Term Structure Modeling under Volatility Uncertainty

A Forward Rate Model driven by G-Brownian Motion

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12th European Summer School in Financial Mathematics,
September 2 - 6, 2019

Motivation

- Standard Models in Finance: Constant Volatility
  - → Unrealistic
  - → Statistical Uncertainty
- New Approach: Volatility Uncertainty
  - → More "Realistic"
  - → Robust

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Literature

- Mathematical Approaches to Volatility Uncertainty:
  - → Denis and Martini (2006)
  - → Peng (2010)
  - → Soner, Touzi, and Zhang (2011)
- Volatility Uncertainty in Asset Markets:
  - → Avellaneda, Levy, and Parás (1995)
  - → Lyons (1995)
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Historical Overview

- Short Rate Models:

$$r_t = r_0 + \int_0^t \mu(s, r_s)ds + \int_0^t \sigma(s, r_s)dB_s$$

→ Vasicek (1977), Cox, Ingersoll Jr, and Ross (1985), Ho and Lee (1986), Hull and White (1990)

- Forward Rate Models:

$$f(t, T) = f(0, T) + \int_0^t \alpha(s, T)ds + \int_0^t \beta(s, T)dB_s,$$

→ Heath, Jarrow, and Morton (1992), ...

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Model Framework

Forward Rate Model

Examples

Conclusion

Construction of the Set of Beliefs

- Let $(\Omega, \mathcal{F}, P_0)$ be a probability space such that $\Omega = C_0(\mathbb{R}_+)$, $\mathcal{F} = \mathcal{B}(\Omega)$, and P_0 is the Wiener measure and let $(B_t)_t$ be the canonical process.
- For all $[\underline{\sigma}, \overline{\sigma}]$-valued, $(\mathcal{F}_t)_t$-adapted processes $\sigma = (\sigma_t)_t$, where $\overline{\sigma} \geq \underline{\sigma} > 0$, we define the process

$$B_t^\sigma := \int_0^t \sigma_s dB_s$$

and the measure P^σ by

$$P^\sigma := P_0 \circ (B^\sigma)^{-1}.$$

- Denote by $\mathcal{P}$ the closure of all such measures under the topology of weak convergence.

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G-Expectation and G-Brownian Motion

- Now we define the sublinear expectation

$$\hat{\mathbb{E}}[X] := \sup_{P\in\mathcal{P}} \mathbb{E}_P[X].$$

- By Denis, Hu, and Peng (2011), $\hat{\mathbb{E}}$ corresponds to the G-expectation on $L_G^1(\Omega)$ and $(B_t)_t$ is a G-Brownian motion under $\hat{\mathbb{E}}$.
- The G-Brownian motion has an uncertain volatility, which implies

$$\underline{\sigma}^2 t \leq \langle B\rangle_t \leq \overline{\sigma}^2 t.$$

- Henceforth, statements hold quasi-surely, i.e., P-a.s. for all $P \in \mathcal{P}$.

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Space of Admissible Integrands

- From now on we fix a finite time horizon $\tau < \infty$.
- Let $\tilde{M}_G^{p,0}(0,T)$ be the space of all processes ϕ of the form

$$\phi(t,s) = \sum_{i=0}^{N-1} \varphi_t^i 1_{[s_i,s_{i+1})}(s)$$

  for $0 = s_0 < s_1 < ... < s_N = \tau$ and $\varphi^i \in M_G^p(0,T)$.
- Denote by $\tilde{M}_G^p(0,T)$ the completion of $\tilde{M}_G^{p,0}(0,T)$ under the norm

$$\| \phi \|_{\tilde{M}_G^p(0,T)} := \Big(\int_0^\tau \hat{\mathbb{E}}\Big[\int_0^T |\phi(t,s)|^p dt \Big] ds \Big)^{\frac{1}{p}}.$$

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Stochastic Integrals and Fubini's Theorem

- For processes $\phi \in \tilde{M}_G^2(0,T)$, we can define the integrals

$$\int_0^T \int_0^\tau \phi(t,s) ds dB_t, \quad \int_0^\tau \int_0^T \phi(t,s) dB_t ds,$$

and

$$\int_0^T \phi(t,s) dB_t \quad \text{for almost every } s \in [0,\tau].$$

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Let $\phi \in \tilde{M}_G^2(0,T)$. Then it holds

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Sufficient Conditions

- Let $\phi : [0, T] \times [0, \tau] \to \mathbb{R}$ be a (deterministic) function such that

$$\phi \in \mathcal{L}^2([0, T] \times [0, \tau]).$$

Then we have $\phi \in \tilde{M}_G^2(0, T)$.

- Let

$$\phi(t, s) = \eta_t \psi(s)$$

for $\eta \in M_G^2(0, T)$ and $\psi \in \mathcal{L}^2([0, \tau])$. Then it holds $\phi \in \tilde{M}_G^2(0, T)$.

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Forward Rate

- For $t \leq T \leq \tau$, we denote the forward rate by $f(t, T)$.
- The evolution of the forward rate is described by the dynamics

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \beta(s,T)dB_s + \int_0^t \gamma(s,T)d\langle B\rangle_s$$

for some initial integrable forward curve $T \to f(0, T)$.
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Bond Market

- The market offers zero-coupon bonds for all maturities $T \in [0, \tau]$.
- The price at time $t \leq T$ of such a bond is given by

$$P(t,T) := \exp\Big(-\int_t^T f(t,s)ds\Big).$$

- In addition, there is the money-market account

$$M_t := \exp\Big(\int_0^t r_s ds\Big).$$

- The money-market account is mainly used for discounting, i.e.,

$$\tilde{P}(t,T) := M_t^{-1} P(t,T).$$

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Assumptions

Assumption 1 (Regularity of the Forward and Short Rate)

We assume that $\alpha, \beta, \gamma \in \tilde{M}_G^2(0, \tau)$.

- Furthermore, we define the processes a, b, and c by

$$a(t,T) := \int_t^T \alpha(t,s)ds, \quad b(t,T) := \int_t^T \beta(t,s)ds$$

$$c(t,T) := \int_t^T \gamma(t,s)ds.$$

- We have $a(\cdot, T), b(\cdot, T), c(\cdot, T) \in M_G^2(0, \tau)$ for all $T \in [0, \tau]$.

Assumption 2 (Regularity of the Discounted Bonds)

We assume that $b(\cdot, T)^2 \in M_G^2(0, \tau)$ for all $T \in [0, \tau]$.

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Dynamics of the Discounted Bond

Lemma 2.1

The integral of the forward rate satisfies the dynamics

$$\int_t^T f(t,u)du = \int_0^T f(0,u)du + \int_0^t \big(a(u,T) - r_u\big)du + \int_0^t b(u,T)dB_u + \int_0^t c(u,T)d\langle B\rangle_u.$$

Proposition 2.1

The discounted bond price process satisfies the G-SDE

$$\tilde{P}(t,T) = \tilde{P}(0,T) - \int_0^t a(u,T)\tilde{P}(u,T)du - \int_0^t b(u,T)\tilde{P}(u,T)dB_u - \int_0^t \big(c(u,T) - \tfrac{1}{2}b(u,T)^2\big)\tilde{P}(u,T)d\langle B\rangle_u.$$

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Market Structure

Definition 2.1

An admissible market strategy is a process $\pi \in \tilde{M}^2_G(0,\tau)$ such that $\pi a \tilde{P} \in \tilde{M}^1_G(0,\tau)$, $\pi b \tilde{P} \in \tilde{M}^2_G(0,\tau)$, and $\pi(c - \frac{1}{2}b^2)\tilde{P} \in \tilde{M}^1_G(0,\tau)$. The corresponding portfolio value process $(\tilde{v}_t(\pi))_t$ is given by

$$\tilde{v}_t(\pi) = \int_0^{\tau} \int_0^{t \wedge T} \pi(s,T) d\tilde{P}(s,T) dT.$$

Definition 2.2

An admissible market strategy π is called arbitrage strategy if it holds

$$\tilde{v}_\tau(\pi) \geq 0 \quad \text{q.s.} \quad \text{and} \quad P\big(\tilde{v}_\tau(\pi) > 0\big) > 0 \quad \text{for at least one } P \in \mathcal{P}.$$

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Girsanov Transformation

- Let $(B_t, \tilde{B}_t)_t$ be a 2-dimensional G-Brownian motion on the extended $\tilde{G}$-expectation space $(\tilde{\Omega}_\tau, L^1_{\tilde{G}}(\tilde{\Omega}_\tau), \hat{\mathbb{E}}^{\tilde{G}})$ such that

$$\langle B, \tilde{B} \rangle_t = t.$$

- By Hu, Ji, Peng, and Song (2014), we know that

$$\bar{B}_t := B_t - \int_0^t \kappa_s ds - \int_0^t \lambda_s d\langle B \rangle_s$$

is a G-Brownian motion under $\tilde{\mathbb{E}}$, where $\tilde{\mathbb{E}}(\cdot) := \hat{\mathbb{E}}^{\tilde{G}}(\mathcal{E} \cdot)$ and

$$\mathcal{E} = \exp\Big(\int_0^\tau \lambda_t dB_t + \int_0^\tau \kappa_t d\tilde{B}_t - \frac{1}{2} \int_0^\tau \lambda_t^2 d\langle B \rangle_t - \int_0^\tau \lambda_t \kappa_t dt - \frac{1}{2} \int_0^\tau \kappa_t^2 d\langle \tilde{B} \rangle_t \Big).$$

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Drift Condition

Theorem 2.1

Suppose that the processes κ and λ satisfy the drift condition

$$a(t,T) + b(t,T)\kappa_t = 0,$$
$$c(t,T) - \tfrac{1}{2}b(t,T)^2 + b(t,T)\lambda_t = 0.$$

Then the discounted bond price process $(\tilde{P}(t,T))_t$ is a symmetric G-martingale under $\tilde{\mathbb{E}}$ and the forward rate satisfies

$$f(t,T) = f(0,T) + \int_0^t \beta(s,T)\,d\bar{B}_s + \int_0^t \beta(s,T)b(s,T)\,d\langle B\rangle_s.$$

Corollary 2.1

If the drift condition is satisfied, then the market is arbitrage-free.

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$$f(t,T) = f(0,T) + \int_0^t \beta(s,T)d\bar{B}_s + \int_0^t \beta(s,T)b(s,T)d\langle B\rangle_s.$$

Corollary 2.1

If the drift condition is satisfied, then the market is arbitrage-free.

Consistency Check

- If there is no volatility uncertainty, that is, $\overline{\sigma} = 1 = \underline{\sigma}$, we have

$$\langle B \rangle_t = t$$

and $(B_t)_t$ is a standard Brownian motion.

- Then the dynamics of the forward rate are given by

$$f(t,T) = f(0,T) + \int_0^t \beta(s,T)dB_s + \int_0^t \left(\alpha(s,T) + \gamma(s,T)\right)ds.$$

- The drift condition implies

$$a(t,T) + c(t,T) - \tfrac{1}{2}b(t,T)^2 + b(t,T)(\kappa_t + \lambda_t) = 0.$$

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Model Framework

Forward Rate Model

Examples

Conclusion

Ho-Lee Model

- Suppose that $\beta(t, T) = \sigma$.
- Then the risk-neutral dynamics of the forward rate are given by

$$f(t, T) = f(0, T) + \sigma \bar{B}_t + \sigma^2 \int_0^t (T - s) d\langle B \rangle_s.$$

- Moreover, we can derive the related short rate dynamics,

$$r_t = r_0 + \int_0^t \big(\partial_u f(0, u) + \sigma^2 \langle B \rangle_u\big) du + \sigma \bar{B}_t.$$

- In this case, the bond prices are given by

$$P(t, T) = \exp\Big(- \int_t^T f(0, u) du + (T - t) f(0, t) + \frac{\sigma^2}{2} \langle B \rangle_t (T - t)^2 - (T - t) r_t \Big).$$

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Hull-White Model

- Suppose that $\beta(t, T) = \sigma e^{-\theta(T-t)}$.
- Then we obtain

$$r_t = r_0 + \int_0^t \Big(\partial_u f(0,u) + \theta f(0,u) + \sigma^2 \int_0^u e^{-2\theta(u-s)} d\langle B \rangle_s - \theta r_u \Big) du + \sigma \bar{B}_t.$$

- The bond prices are now given by

$$P(t,T) = \exp\Big(- \int_t^T f(0,u) du + \tfrac{1}{\theta}\big(1 - e^{-\theta(T-t)}\big) f(0,t) - \tfrac{\sigma^2}{2\theta^2}\big(1 - e^{-\theta(T-t)}\big)^2 \int_0^t e^{-2\theta(t-s)} d\langle B \rangle_s - \tfrac{1}{\theta}\big(1 - e^{-\theta(T-t)}\big) r_t \Big).$$

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Vasicek Model

- The Vasicek model has the same volatility structure but a constant mean reversion level.
- Hence, we set $\beta(t, T) = \sigma e^{-\theta(T-t)}$ and, for $\mu \in \mathbb{R}$, it has to hold

$$\partial_t f(0,t) + \theta f(0,t) + \sigma^2 \int_0^t e^{-2\theta(t-s)} d\langle B\rangle_s = \mu.$$

- Therefore, we cannot obtain the classical Vasicek model, since

$$\sigma^2 \int_0^t e^{-2\theta(t-s)} d\langle B\rangle_s \neq \mu - \partial_t f(0,t) - \theta f(0,t).$$

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  - → Integration w.r.t. Two Integrators
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