of the problem, keeping **FIXED** the induced distribution $\mathbb{P}_0 := \mathbb{P} \circ (f, L)^{-1} \in \mathcal{P}(\mathcal{C}_+^N \times \mathcal{D})$.

Definition (Weak Nash Equilibrium)

We say that the 6-tuple $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{Q}}, \bar{f}, \bar{L}, \bar{A})$ is a weak Nash equilibrium if:

(i) $(\bar{f}, \bar{L}, \bar{\mathbf{A}}) : \bar{\Omega} \to \mathcal{C}_+^N \times \mathcal{D} \times \mathcal{D}_\uparrow^N$

(ii) $\bar{\mathbb{Q}} \circ (\bar{f}, \bar{L})^{-1} = \mathbb{P}_0$

(iii) for every $i = 1, ..., N$ and every process $V : \bar{\Omega} \to \mathcal{D}_\uparrow$ we have

$$\mathcal{J}^i_{\bar{\mathbb{Q}}}(\bar{A}^i, \bar{A}^{-i}) \leq \mathcal{J}^i_{\bar{\mathbb{Q}}}(V, \bar{A}^{-i}).$$

Where $\mathcal{D}_\uparrow := \{A : [0, T] \to \mathbb{R} : A$ is càdlàg, increasing, with $A_0 \geq 0\}$ and

$$\mathcal{J}^i_{\mathbb{Q}}(A^i, A^{-i}) := \mathbb{E}^{\mathbb{Q}}\left[\int_0^T h^i(L_t, A^i_t, A^{-i}_t)\, dt + g^i(L_T, A^i_T, A^{-i}_T) + \int_{[0,T]} f^i_t\, dA^i_t\right]$$

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Theorem (Existence and Approximation of Weak NE)

- *The sequence of laws $\mathbb{P} \circ (\mathbf{A}^n)^{-1}$ is relatively compact in the Meyer-Zheng topology.*
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Corollary (Existence of Lipschitz ε-Nash equilibria)

- *For each $\varepsilon > 0$, there exists n_ε such that the Nash equilibrium $(A^{1,n_\varepsilon}, ..., A^{N,n_\varepsilon}) \in \mathcal{L}(n_\varepsilon)^N$ of the n_ε-Lipschitz game is an ε-Nash equilibrium (in the strong sense) of the Monotone-Follower game; that is, for each i,*

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Concluding:

- existence of NE (Tarski's fixed point theorem)
- application to differential games, whenever a certain structure is preserved by the dynamics
- algorithm to construct the least NE, solving minimization problems
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Introduction

- The Monotone-Follower problem
  - [Bather Chernoff 1967] spacecraft control
  - [Karatzas Shreve 1984] link with optimal stopping
  - [Li, Žitković 2017] weak formulation
- Games of Monotone-Follower individual optimization
  - [Steg 2012] game with symmetric payoff
  - [Ferrari, Riedel, Steg 2016] include classical controls
- The first order condition (FOCs) characterization
  - [Bank Riedel 2001,2003; Bank 2005; Chiarolla Ferrari 2014; Ferrari, Riedel, Steg 2016; etc. etc.]
- The idea of the approximation
  - [Li, Žitković 2017] approximation of weak solution of the Monotone-Follower problem throught Lipschitz controls