Optimal stopping contract for Public Private Partnerships under moral hazard.

Ishak HAJJEJ
ENIT/LAMSIN && ENSAE/CREST

Joint work with **Caroline Hillairet** and **Mohamed Mnif**
(Ensae ParisTech, ENIT-Tunis)

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Outline of the talk

1. PPP : political and economic framework
2. Formulation of the optimization problem: Weak formulation
   - Incentive compatible contract
   - Hamilton Jacobi Bellman Variational Inequality
   - Verification theorem
3. Numerical results
4. Conclusion and perspectives

OUTLINE

INTRODUCTION

Public-**P**rivate **P**artnership (**PPP**) is defined as a long-term contract between a private party and a public entity, for the management of an asset or public service.

- The public outsources the construction and the maintenance of an equipment (hospital, university, prison ...).
- The consortium takes the risks and a great responsibility to manage the project.

The goal of PPP : to transfer the risk to the consortium,to ensure a better value for money in the use of public funds.

Principal-agent problem with moral hazard

- The problem of this contract is the asymmetry of information between the two parties :Not the same information. Consortium's effort not observable by the public

Aim **characterizing a optimal PPP contract in this setting of asymmetric information between both partners : this is a principal-agent problem with moral hazard.**

- The public pays the consortium continuously. The public could end the contract at the date τ.
- The first paper on principal-agent problems is the paper of Holmstrom and Milgrom [3].
- Book of Cvitanic et al. [4] a general theory can be used to solve these problems, by means of forward-backward stochastic differential equations.
- This work is build on the literature on dynamic contracting using recursive methods, and in particular the seminal paper of **Sannikov** (2008)[6].

OUTLINE

Let W be a standard Brownian motion under some probability space with probability measure $\mathbb{P}$, $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be the information filtration generated by W.

- The social value of the project that is observed by the public

$$X_t := X_0 + \sigma W_t$$

where

- $X_0 > 0$ is the initial value of the project.
- $\sigma > 0$ is the volatility of the operational cost of the infrastructure maintenance, that is assumed to be constant.

We consider a weak formulation: The consortium's effort A changes the distribution of the process X + add a drift $\varphi(A_t)$.
We define the process $\gamma^A = (\gamma_t^A)_{t\geq 0}$ by

$$\gamma_t^A := \exp\left[\int_0^t \frac{\varphi(A_s)}{\sigma} dW_s - \frac{1}{2}\int_0^t \left(\frac{\varphi(A_s)}{\sigma}\right)^2 ds\right] \; dt \otimes d\mathbb{P} a.s.$$

We consider:

$$\mathcal{A} := \{(A_s)_{s\geq 0}\ \mathbb{F}\text{-progressively measurable process, } A_s \geq 0\ ds\otimes d\mathbb{P}\ a.e.\ \text{such that } \sup_{\tau\in\mathcal{T}} \mathbb{E}[(\gamma_\tau^A)^p] < \infty,\ \forall\, p>1\}.$$

where $\mathcal{T}$ is the set of all $\mathbb{F}$-finite stopping times.
The probability measure $\mathbb{P}^A$ is defined by $\frac{d\mathbb{P}^A}{d\mathbb{P}}|_{\mathcal{F}_\tau} = \gamma_\tau^A$. The process $(W_t^A)_{t\geq 0}$ defined by

$$W_t^A = W_t - \int_0^t \frac{\varphi(A_s)}{\sigma} ds,\ \text{for } t \geq 0$$

is a $\mathbb{P}^A$-Brownian motion.
The social value of the project is given under $\mathbb{P}^A$ by:

$$X_t = X_0 + \int_0^t \varphi(A_s)ds + \sigma W_t^A,\ t\geq 0\ dt\otimes d\mathbb{P}a.s$$

Moral hazard as a Stackelberg leadership model

- Asymmetric information: the public observes the social value X but not the effort A.
  - The public chooses the rent $(R_s)_{s\geq 0}$ $\mathbb{F}$-adapted, he will pay to the consortium to compensate him for his efforts.
  - The public could end the contract at the date τ, where τ is a random time in $\mathcal{T}$ (the set of all $\mathbb{F}$-finite stopping times).
- A contract is a triplet $\Gamma = ((R_t)_t, \tau, \xi)$ where R is non negative $\mathbb{F}$ adapted process, $\tau \in \mathcal{T}$, and ξ is non negative $\mathcal{F}_\tau$ measurable random variable which represents the cost of stopping the contract.
- Stackelberg leadership model
  - Principal is the leader by offering a contract Γ.
  - Agent gives a best response in terms of effort A.

THE OPTIMIZATION PROBLEMS FOR THE CONSORTIUM AND THE PUBLIC

ASSUMPTION 1

- *φ is the function that models the marginal impact of the consortium's efforts on the social value, φ is C^1 concave, bounded, increasing, $\varphi > 0$ and $\frac{\|\varphi\|_\infty}{\sigma} < 1$.*
- *U is the utility function of the consortium, strictly concave increasing and satisfying Inadas conditions $U'(\infty) = 0$, $U'(0) = \infty$.*
- *h is the cost of the effort for the consortium; h is C^1, convex increasing, $h(0) = 0$.*

- *Agent best response*

$$A^* \in \arg\max_{A \in \mathcal{A}^C} \mathbb{E}^A \left(\int_0^\tau e^{-\lambda s}(U(R_s) - h(A_s))ds + e^{-\lambda\tau}\xi \right)$$

where

$$\begin{aligned} \mathcal{A}^C \quad := \quad & \{(A_s)_{s\geq 0} \in \mathcal{A}, \text{ such that } \mathbb{E}^{\mathbb{P}}[(\int_0^\infty e^{-\lambda s}|h(A_s)|^p ds)] < \infty, \\ & \mathbb{E}^{\mathbb{P}}[(\int_0^\infty e^{-\lambda s}|\varphi(A_s)|^p ds)] < \infty \; \forall p > 1\}. \end{aligned}$$

The *objective function* at time t for the consortium is $\mathbb{P}^A$-a.s.

$$J_t^C(\Gamma, A) := \mathbb{E}^A \left(\int_t^\tau e^{-\lambda(s-t)}(U(R_s) - h(A_s))ds + e^{-\lambda(\tau-t)}\xi | \mathcal{F}_t \right).$$

Given the best response of the agent the principal problem is formulated by

- *Principal problem*

$$\sup_{\Gamma \in \mathcal{A}^P} \sup_{P^{A^*} \in \mathcal{P}} \mathbb{E}^{A^*}\left[\int_0^\tau e^{-\delta s}(\varphi(A_s^*) - R_s)ds - e^{-\delta\tau}\xi\right]$$

subject to the reservation constraint

$$\mathbb{E}^{A^*}\left(\int_0^\tau e^{-\lambda s}(U(R_s) - h(A_s^*))ds + e^{-\lambda\tau}\xi\right) \geq \underline{x}$$

$$\begin{aligned}\mathcal{A}^P \quad := \quad & \{((R_s)_{s\geq 0}, \tau, \xi) \text{ such that } R\ \mathbb{F}\text{-progressively measurable } R_s \geq 0\ ds \otimes d\mathbb{P}\ a.s. \text{ and } \forall p \geq 1 \\ & \mathbb{E}^{\mathbb{P}}[\int_0^\infty e^{-\delta s}(R_s)^p ds] < \infty, \tau \in \mathcal{T},\ \xi\ \mathcal{F}_\tau\text{-measurable such that } \mathbb{E}^{\mathbb{P}}(e^{-\lambda\tau}\xi)^p < \infty\}.\end{aligned}$$

and

$$\mathcal{P} = \{\mathbb{P}^{A^*} \sim \mathbb{P}, A^* \in \mathcal{A}^C\}.$$

The *objective function* at time t for the public is $\mathbb{P}^{A^*}$-a.s.

$$J_t^P(\Gamma, A^*) := \mathbb{E}^{A^*}\left(\int_t^\tau e^{-\delta(s-t)}(\varphi(A_s^*) - R_s)ds - e^{-\delta(\tau-t)}\xi|\mathcal{F}_t\right).$$

- The stochastic control problem is nonstandard.
- Asymmetry of information: The public does not observe the effort of the consortium A, but she observes only his impact on the social value X, which is the state process of the optimization control problem.
- **The trick** (Sannikovs idea): reformulate the optimization problems in terms of the consortium objective function J^C.
  - J^C = new state process.
  - The consortium objective function is related to the solution of the following BSDE with a random time horizon τ.

$$Y_t = \zeta + \int_t^\tau g(s, \omega, Z_s) ds - \int_t^\tau Z_s dW_s \tag{1}$$

  - Chen [2] considers a random horizon which could be infinite and assumes that the constant of Lipschitz is time dependent and square integrable on $[0, \infty]$.
  - Darling and Pardoux [1] studied a BSDE with random horizon. They assumed that the generator depends on (y, z).

That is not satisfied in our case.

BSDEs WITH RANDOM TERMINAL TIME

(H1) For any $z \in \mathbb{R}$, $g(., w, z)$ is a progressively measurable process such that

$$\mathbb{E}\left(\int_0^\tau |g(s,w,z)|ds\right)^2 < \infty.$$

(H2) g satisfies the following contraction condition, i.e. there exist a constant $0 \leq c < 1$ such that

$$|g(s,w,z_1) - g(s,w,z_2)| \leq c|z_1 - z_2| \; ds \otimes d\mathbb{P} \; a.s.$$

We introduce the following spaces for a fixed stopping time $\tau \in \mathcal{T}$:

$$\begin{aligned}
\mathcal{S}^2(\tau) : &= \{Y : Y \; \mathbb{F}\text{progressively measurable such that } ||Y||_{\mathcal{S}^2(\tau)} := \left(\mathbb{E}^{\mathbb{P}} \sup_{0\leq s\leq \tau} |Y_s|^2\right)^{\frac{1}{2}} < \infty\},\\
\mathcal{H}^2(\tau) : &= \{Z : Z \; \mathbb{F}\text{-progressively measurable such that } ||Z||_{\mathcal{H}^2(\tau)} := \left(\mathbb{E}^{\mathbb{P}} \int_0^\tau |Z_s|^2 ds\right)^{\frac{1}{2}} < \infty\},\\
L^2(\mathcal{F}_\tau) : &= \{\zeta \; \mathcal{F}_\tau\text{-measurable random variable such that } \mathbb{E}|\zeta|^2 < \infty\}.
\end{aligned}$$

BSDEs with random terminal time

Proposition 1

Let τ be a stopping time in $\mathcal{T}$, $\xi \in L^2(\mathcal{F}_\tau)$ and g satisfies (H1) and (H2), then:

1. *There exists a unique solution $(Y,Z) \in \mathcal{S}^2(\tau) \times \mathcal{H}^2(\tau)$ to the BSDE (τ, ξ, g) (1).*
2. *Comparison theorem: if (Y,Z) (resp. (Y',Z')) is the solution of the BSDE (τ, ξ, g) (resp. BSDE (τ, ξ, g')) with generators satisfying (H1) and (H2), and $g(t,w,z) \leq g'(t,w,z)$, $t \in [\![0,\tau]\!]$, $dt \otimes d\mathbb{P}$ a.e. Then*

$$Y_t \leq Y'_t \text{ for all } t \in [\![0,\tau]\!] \text{ a.s.}$$

- *Key for the proof*: *By the fixed point theorem.*
- Proposition 1 is used to determine the incentive compatible contract and to provide the dynamics of the consortium objective function J^C.

The public must propose a contract to the consortium, the agent chooses the contract which maximizes his expected utility.

Lemma 1

Suppose Assumption 1. For any admissible contract $\Gamma \in \mathcal{A}^P$, and for any $A \in \mathcal{A}^C$, there exists $Z^A \in \mathcal{H}^2(\tau)$ such that the dynamics of the consortium objective function evolves according to the BSDE with random terminal condition

$$dJ_t^C(\Gamma, A) = \left(\lambda J_t^C(\Gamma, A) - U(R_t) - \psi(A_t, Z_t^A)\right) dt + Z_t^A dW_t, \; J_\tau^C(\Gamma, A) = \xi \tag{2}$$

where

$$\psi(a, z) := -h(a) + z\frac{\varphi(a)}{\sigma}. \tag{3}$$

If there exists $A^* \in \mathcal{A}^C$ such that $\psi(A_t, Z_t^A) \leq \psi(A_t^*, Z_t^A), \; \forall t \in [\![0, \tau[\![\; dt \otimes d\mathbb{P}$, then

$$J_t^C(\Gamma, A) \leq J_t^C(\Gamma, A_t^*), \; \forall t \in [\![0, \tau[\![\; dt \otimes d\mathbb{P}.$$

Lemma 2

Suppose Assumption 1. Let z be a real number. We define, $A^*(z) := \arg\max_{a\geq 0} \psi(a,z)$. *If* $z > \sigma \frac{h'(0)}{\varphi'(0)}$, *then* $A^*(z) = (\frac{h'}{\varphi'})^{-1}(\frac{z}{\sigma})$ *and if* $z \leq \sigma \frac{h'(0)}{\varphi'(0)}$, *then* $A^*(z) = 0$.

The control Z is a control variable chosen by the public. The control variables for the public are $\Gamma = (R, \tau, \xi)$ and $(Z_t^A)_{t\geq 0}$.

- As usually in the literature (Sannikov [6]), the optimisation problem consists in maximizing a certain criterion with controls variables that are Γ and $(Z_t^A)_{t\geq 0}$.
- In this paper, we keep the variable of the explicit control $(A_t)_{t\geq 0}$ which represents a physical quantity.
- There exits a bijection between the process $(Z_t^A)_{t\geq 0}$ and the optimal effort $(A_t^*)_{t\geq 0}$, the bijection is given by

$$A_t^* = A^*(Z_t^A) = (\frac{h'}{\varphi'})^{-1}(Z_t^A\sigma^{-1})\mathbf{1}_{\{Z_t^A>0\}}.$$

PROPOSITION 2

Suppose Assumption 1. The dynamics of J^C for any incentive compatible contract $(\Gamma, A^*(Z))$ is given by the BSDE with random terminal condition

$$dJ_t^C(\Gamma, A^*(Z)) = \Big(\lambda J_t^C(\Gamma, A^*(Z_t)) - U(R_t) - \psi(A^*(Z_t), Z_t)\Big)\, dt + Z_t dW_t,\ J_\tau^C(\Gamma, A^*(Z)) = \xi \tag{4}$$

where $A^*(Z)$ is defined in Lemma 2.

- Using the characterization of the incentive compatible contract, the optimization problem of the public is a standard stochastic control problem. The state process is the consortium objective function J^C.
- The value function is formulated as

$$v(x) := \sup_{(R,\tau,A^*(Z))\in\mathcal{Y}} \mathbb{E}_x^{A^*(Z)}\left(\int_0^\tau e^{-\delta s}(\varphi(A^*(Z_s)) - R_s)ds - e^{-\delta\tau}J_\tau^C(x,R,\tau,A^*(Z))\right) \quad (5)$$

$$\mathcal{Y} := \{(R,\tau,A^*(Z))\ R \geq 0\ \mathbb{F}\text{-progressively measurable process such that } \mathbb{E}^{\mathbb{P}}[\int_0^\infty e^{-\delta s}(R_s)^p ds] < \infty, \forall p > 1,\ \tau \in \mathcal{T}, A^*(Z) \in \mathcal{A}^C\}.$$

$\mathcal{P} := \{x : v(x) \leq -x\}$ is called the stopping region. Its complement $\mathcal{P}^c$ is called the continuation region.

- HJBVI

$$\min\left\{\delta v(x) - \sup_{(r,a)\in\mathbb{R}^+\times\mathbb{R}^+}[\mathcal{L}^{a,r}v(x) + \varphi(a) - r], v(x) + x\right\} = 0 \quad (6)$$

where the second order differential operator $\mathcal{L}^{a,r}$ is defined by

$$\mathcal{L}^{a,r}v(x) := \frac{1}{2}(\sigma\frac{h'(a)}{\varphi'(a)})^2\mathbf{1}_{\{a>0\}}v''(x) + [\lambda x - U(r) + h(a)]v'(x).$$

Boundary condition

Lemma 3

The function v satisfies

$$v(0) = \max\left\{\frac{1}{\delta}\sup_{y\geq 0}\{\varphi\circ h^{-1}\circ U(y) - y\}, 0\right\}.$$

Lemma 4

There exists a positive constant K such that for all $x \geq 0$,

$$|v(x)| \leq K(1+|x|)$$

VERIFICATION THEOREM

PROPOSITION 1

Let $w \in C^2(\mathbb{R}^+)$, satisfying a linear growth condition, and we assume that

$$\sup_{(R,\tau,A^*(Z))\in\mathcal{Y}} \mathbb{E}[|J^C_\tau(x,R,\tau,A^*(Z))|^q] < \infty \; \forall q > 1. \tag{7}$$

Then we have:

(I) *For $x \geq 0$, if w satisfies $\delta w(x) \geq \sup_{(r,a)\in\mathbb{R}^+\times\mathbb{R}^+} \{\mathcal{L}^{a,r}w(x) + \varphi(a) - r\}$ and $w(x) \geq -x$, then $w(x) \geq v(x)$.*

Suppose that there exists two measurable non-negative functions (a^*, r^*) defined on $(0, \infty)$ s.t.

$$\sup_{(r,a)\in\mathbb{R}^+\times\mathbb{R}^+} \{\mathcal{L}^{a,r}w(x) + \varphi(a) - r\} = \mathcal{L}^{a^*(x),r^*(x)}w(x) + \varphi(a^*(x)) - r^*(x),$$

the SDE

$$dJ_t^C = \left(\lambda J_t^C - U(r^*(J_t^C)) + h(a^*(J_t^C))) - Z_t \frac{a^*(J_t^C)}{\sigma}\right) dt + Z_t dW_t,\ J_0^C = x$$

admits a unique solution $\widehat{J_t^C}$, and $(r^*(\widehat{J_t^C}), \tau, a^*(\widehat{J_t^C}))$ lies in $\mathcal{Y}$.
If w is a solution of HJBVI, then

(II) $w = v$ and

$$\tau^* := \inf\{t \geq 0 \ : \ w(\widehat{J_t^C}) \leq -\widehat{J_t^C}\} \tag{8}$$

is an optimal stopping time of the problem (5).

(III) The optimal rent is given by $r^*(x) = (U')^{-1}(-\frac{1}{w'(x)})\mathbf{1}_{w'(x)<0}$.

OUTLINE

- The Hamilton Jacobi Bellman Variational Inequality is written as follows

$$\min[\delta v(x) - \sup_{(r,a)\in\mathbb{R}^+\times\mathbb{R}^+}\{[\lambda x - U(r) + h(a)]v'(x) + \frac{1}{2}(\sigma\frac{h'(a)}{\varphi'(a)})^2 v''(x) - r + \varphi(a)\}, v(x) + x] = 0 \quad (9)$$

- $v(0) = \max\{\frac{1}{\delta}\sup_y\{\varphi\circ h^{-1}\circ U(y) - y\}, 0\},\ v(\bar{x}) = -\bar{x}.$
- The solution of (9) can be approximated by the following numerical method:
  - Reduction to a bounded domain. We have to replace $[0,\infty)$ by a bounded domain $[0,\bar{x}]$.
  - We use finite difference approximation to approximate the variational inequality (9).
  - We use Howard algorithm to solve the discrete equation.

- We choose the functions
  - $U(x)$= power utility.
  - $\varphi(x)$ and h= exponential functions.
- We compute
  - sensitivity with respect to σ.
  - optimal rent as a function of the effort.

SENSITIVITY WITH RESPECT TO σ

$$\sigma = 1.2, 1.65 \text{ or } 2.2.$$

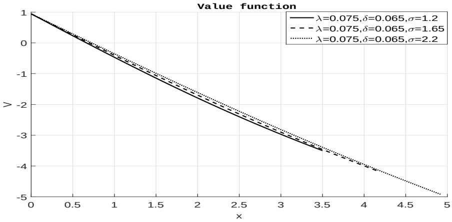
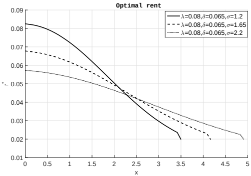


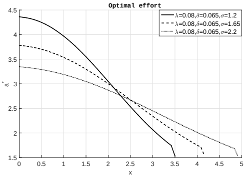
FIGURE: Value function v for different σ.

The optimal public value function v is increasing with respect to σ: the risk is supported by the consortium.

FIGURE: Optimal rent and optimal effort



optimal rent R^*



optimal effort A^*

The consortium is subject to volatility risk. A significant volatility crushes the impact of wealth: in this case, there is more risk for the consortium which must make efforts even if x promised is large enough. And if x is small, the consortium is not ready to provide more effort (compared to a lower volatility).

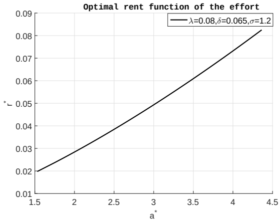


FIGURE: Optimal rent r^* function of the effort a^*.

The optimal rent is an increasing convex function of the optimal effort.

OUTLINE

CONCLUSION AND PERSPECTIVES

- This paper provides a characterisation of optimal public private partnership contracts in a moral hazard framework.
  - Using martingale methods and stochastic control.
- Perspective :
  - Strong formulation.
  - First best/second best.
  - Adding the possibility of penalty imposed on the consortium, in case of non-compliance with contract terms.

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