Stochastic Volatility Models for VIX Option Pricing

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Motivation and Context

- Challenging to price VIX options
  VIX: mean reversion, jump and stochastic volatility
  VIX options: upward volatility skew
- Absence of a widely accepted model for VIX option pricing: relatively large bid-ask spread
- Stochastic volatility models are needed
- Adopt the framework proposed by Ballotta and Rayée (2017) based on time changed Lévy process (Carr and Wu 2004), and examine the performance of six representative models on VIX options. These models vary in:
  1. Number of stochastic volatility factors
  2. Jump-diffusion structure
  3. Sources of leverage effect

Outline

Model Framework

- Consider VIX of the form $S(t) = S(0)e^{(r-q)t+X(t)}$
- X(t) can be composed into time-unchanged and time-changed parts: $X(t) = L_0(t) + L_1(T(t))$
- Assume no other time-unchanged part apart from the instantaneous drift: $L_0(t) = 0$
- Stochastic clock: $T_i(t) = \int_0^t v_i(s)ds$, $i = 1, 2$
  where $v_i(s)$ is the activity rate process.

Model Specifications (1)

- Heston (Heston 1993)

$$dv(t) = k(\theta - v(t))dt + \eta\sqrt{v(t)}dZ_t$$

$$dX(t) = -\frac{1}{2}v(t)dt + \sqrt{v(t)}dW_t$$

$$d\langle W_t, Z_t\rangle = \rho dt$$

- Two-factor Stochastic Volatility Model (TFSV, Christoffersen, Heston, and Jacobs 2009)

$$dv_i(t) = k_i(\theta_i - v_i(t))dt + \eta_i\sqrt{v_i(t)}dZ_{it}$$

$$dX(t) = -\frac{1}{2}v_1(t)dt - \frac{1}{2}v_2(t)dt + \sqrt{v_1(t)}dW_{1t} + \sqrt{v_2(t)}dW_{2t}$$

$$d\langle W_i, Z_i\rangle = \rho_i dt, \quad i = 1, 2$$

Model Specifications (2)

- VGDSV3 (Huang and Wu 2004)

$$dv(t) = k(\theta - v(t))dt + \eta\sqrt{v(t)}dZ_t$$

$$dX(t) = -[\frac{1}{2} + \varphi_J(-i)]v(t)dt + \sqrt{v(t)}dW_t + dJ_{T(t)}$$

$$d\langle W_t, Z_t\rangle = \rho dt$$

- VGDSV4 (Huang and Wu 2004)

$$dv_i(t) = k_i(\theta_i - v_i(t))dt + \eta_i\sqrt{v_i(t)}dZ_{it}$$

$$dX(t) = -\frac{1}{2}v_1(t)dt - \varphi_J(-i)v_2(t)dt + \sqrt{v_1(t)}dW_t + dJ_{T_2(t)}$$

$$d\langle W_i, Z_i\rangle = \rho_i dt, \quad i = 1, 2$$

Model Specifications (3)

- JH (Ballotta and Rayée, 2017)

$$dv(t) = k(\theta - v(t))dt + \eta_J dJ_{+,T(t)}$$

$$dX(t) = -(\varphi_{J,+}(-i) + \varphi_{J,-}(-i))v(t)dt + dJ_{T(t)}$$

- 1SVFSE (Ballotta and Rayée, 2017)

$$dv(t) = k(\theta - v(t))dt + \eta_D \sqrt{v(t)}dZ_t + \eta_J dJ_{+,T(t)}$$

$$dX(t) = -(\frac{1}{2} + \varphi_{J,+}(-i) + \varphi_{J,-}(-i))v(t)dt + \sqrt{v(t)}dW_t + dJ_{T(t)}$$

$$d\langle W_t, Z_t\rangle = \rho dt$$

Model Comparison

Model	Volatility Factor(s)	Sources of SV	Sources of LE
Heston	Diffusion	1	Diffusion
TFSV	Diffusion+ Diffusion	2	Diffusion
VGDSV3	Diffusion+Jump	1	Diffusion
VGDSV4	Diffusion+Jump	2	Diffusion
JH	Jump	1	Jump
1SVFSE	Diffusion+Jump	1	Diffusion+Jump

Table: Comparison between different models

COS Method: Characteristic Function

For option pricing, we apply COS method developed by Fang and Oosterlee (2008).

- Characteristic function
  1. For Heston, TFSV, VGDSV3 and VGDSV4, characteristic functions are available in closed form
  2. For JH and 1SVFSE, Riccati-type ODEs have to be solved numerically

COS Method: Truncation Range

- For Heston and TFSV model, cumulants are in closed form, and we use
$$[a, b] = [c_1 - L\sqrt{|c_2|}, c_1 + L\sqrt{|c_2|}], \quad where\ L = 12.$$
- For VGDSV3, VGDSV4, JH and 1SVFSE, we use

$$[a, b] = [c_1 - L\sqrt{|c_2| + \sqrt{|c_4|}}, c_1 + L\sqrt{|c_2| + \sqrt{|c_4|}}], \quad where\ L = 10.$$

where $c_n = i^{-n} \frac{\partial^n ln\phi_X(u)}{\partial u^n}|_{u=0}$.
Use centred difference method to approximate derivatives

Estimation Methodology

- Two groups of parameters: spot variance ($\{V\}$) and structural parameters ($\{\Theta\}$)
- Loss function: implied volatility root mean squared error (IVRMSE)

$$IVRMSE \equiv \sqrt{\frac{1}{N}\sum_{j=1}^{N}(C_{j,t} - C(\Theta, V_i))^2 / Vega_{j,t}^2}$$

- Iterative two-step method:
  1. $\{\hat{V}\} = argmin \sum_{j=1}^{N}(C_{j,t} - C(\Theta, V_i))^2 / Vega_{j,t}^2$
  2. $\{\hat{\Theta}\} = argmin \sum_{j=1}^{N}(C_{j,t} - C(\Theta, V_i))^2 / Vega_{j,t}^2$
- Local optimisation method, randomize initial parameters and conduct calibration repeatedly

Estimation Results: Heston and TFSV

Parameter	Heston Estimates	Parameter	TFSV Estimates
V	0.593	(V_1, V_2)	$(0.479, 2.977 \times 10^{-6})$
η	4.175	(η_1, η_2)	$(4.05, 3.642)$
ρ	0.942	(ρ_1, ρ_2)	$(0.956, 1.000)$
k	9.235	(k_1, k_2)	$(4.357, 17.029)$
θ	0.381	(θ_1, θ_2)	$(4.892 \times 10^{-15}, 0.256)$
IVRMSE	0.020	IVRMSE	0.0157

Table: Results for Heston and TFSV

Density Recovery: Heston and TFSV

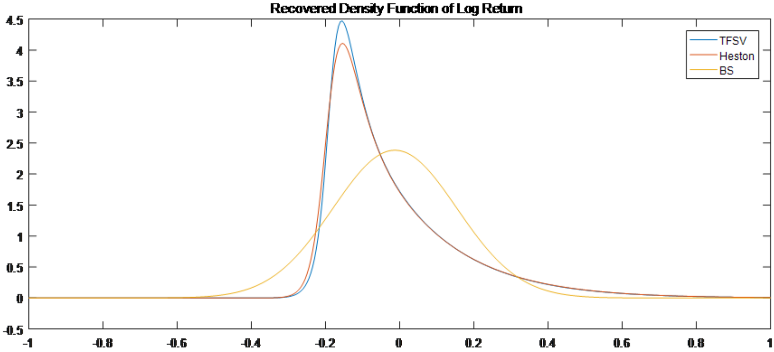

Figure: Recovered Density Function for TFSV, Heston and BS

Estimation Results: VGDSV3 and VGDSV4

Parameter	VGDSV3 Estimates	Parameter	VGDSV4 Estimates
V	0.1916	(V_1, V_2)	$(0.265, 0.066)$
η	0.6952	(η_1, η_2)	$(1.971, 0.007)$
ρ	0.9951	ρ	1.000
θ	0.0634	(θ_1, θ_2)	$(0.214, 6.494 \times 10^{-4})$
k	6.3870	(k_1, k_2)	$(7.201, 6.160)$
α	0.3505	α	0.583
σ_j	0.0436	σ_j	0.101
λ	16.3208	λ	12.425
IVRMSE	0.0121	IVRMSE	0.0124

Table: Results for VGDSV3 and VGDSV4

Density Recovery: VGDSV3 and VGDSV4

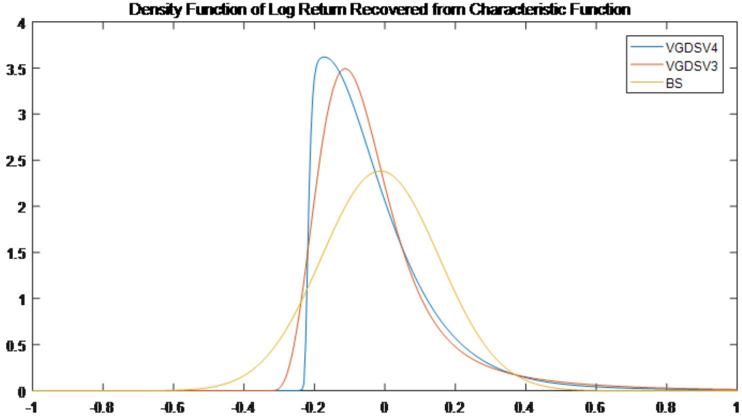

Figure: Recovered Density Function for VGDSV3, VGDSV4 and BS

Estimation Results: JH and 1SVFSE

Parameter	JH Estimates	Parameter	1SVFSE Estimates
V	0.8175	V	0.1254
η_J	2.0329	(η_D, η_J)	$(0.4120, 0.0414)$
k	12.7204	k	5.4806
θ	0.1284	θ	0.0315
C	23.2738	C	35.3096
M	5.9798	M	3.1906
G	33.3420	G	59.1172
		ρ	1.0000
IVRMSE	0.0159	IVRMSE	0.0128

Table: Results for JH and 1SVFSE

Density Recovery: JH and 1SVFSE

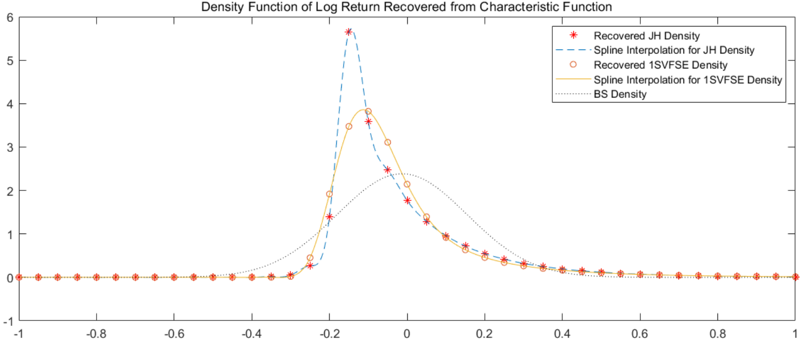

Figure: Recovered Density Function for JH, 1SVFSE and BS

Out-of-sample Results: Overall

Model	Out-of-sample IVRMSE
1SVFSE	0.0217
VGDSV3	0.0232
JH	0.0280
VGDSV4	0.0303
Heston	0.0385
TFSV	0.0460

Table: Overall Out-of-sample IVRMSE (date: July 20, 2017)

Out-of-sample Results: Moneyness

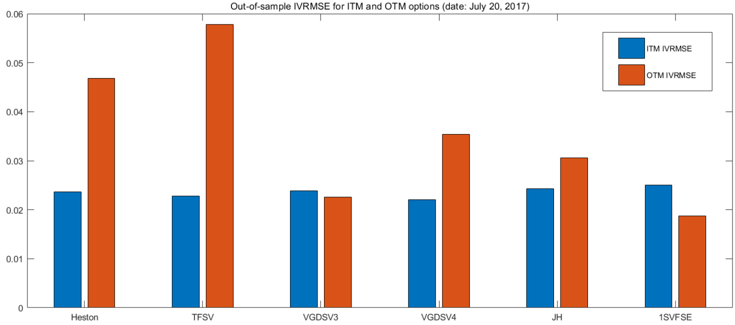

Figure: Out-of-sample IVRMSE for ITM and OTM options

Out-of-sample Results: Maturity

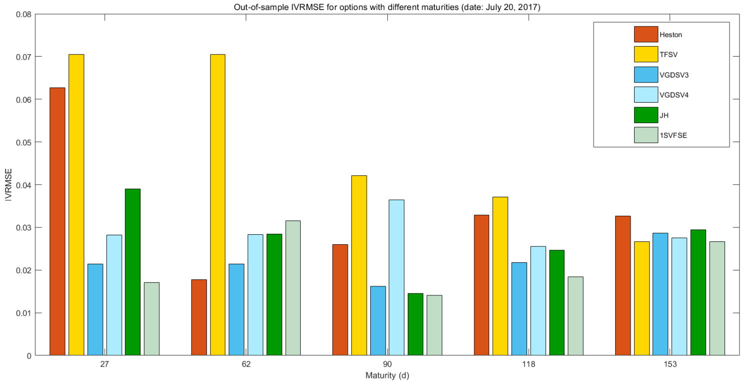

Figure: Out-of-sample IVRMSE for options with different maturities

Conclusion

- Both jump and diffusion are necessary in the underlying log return process, a second diffusion component is redundant.
- Pure-jump model is feasible, and better than its pure-diffusion counterpart. But jump-diffusion models can be even better.
- One source of stochastic volatility is enough for pricing VIX options.
- Since most of the leverage effect is generated by correlated diffusion parts, 1SVFSE model is quite close to VGDSV3 model.
- Due to the difficulty in calibrating 1SVFSE model, we would recommend VGDSV3 model, which is characterized by one stochastic volatility factor, risk factors of both diffusive and jump nature, and dependence between diffusions.

A few selected references

Bakshi, G., C. Cao, and Z. Chen (1997)
Empirical Performance of Alternative Option Pricing Models
The Journal of Finance 52(5), p. 2003.

Ballotta, L. and G. Rayée (2017)
Smiles & Smirks: A Tale of Factors
SSRN Electronic Journal.

Carr, P., H. Geman, et al. (2003)
Stochastic volatility Lévy processes
Mathematical Finance 13, pp. 345-382.

Carr, P. and L. Wu (2004)
Time-changed Lévy processes and option pricing
Journal of Financial Economics 71(1), pp. 113-141.

CBOE (2009)
VIX, White Paper
Chicago Board Options Exchange.

A few selected references

- Christoffersen, P., S. Heston, and K. Jacobs (2009)
  The shape and term structure of the index option smirk: Why multifactor stochastic volatility models work so well
  Management Science 55, pp. 1914-1932.

- Fang, F. and C.W. Oosterlee (2008)
  A Novel Pricing Method for European Options Based on Fourier-Cosine Series Expansions
  SIAM Journal on Scientific Computing 31, pp. 826-848.

- Huang, J. and L. Wu (2004)
  Specification Analysis of Option Pricing Models Based on Time-Changed Lévy Processes
  The Journal of Finance 59(3), pp. 1405-1439.

- Madan, D., P. Carr, and E. Chang (1998)
  The Variance Gamma Process and Option Pricing
  Review of Finance 2(1), pp. 79-105.

A few selected references

Marabel Romo, J. (2015)
Pricing volatility options under stochastic skew with application to the VIX index
Pricing volatility options under stochastic skew with application to the VIX index 23(4), pp. 353-374.

Sepp, A. (2008)
VIX Option Pricing in a Jump-diffusion Model
Risk 21, pp. 84-89.

Wetterau, D. and J. Kienitz (2013)
Financial modelling
Hoboken, N.J.: Wiley.