High Frequency Data: Theory and Applications

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Plan of the course

- Chapter 0: Motivation and overview
- Chapter 1: Definition of stable convergence and its properties
- Chapter 2: Some basic facts about semimartingale theory
- Chapter 3: Stable convergence theorem of Jacod
- Chapter 4: High frequency observations of Itô semimartingales
- Chapter 5: Optimal estimation of volatility functionals
- Chapter 6: New optimality results for supremum, local times and occupation time measure

Chapter 0: Motivation and overview

The setting

- In this course we are focusing on 1-dimensional **Itô semimartingales** of the form

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s + J_t \qquad t \geq 0$$

  defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Here
  - W is a standard Brownian motion
  - a is the **drift process**
  - σ is the **volatility process**
  - J is a **compound Poisson process**
- We observe **high frequency data**

$$X_0, X_{\Delta_n}, X_{2\Delta_n}, \ldots, X_{\lfloor T/\Delta_n \rfloor \Delta_n}$$

  where $T > 0$ is **fixed** and $\Delta_n \to 0$.

Example: Asset prices in financial markets

Low vs. high frequency data

Low frequency data	*High frequency data*
Observed data	***Observed data***
$X_1, X_2, ..., X_n \;\; i.i.d. \sim F$	$X_0(\omega), X_{\Delta_n}(\omega), ..., X_{\lfloor T/\Delta_n \rfloor \Delta_n}(\omega)$
Asymptotic knowledge	***Asymptotic knowledge***
distribution function F	$(X_t(\omega))_{t \in [0,T]}$
Identifiable objects	***Identifiable objects***
functionals of F	functionals of $(X_t(\omega))_{t \in [0,T]}$

A list of classical estimation and testing problems

- How to estimate the **quadratic variation**

$$[X]_T = \int_0^T \sigma_s^2 ds + \sum_{s \in [0,T]} (\Delta J_s)^2 \qquad \Delta J_s := J_s - J_{s-}$$

  of X?

- How to distinguish the volatility and the jump part of the quadratic variation?
- Are jumps present in the price process?
- Is the Brownian part present in the price process?
- What are jump robust measures of the volatility?
- What are **optimal** measures of the volatility?

A toy example

- Let us consider a simple model

$$X_t = X_0 + at + \sigma W_t + J_s$$

where $a \in \mathbb{R}$, $\sigma > 0$ are constants and J is a Poisson process with arrival rate $\lambda > 0$. Assume that we observe the process X on the interval $[0, 1]$.

- We can identify: the volatility σ^2, the realised jumps $(\Delta J_s)_{s\in[0,1]}$ and the quadratic variation process $([X]_s)_{s\in[0,1]}$.

- We can **not** identify: the drift a and the arrival rate λ.

Main goals of the course

- In this lecture we will study asymptotic behaviour of statistics of the type

$$V(X, f, \Delta_n)_t = a_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} f\left(b_n(X_{i\Delta_n} - X_{(i-1)\Delta_n})\right)$$

where $f : \mathbb{R} \to \mathbb{R}$ is a measurable function and a_n, b_n are certain deterministic sequences. A particularly important case is $f(x) = |x|^p$, which is often referred to as **power variation**.

- The probably most famous example is the following theorem:

Theorem

Assume that X is continuous. Then it holds that

$$\Delta_n^{-1/2}\left(\sum_{i=1}^{\lfloor t/\Delta_n \rfloor}\left(X_{i\Delta_n} - X_{(i-1)\Delta_n}\right)^2 - [X]_t\right) \xrightarrow{d_{st}} \sqrt{2}\int_0^t \sigma_s^2 dW'_s$$

where W' is a new Brownian motion independent of $\mathcal{F}$.

Selected milestones in the theory of high frequency data

- Jacod (94): *Limit of random measures associated with the increments of a Brownian semimartingale.*
- Jacod (97): *On continuous conditional Gaussian martingales and stable convergence in law.*
- Mancini (01): *Disentangling the jumps of the diffusion in a geometric jumping Brownian motion.*
- Barndorff–Nielsen & Shephard (02): *Econometric analysis of realised volatility and its use in estimating stochastic volatility models.*
- Andersen, Bollerslev, Diebold & Labys (03): *Modeling and forecasting realized volatility.*
- Barndorff–Nielsen, Graversen, Jacod, P. & Shephard (06): *A central limit theorem for realised power and bipower variations of continuous semimartingales.*

Chapter 1: Definition of stable convergence and its properties

Definition

We say that a random variable Y has a **mixed normal distribution** with conditional mean 0 and conditional variance $V^2 > 0$ if

$$Y = V \cdot N \quad \text{with} \quad N \sim \mathcal{N}(0, 1) \quad \text{and} \quad V \perp\!\!\!\perp N$$

In this case we write $Y \sim \mathcal{MN}(0, V^2)$.

- In high frequency setting we often obtain convergence in distribution

$$Y_n \xrightarrow{d} Y \sim \mathcal{MN}(0, V^2)$$

To construct confidence regions we would like to have that

$$Y_n / V \xrightarrow{d} \mathcal{N}(0, 1)$$

- This does not work in general, since

$$Y_n \xrightarrow{d} Y \quad \not\Rightarrow \quad (Y_n, V) \xrightarrow{d} (Y, V)$$

Another example

- When estimating a **random parameter** Q, we will often show that

$$a_n(Q_n - Q) \xrightarrow{d} Y \qquad \text{with} \qquad a_n \to \infty$$

Assume that we are rather interested in $g(Q)$ for some function $g \in C^1(\mathbb{R})$. In this case we want to obtain the δ-**method**, i.e.

$$a_n\left(g(Q_n) - g(Q)\right) \xrightarrow{d} g'(Q)Y$$

- However, this again does not work in general, since

$$(a_n(Q_n - Q), Q) \xrightarrow{d} (Y, Q)$$

is not guaranteed!

Our goal

Define a new weak type convergence $Y_n \to Y$ that guarantees the joint convergence

$$(Y_n, Z) \xrightarrow{d} (Y, Z)$$

for all $\mathcal{F}$-measurable random variables Z.

Stable convergence

Definition (Renyi 63)

Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of real valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We say that $(Y_n)_{n\in\mathbb{N}}$ converges **stably in law** towards Y, defined on the extended space $(\Omega', \mathcal{F}', \mathbb{P}')$, if

$$\mathbb{P}(\{Y_n \leq y\} \cap B) \to \mathbb{P}'(\{Y \leq y\} \cap B) \qquad \text{as } n \to \infty$$

holds for any $B \in \mathcal{F}$ and any dense countable set of points $y \in \mathbb{R}$. We write

$$Y_n \xrightarrow{d_{st}} Y$$

- Stable convergence is a property of Y_n not only $\mathbb{P}^{Y_n}$! Let $Y_n = Y \sim \mathcal{N}(0,1)$, then

$$Y_n \xrightarrow{d} -Y \qquad \text{but} \qquad Y_n \overset{d_{st}}{\not\to} -Y$$

Corollary

It holds that

(i) $Y_n \xrightarrow{d_{st}} Y$ *implies* $Y_n \xrightarrow{d} Y$.

(ii) $Y_n \xrightarrow{\mathbb{P}} Y$ *implies* $Y_n \xrightarrow{d_{st}} Y$.

Proof: To show (i) take $B = \Omega$. To prove (ii) observe that

$$(Y_n, 1_B) \xrightarrow{\mathbb{P}} (Y, 1_B)$$

and hence we readily deduce that $(Y_n, 1_B) \xrightarrow{d} (Y, 1_B)$. This immediately implies that $Y_n \xrightarrow{d_{st}} Y$.

□

A useful characterisation

Proposition

The following statements are equivalent

(i) $Y_n \xrightarrow{d_{st}} Y$.

(ii) $(Y_n, Z) \xrightarrow{d} (Y, Z)$ *for all* $\mathcal{F}$*-measurable rv's* Z.

(iii) $(Y_n, Z) \xrightarrow{d_{st}} (Y, Z)$ *for all* $\mathcal{F}$*-measurable rv's* Z.

Proof: (i)⇒(ii) Let A be a measurable set. Then it holds that

$$\mathbb{P}(Y_n \leq y, Z \in A) \rightarrow \mathbb{P}'(Y \leq y, Z \in A)$$

and hence $(Y_n, Z) \xrightarrow{d} (Y, Z)$. □

(ii)⇒(i) Assume that $(Y_n, Z) \xrightarrow{d} (Y, Z)$ for all $\mathcal{F}$-measurable rv's Z. For any $A \in \mathcal{F}$, set $Z = 1_A$. Then it holds that

$$\mathbb{P}(Y_n \leq y, \{Z = 1\}) \rightarrow \mathbb{P}'(Y \leq y, \{Z = 1\})$$

which implies the stable convergence $Y_n \xrightarrow{d_{st}} Y$. □

End of the proof

(ii)⇒(iii) Let Z', Z'' be $\mathcal{F}$-measurable rv's and set $Z = (Z', Z'')$. By assumption we know that

$$((Y_n, Z'), Z'') \xrightarrow{d} ((Y, Z'), Z'')$$

But we just showed that this implies the stable convergence

$$(Y, n, Z') \xrightarrow{d_{st}} (Y, Z')$$

which completes the proof.
□

Why do we need an extension of the probability space?

When we have a true stable convergence we always need to extend the original probability space, as the next lemma demonstrates.

Lemma

Assume that $Y_n \xrightarrow{d_{st}} Y$ and the rv Y is $\mathcal{F}$-measurable. Then it holds that

$$Y_n \xrightarrow{\mathbb{P}} Y$$

Proof: Since Y is $\mathcal{F}$-measurable, we conclude by the previous proposition

$$(Y_n, Y) \xrightarrow{d} (Y, Y)$$

and hence by continuous mapping theorem

$$Y_n - Y \xrightarrow{d} 0$$

But this implies the convergence $Y_n \xrightarrow{\mathbb{P}} Y$.

□

Classical central limit theorem

To better understand the nature of stable convergence, we consider the classical central limit theorem. Assume that $(X_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. rv's with $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1^2] = 1$. Define the statistic

$$S_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \xrightarrow{d} \mathcal{N}(0,1)$$

Theorem

Assume that $\mathcal{F} = \sigma\{X_n : \ n \in \mathbb{N}\}$. *Then it holds that*

$$S_n \xrightarrow{d_{st}} S \sim \mathcal{N}(0,1)$$

where S *is* ***independent*** *of* $\mathcal{F}$.

Proof: Recall that it suffices to show that

$$(S_n, Z) \xrightarrow{d} (S, Z)$$

for all $\mathcal{F}$-measurable rv's Z. Since $\mathcal{F} = \sigma\{X_n : \; n \in \mathbb{N}\}$ it is enough to prove the joint convergence

$$(S_n, X_1, \ldots, X_k) \xrightarrow{d} (S, X_1, \ldots, X_k)$$

for any fixed $k \in \mathbb{N}$. Define

$$S_n^k := n^{-1/2} \sum_{i=k+1}^{n} X_i$$

Then $S_n^k \perp\!\!\!\perp (X_1, \ldots, X_k)$ and we readily deduce that

$$(S_n^k, X_1, \ldots, X_k) \xrightarrow{d} (S, X_1, \ldots, X_k)$$

On the other hand, we have that $S_n - S_n^k \xrightarrow{\mathbb{P}} 0$, which completes the proof of the theorem.

□

Note that we need the assumption $\mathcal{F} = \sigma\{X_n : \; n \in \mathbb{N}\}$ to perform the last proof. What if the σ-field $\mathcal{F}$ is much richer? As the next lemma shows, the main result remains valid.

Lemma (Aldous & Eagleson 78)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an arbitrary probability space. If the convergence

$$(Y_n, Y_1, \ldots, Y_k) \xrightarrow{d} (Y, Y_1, \ldots, Y_k)$$

holds for all $k \in \mathbb{N}$ and $Y \perp\!\!\!\perp (Y_1, Y_2, \ldots)$, then we obtain the stable convergence

$$Y_n \xrightarrow{d_{st}} Y$$

and Y is independent of $\mathcal{F}$.

Theorem (Tukey 38)

Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of $x \in \mathbb{R}$. Consider a rv T with a Lebesgue density f. Then it holds that

$$\{nT\} \overset{d_{st}}{\to} U \sim \mathcal{U}(0,1)$$

and $U \perp\!\!\!\perp \mathcal{F}$.

Sketch of proof: According to the previous lemma, it suffices to prove that

$$\mathbb{P}(T \in [a,b], \{nT\} \in [0,c]) \to c\mathbb{P}(T \in [a,b])$$

for any $c \in [0,1]$. We have

$$\mathbb{P}(T \in [a,b], \{nT\} \in [0,c]) = \int_a^b 1_{[0,c]}(\{nx\}) f(x) dx$$

$$= \sum_{i=\lfloor na \rfloor}^{\lfloor nb \rfloor} \int_{\frac{i-1}{n}}^{\frac{i-1+c}{n}} f(x) dx + o(1)$$

Cont.

$$= \frac{c}{n} \sum_{i=\lfloor na \rfloor}^{\lfloor nb \rfloor} f(\xi_{i,n}) dx + o(1) \to c \int_a^b f(x) dx = c\mathbb{P}(T \in [a, b]).$$

□

Proposition

Assume that $Y_n \xrightarrow{d_{st}} Y$.

(i) *If* $Z_n \xrightarrow{\mathbb{P}} Z$ *then it holds that*

$$(Y_n, Z_n) \xrightarrow{d_{st}} (Y, Z)$$

(ii) *Continuous mapping theorem: If* $g \in C(\mathbb{R})$ *then it holds*

$$g(Y_n) \xrightarrow{d_{st}} g(Y)$$

Proof: (i) For any $\mathcal{F}$-measurable rv X we have $(Y_n, Z, X) \xrightarrow{d} (Y, Z, X)$. Since $Z_n \xrightarrow{\mathbb{P}} Z$, we deduce that

$$(Y_n, Z_n, X) \xrightarrow{d} (Y, Z, X)$$

But this means $(Y_n, Z_n) \xrightarrow{d_{st}} (Y, Z)$. □

Cont.

(ii) Similarly, for any $\mathcal{F}$-measurable rv X, we have $(Y_n, X) \xrightarrow{d} (Y, X)$. Applying the classical continuous mappring theorem for convergence in law, we deduce that

$$(g(Y_n), X) \xrightarrow{d} (g(Y), X)$$

Hence, we obtain $g(Y_n) \xrightarrow{d_{st}} g(Y)$. $\square$

- In practical applications we will often obtain the stable convergence

  $$Y_n \xrightarrow{d_{st}} Y \sim \mathcal{MN}(0, V^2)$$

  If we find a rv $V_n > 0$ such that $V_n^2 \xrightarrow{\mathbb{P}} V^2$, then by the previous proposition we conclude that

  $$Y_n / V_n \xrightarrow{d} \mathcal{N}(0, 1)$$

  The latter result is very important for statistical inference.

Theorem

Assume that

$$a_n(X_n - X) \stackrel{d_{st}}{\to} Y$$

for some sequence $a_n \to \infty$ *and let* $g \in C^1(\mathbb{R})$. *Then it holds that*

$$a_n\left(g(X_n) - g(X)\right) \stackrel{d_{st}}{\to} g'(X)Y$$

Proof: The mean value theorem implies the identity

$$a_n\left(g(X_n) - g(X)\right) = a_n g'(Z_n)(X_n - X)$$

with $|Z_n - X| \leq |X_n - X|$, and hence $Z_n \stackrel{\mathbb{P}}{\to} X$. Hence, we deduce the convergence

$$(a_n(X_n - X), Z_n) \stackrel{d_{st}}{\to} (Y, X)$$

But this implies the desired stable convergence by the continuous mapping theorem. $\square$

Definition

Let $(Y^n)_{n\in\mathbb{N}}$ be a sequence of stochastic processes taking values in a *Polish space* $(E,\mathcal{E})$ (a separable completely metrizable space). We say that $(Y^n)_{n\in\mathbb{N}}$ converges stable in law towards Y, defined on an extended probability space $(\Omega',\mathcal{F}',\mathbb{P}')$, if

$$\lim_{n\to\infty}\mathbb{E}[g(Y^n)Z]=\mathbb{E}'[g(Y)Z]$$

holds for any continuous bounded function $g:E\to\mathbb{R}$ and any bounded $\mathcal{F}$-measurable rv Z. In this case we write $Y^n\stackrel{d_{st}}{\to}Y$.

We will often deal with the space $(C[0,T],\|\cdot\|_\infty)$. In this situation the stable convergence $Y^n\stackrel{d_{st}}{\to}Y$ is equivalent to finite dimensional convergence

$$\left(Y^n_{t_1},\ldots,Y^n_{t_k}\right)\stackrel{d_{st}}{\to}\left(Y_{t_1},\ldots,Y_{t_k}\right)\qquad\forall t_j\in[0,T]$$

and tightness of the sequence $(Y^n)_{n\in\mathbb{N}}$.

Example: Donsker's invariance principle

Theorem

Assume that $(X_n)_{n\in\mathbb{N}}$ *is a sequence of i.i.d. rv's with* $\mathbb{E}[X_1] = 0$ *and* $\mathbb{E}[X_1^2] = 1$*. Define the statistic*

$$S_t^n := \frac{1}{\sqrt{n}} \left(\sum_{i=1}^{\lfloor nt \rfloor} X_i + (nt - \lfloor nt \rfloor) X_{\lfloor nt \rfloor + 1} \right)$$

Then it holds that

$$S^n \overset{d_{st}}{\to} W' \qquad \text{on } (C[0,T], \|\cdot\|_\infty)$$

where W' *is a Brownian motion defined on an extended space* $(\Omega', \mathcal{F}', \mathbb{P}')$ *and independent of* $\mathcal{F}$*.*

Proof: We know that the sequence $(S^n)_{n\in\mathbb{N}}$ is tight. On the other hand we have proved earlier that

$$\left(S_{t_1}^n, \ldots, S_{t_k}^n\right) \overset{d_{st}}{\to} \left(W'_{t_1}, \ldots, W'_{t_k}\right) \qquad \forall t_j \in [0,T]$$

Hence, we obtain the desired stable convergence $S^n \overset{d_{st}}{\to} W'$. $\square$

Chapter 2: Some basic facts about semimartingale theory

Definition

Let $\mathcal{C}$ be a class of stochastic processes defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Then $\mathcal{C}_{\text{loc}}$ denotes the **localised class**, i.e. $X \in \mathcal{C}_{\text{loc}}$ if there exists an increasing sequence of $(\mathcal{F}_t)$-stopping times $(T_n)_{n\in\mathbb{N}}$ such that

$$X^{T_n} := (X_{t\wedge T_n})_{t\geq 0} \in \mathcal{C} \qquad \text{and} \qquad T_n \overset{a.s.}{\to} \infty$$

In particular, we denote by $\mathcal{M}_{\text{loc}}$ the class of local martingales and by $\mathcal{B}_{\text{loc}}$ the class of locally bounded process.

It is well known that e.g. **cáglád processes** are locally bounded. **Cádlág processes** are not necessarily locally bounded.

Itô integral in a nutshell

Let W denote a standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Definition

(a) For any **basic** stochastic process $(H_t)_{t\geq 0}$ of the form

$$H_t = A \cdot 1_{(a,b]}(t) \qquad \text{where } A \text{ is bounded } \mathcal{F}_a\text{-measurable}$$

the integral $\int_0^t H_s dW_s$ is defined via

$$\int_0^t H_s dW_s := A(W_{b\wedge t} - W_{a\wedge t})$$

(b) The definition is directly extended to linear combination of basic processes by linearity.

(c) By an approximation argument the Itô integral can be defined for any **progressively measurable** stochastic process $(H_t)_{t\geq 0}$ satisfying $\int_0^\infty H_s^2 ds < \infty$ almost surely.

Some basic properties

Proposition

Let $(H_t)_{t\geq 0}$ and $(G_t)_{t\geq 0}$ denote progressively measurable stochastic processes satisfying $\mathbb{E}\int_0^\infty H_s^2 ds < \infty$ and $\mathbb{E}\int_0^\infty G_s^2 ds < \infty$.

(a) *The stochastic process $(\int_0^t H_s dW_s)_{t\geq 0}$ is an $(\mathcal{F}_t)$-martingale.*

(b) *It holds that*

$$\mathbb{E}\left[\int_0^t H_s dW_s \cdot \int_0^t G_s dW_s\right] = \mathbb{E}\left[\int_0^t H_s G_s ds\right]$$

(c) *For any continuous local martingales X, Y the* ***covariation process*** *$([X,Y]_t)_{t\geq 0}$ is defined as*

$$[X,Y]_t := \mathbb{P} - \lim_{n\to\infty} \sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}})$$

where $(t_i)_{0\leq i\leq n}$ is a partition of $[0,t]$ with $\max_i |t_i - t_{i-1}| \to 0$. $[X] := [X,X]$ is called the ***quadratic variation process*** *of X.*

The following identity is straightforward

$$\left[\int_0^\cdot H_s dW_s, \int_0^\cdot G_s dW_s\right]_t = \int_0^t H_s G_s ds$$

Definition

A stochastic process $(X_t)_{t\geq 0}$ is called a **continuous semimartingale** if it admits a decomposition

$$X_t = X_0 + A_t + M_t$$

where $(A_t)_{t\geq 0}$ has finite total variation, $(M_t)_{t\geq 0}$ is a continuous local martingale and $A_0 = M_0 = 0$. The process $(X_t)_{t\geq 0}$ is called a **continuous Itô semimartingale** if it admits a decomposition

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s$$

Important inequality

Theorem (Burkholder-Davis-Gundy inequality)

Let $(X_t)_{t\geq 0}$ be a continuous local martingale defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Then, for any $p > 0$, there exist constants $c_p, C_p > 0$ such that

$$c_p \mathbb{E}\left[[X]_t^{p/2}\right] \leq \mathbb{E}\left[\sup_{s\in[0,t]} |X_s|^p\right] \leq C_p \mathbb{E}\left[[X]_t^{p/2}\right]$$

Example: Let $X_t = \int_0^t H_s dW_s$ and assume that H is a bounded process. Then, for any $\Delta > 0$, it holds that

$$\mathbb{E}\left[|X_{t+\Delta} - X_t|^p\right] \leq C_p \mathbb{E}\left[\left(\int_t^{t+\Delta} H_s^2 ds\right)^{p/2}\right] = O(\Delta^{p/2})$$

In other words, the increments of X have the same order as the increments of the driving Brownian motion W.

Chapter 3: Stable convergence theorem of Jacod

- Despite the availability of the notion of stable convergence, "ready to use" limit theorems for the high frequency setting were not known for a long time.
- In a seminal paper of 97 Jean Jacod has established a general theorem, which perfectly fits the high frequency framework. Until now it remains the only general theorem in this setting.
- To demonstrate the result, we consider a sequence $(Y^n)_{n\in\mathbb{N}}$ of continuous local martingales defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We also consider another continuous local martingale $(M_t)_{t\geq 0}$, which we refer to as **reference martingale**. Roughly speaking, M is a "major martingale" determining Y^n.

Jacod's theorem: Continuous time version

Theorem (Jacod 97)

Assume that there exist $(\mathcal{F}_t)$-adapted processes u, v and w such that the following holds for all $t \in [0, T]$:

(i) $[Y^n]_t \xrightarrow{\mathbb{P}} F_t = \int_0^t (v_s^2 u_s^2 + w_s^2) ds$

(ii) $[Y^n, M]_t \xrightarrow{\mathbb{P}} G_t = \int_0^t v_s u_s^2 ds$

(iii) $[Y^n, N]_t \xrightarrow{\mathbb{P}} 0$

where $[M]_t = \int_0^t u_s^2 ds$ and the last convergence holds for all bounded continuous martingales N with $[M, N] = 0$. Then it holds that $Y^n \xrightarrow{d_{st}} Y$ on $(C[0, T], \| \cdot \|_\infty)$, and

$$Y_t = \int_0^t v_s dM_s + \int_0^t w_s dW_s'$$

where W' is a Brownian motion defined on an extended space $(\Omega', \mathcal{F}', (\mathcal{F}_t')_{t \geq 0}, \mathbb{P}')$ and independent of $\mathcal{F}$.

Some remarks

- The intuition behind the limit theorem is similar to orthogonal representation of vectors in linear spaces.
- Condition (ii) identifies the part of the quadratic variation of Y, which can be attributed to the reference martingale M.
- Condition (iii) says that no other continuous local martingales N defined on the original space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ contribute to the quadratic variation of Y.
- Finally, condition (i) determines the portion of the quadratic variation of Y, which needs to be explained by the new Brownian motion W' defined on the extended space $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, \mathbb{P}')$.

Question

Is there a discretized version of the stable limit theorem?

The discrete setting

- We consider statistics of the form

$$Y_t^n = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} X_{in}$$

where $\Delta_n \to 0$ and X_{in}'s are $\mathcal{F}_{i\Delta_n}$-measurable square integrable rv's.

- As before M denotes the reference (local) martingale with $[M]_t = \int_0^t u_s^2$. We define

$$M_b^\perp = \Big\{ \text{bounded } (\mathcal{F}_t)\text{-martingales } N \text{ with } [M,N] = 0 \Big\}$$

- In the following we write $Z^n \overset{u.c.p.}{\to} Z$ to denote the uniform convergence

$$\sup_{t \in [0,T]} |Z_t^n - Z_t| \overset{\mathbb{P}}{\to} 0$$

for any $T > 0$.

Theorem (Jacod 97)

Assume that there exist $(\mathcal{F}_t)$*-adapted processes* u, v, w *and a finite variation process* B *such that the following holds for all* $t \in [0, T]$*:*

(i) $\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[X_{in} | \mathcal{F}_{(i-1)\Delta_n}] \overset{u.c.p.}{\to} B_t$

(ii) $\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\mathbb{E}[X_{in}^2 | \mathcal{F}_{(i-1)\Delta_n}] - \mathbb{E}[X_{in} | \mathcal{F}_{(i-1)\Delta_n}]^2) \overset{\mathbb{P}}{\to} F_t = \int_0^t (v_s^2 u_s^2 + w_s^2) ds$

(iii) $\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[X_{in}(M_{i\Delta_n} - M_{(i-1)\Delta_n}) | \mathcal{F}_{(i-1)\Delta_n}] \overset{\mathbb{P}}{\to} G_t = \int_0^t v_s u_s^2 ds$

(iv) $\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[X_{in}(N_{i\Delta_n} - N_{(i-1)\Delta_n}) | \mathcal{F}_{(i-1)\Delta_n}] \overset{\mathbb{P}}{\to} 0 \qquad \forall N \in M_b^\perp$

(v) $\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[X_{in}^2 1_{\{|X_{in}| > \epsilon\}} | \mathcal{F}_{(i-1)\Delta_n}] \overset{\mathbb{P}}{\to} 0 \qquad \forall \epsilon > 0$

Then it holds that $Y^n \overset{d_{st}}{\to} Y$ *on* $(C[0, T], \| \cdot \|_\infty)$*, and*

$$Y_t = B_t + \int_0^t v_s dM_s + \int_0^t w_s dW'_s$$

where W' *is a new Brownian motion independent of* $\mathcal{F}$*.*

Remarks

- Conditions (ii)-(iv) are discrete versions of the covariation assumptions in the continuous case.
- Condition (i) determines the finite variation part of the semimartingale Y.
- Condition (v) is a standard assumption, which ensures that there are no jumps in the limiting process Y.
- In most applications the reference martingale M is the Brownian motion W, which is "driving" the rv's X_{in}.

Example 1

- Define the rv's X_{in} by

$$X_{in} = \Delta_n^{-1/2}\sigma^2_{(i-1)\Delta_n}\left((\Delta_i^n W)^2 - \Delta_n\right) \qquad \Delta_i^n W := W_{i\Delta_n} - W_{(i-1)\Delta_n}$$

where σ is a continuous adapted process. We are interested in the asymptotic behaviour of $Y_t^n = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} X_{in}$. We choose $M = W$.

- To check (i) observe that

$$\mathbb{E}[X_{in}|\mathcal{F}_{(i-1)\Delta_n}] = \Delta_n^{-1/2}\sigma^2_{(i-1)\Delta_n}\mathbb{E}[(\Delta_i^n W)^2 - \Delta_n|\mathcal{F}_{(i-1)\Delta_n}] = 0$$

and hence $B_t = 0$.

- For (ii) we compute

$$\begin{aligned}\mathbb{E}[X_{in}^2|\mathcal{F}_{(i-1)\Delta_n}] &= \Delta_n^{-1}\sigma^4_{(i-1)\Delta_n}\mathbb{E}[\left((\Delta_i^n W)^2 - \Delta_n\right)^2|\mathcal{F}_{(i-1)\Delta_n}] \\ &= 2\Delta_n\sigma^4_{(i-1)\Delta_n}\end{aligned}$$

and thus $F_t = 2\int_0^t \sigma_s^4 ds$.

Cont.

- To prove (iii), observe that the Brownian motion has symmetric distribution. Consequently

$$\mathbb{E}[X_{in}\Delta_i^n W|\mathcal{F}_{(i-1)\Delta_n}] = 0$$

and $G_t = 0$.

- To show (iv) observe the identity

$$(\Delta_i^n W)^2 - \Delta_n = 2\int_{(i-1)\Delta_n}^{i\Delta_n} (W_s - W_{(i-1)\Delta_n})dW_s$$

Hence, we deduce for all $N \in M_b^{\perp}$:

$$\mathbb{E}[X_{in}\Delta_i^n N|\mathcal{F}_{(i-1)\Delta_n}] = 2\Delta_n^{-1/2}\sigma^2_{(i-1)\Delta_n}$$

$$\times \mathbb{E}[\int_{(i-1)\Delta_n}^{i\Delta_n} (W_s - W_{(i-1)\Delta_n})dW_s \cdot \Delta_i^n N|\mathcal{F}_{(i-1)\Delta_n}]$$

$$= 2\Delta_n^{-1/2}\sigma^2_{(i-1)\Delta_n}\mathbb{E}[\int_{(i-1)\Delta_n}^{i\Delta_n} (W_s - W_{(i-1)\Delta_n})d[W,N]_s|\mathcal{F}_{(i-1)\Delta_n}] = 0$$

Cont.

- To prove (v), observe that $\mathbb{E}[X_{in}^4|\mathcal{F}_{(i-1)\Delta_n}] = C\Delta_n^2\sigma_{(i-1)\Delta_n}^8$ for some $C > 0$. Consequently, we obtain that

$$\mathbb{E}[X_{in}^2 1_{\{|X_{in}|>\epsilon\}}|\mathcal{F}_{(i-1)\Delta_n}] \leq \epsilon^{-2}\mathbb{E}[X_{in}^4|\mathcal{F}_{(i-1)\Delta_n}] = C\epsilon^{-2}\Delta_n^2\sigma_{(i-1)\Delta_n}^8$$

  But this means that

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[X_{in}^2 1_{\{|X_{in}|>\epsilon\}}|\mathcal{F}_{(i-1)\Delta_n}] \stackrel{\mathbb{P}}{\to} 0$$

  for all $\epsilon > 0$.

- We now deduce the convergence $Y^n \stackrel{d_{st}}{\to} Y$ with

$$Y_t = \sqrt{2}\int_0^t \sigma_s^2 dW_s' \sim \mathcal{MN}\left(0, 2\int_0^t \sigma_s^4 ds\right)$$

  □

Example 2

- Define the rv's X_{in} by

$$X_{in} = \Delta_n^{1/2} a_{(i-1)\Delta_n} \Big(f(\Delta_n^{-1/2} \Delta_i^n W) - \mathbb{E}[f(\Delta_n^{-1/2} \Delta_i^n W)] \Big)$$

where a is a continuous adapted process. Assume that f is an **even** function with at most polynomial growth. Choose again $M = W$.

- Using exactly the same methods as in Example 1, we obtain the convergence $Y^n \stackrel{d_{st}}{\to} Y$ with

$$Y_t = \lambda \int_0^t a_s dW'_s, \qquad \lambda = \mathrm{var}\left(f(\mathcal{N}(0,1))\right)$$

$\square$

Chapter 4: High frequency observations of Itô semimartingales

Chapter 4.1: Limit theorems in the continuous case

The model and observation scheme

- We consider a 1-dimensional Itô semimartingales of the form

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s \qquad t \geq 0$$

defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Here a and σ are assumed to be **continuous** adapted processes.

- The observation scheme is given as

$$X_0, X_{\Delta_n}, X_{2\Delta_n}, \ldots, X_{\Delta_n \lfloor T/\Delta_n \rfloor}$$

where $T > 0$ is **fixed** and $\Delta_n \to 0$.

- We focus on statistics of the type

$$V(X, f, \Delta_n)_t = \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} f\left(\Delta_n^{-1/2} \Delta_i^n X\right), \qquad \Delta_i^n X := X_{i\Delta_n} - X_{(i-1)\Delta_n}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a measurable function.

Law of large numbers

Theorem (Barndorff-Nielsen, Graversen, Jacod, P. & Shephard (06))

Assume that the function $f : \mathbb{R} \to \mathbb{R}$ is continuous with at most polynomial growth. Then it holds that

$$V(X, f, \Delta_n)_t \overset{u.c.p.}{\to} V(X, f)_t := \int_0^t \rho_{\sigma_s}(f) ds$$

where the map $x \mapsto \rho_x(f)$ is defined as

$$\rho_x(f) = \mathbb{E}[f(xN)], \qquad N \sim \mathcal{N}(0, 1)$$

In the **power variation** setting $f_p(x) := |x|^p$, $p > 0$, we obtain the convergence

$$V(X, f_p, \Delta_n)_t \overset{u.c.p.}{\to} m_p \int_0^t |\sigma_s|^p ds, \qquad m_p := \mathbb{E}[|N|^p]$$

In other words, we can estimate integrated powers of the volatility σ.

Crucial approximation

- First, we write

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s =: X_0 + A_t + M_t$$

and assume for the moment that the processes a and σ are bounded.

- Due to BDG inequality, we observe that

$$\Delta_n^{-1/2}\Delta_i^n X = \underbrace{\Delta_n^{-1/2}\Delta_i^n A}_{\|\cdot\|_{L^p}=O(\Delta_n^{1/2})} + \underbrace{\Delta_n^{-1/2}\Delta_i^n M}_{\|\cdot\|_{L^p}=O(1)} \approx \Delta_n^{-1/2}\Delta_i^n M$$

Hence, we obtain the first order approximation

$$\Delta_n^{-1/2}\Delta_i^n X \approx \alpha_i^n := \Delta_n^{-1/2}\sigma_{(i-1)\Delta_n}\Delta_i^n W$$

This observation is key to all proofs!

Law of large numbers

Theorem (Barndorff-Nielsen, Graversen, Jacod, P. & Shephard (06))

Assume that the function $f : \mathbb{R} \to \mathbb{R}$ is continuous with at most polynomial growth. Then it holds that

$$V(X, f, \Delta_n)_t \overset{u.c.p.}{\to} V(X, f)_t := \int_0^t \rho_{\sigma_s}(f) ds$$

where the map $x \mapsto \rho_x(f)$ is defined as

$$\rho_x(f) = \mathbb{E}[f(xN)], \qquad N \sim \mathcal{N}(0, 1)$$

Road map for the proof

The proof consists of three steps:

Step 1: Show that it is sufficient to prove the theorem in the setting where a and σ are bounded in (ω, t). This step is called localisation procedure.

Step 2: Prove the theorem when the increments $\Delta_n^{-1/2}\Delta_i^n X$ are replaced by the approximation α_i^n.

Step 3: Show that the approximation in the previous step is asymptotically negligible.

Since the processes a and σ are continuous, they are locally bounded. Hence, there exists an increasing sequence of stopping times $(T_k)_{k\in\mathbb{N}}$ with $T_k \overset{a.s.}{\to} \infty$ such that

$$|a_s(\omega)| + |\sigma_s(\omega)| \le k \qquad \forall s \le T_k(\omega)$$

Define

$$X_t^{(k)} := X_0 + \int_0^t a_{s\wedge T_k} ds + \int_0^t \sigma_{s\wedge T_k} dW_s$$

$$V^{(k)}(f)_t^n := V(X^{(k)}, f, \Delta_n)_t$$

$$V^{(k)}(f)_t := \int_0^t \rho_{\sigma_{s\wedge T_k}}(f) ds$$

Note that, for all $t < T_k(\omega)$, it holds that

$$X_t^{(k)} = X_t, \quad V^{(k)}(f)_t^n = V(X, f, \Delta_n)_t, \quad V^{(k)}(f)_t = V(X, f)_t$$

Step 1: Cont.

Assume that we have proven the convergence

$$V^{(k)}(f)_t^n \overset{u.c.p.}{\to} V^{(k)}(f)_t$$

for a fixed k, in which case the drift and volatility are bounded processes. Then we may conclude that

$$\mathbb{P}\left(\sup_{t\in[0,T]} |V(X,f,\Delta_n)_t - V(X,f)_t| > \epsilon\right) \leq \mathbb{P}(T \geq T_k)$$

$$+ \mathbb{P}\left(\sup_{t\in[0,T]} |V^{(k)}(f)_t^n - V^{(k)}(f)_t| > \epsilon, T < T_k\right) \to 0$$

as $n \to \infty$ and then $k \to \infty$. Hence, it is sufficient to prove the main result for bounded processes a and σ. □

Step 2: Limit theorem for the approximation

We define the statistic

$$V'(f)^n_t := \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} f\left(\alpha^n_i\right), \qquad \alpha^n_i = \Delta_n^{-1/2} \sigma_{(i-1)\Delta_n} \Delta^n_i W$$

We will need the following lemma.

Lemma

Let X_{in} be square integrable $\mathcal{F}_{i\Delta_n}$-measurable rv's. Assume that

(i) $\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[X_{in} | \mathcal{F}_{(i-1)\Delta_n}] \overset{u.c.p.}{\to} V_t$ *for some stochastic process* $(V_t)_{t\in[0,T]}$.

(ii) $\sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \mathbb{E}[X_{in}^2 | \mathcal{F}_{(i-1)\Delta_n}] \overset{\mathbb{P}}{\to} 0$

Then it holds that

$$\sup_{t\in[0,T]} \left| \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} X_{in} - V_t \right| \overset{\mathbb{P}}{\to} 0$$

Step 2: Cont.

We apply the previous lemma to the case $X_{in} = \Delta_n f\left(\alpha_i^n\right)$. Note that

$$\mathbb{E}[X_{in}|\mathcal{F}_{(i-1)\Delta_n}] = \Delta_n \rho_{\sigma_{(i-1)\Delta_n}}(f), \qquad \mathbb{E}[X_{in}^2|\mathcal{F}_{(i-1)\Delta_n}] = \Delta_n^2 \rho_{\sigma_{(i-1)\Delta_n}}(f^2)$$

Since σ is continuous and bounded, we deduce that

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[X_{in}|\mathcal{F}_{(i-1)\Delta_n}] \overset{u.c.p.}{\to} V(X,f)_t = \int_0^t \rho_{\sigma_s}(f) ds$$

$$\sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \mathbb{E}[X_{in}^2|\mathcal{F}_{(i-1)\Delta_n}] \overset{\mathbb{P}}{\to} 0$$

Consequently, the limit theorem holds for the approximation α_i^n, i.e.

$$V'(f)^n \overset{u.c.p.}{\to} V(X,f)$$

□

Step 3: Asymptotic negligibility of the approximation

For this step we only give an intuitive argument. We know that the difference

$$\Delta_n^{-1/2}\Delta_i^n X - \alpha_i^n$$

is small with high probability. Since $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, the same is true for the difference

$$f(\Delta_n^{-1/2}\Delta_i^n X) - f(\alpha_i^n)$$

Hence, we may conclude that

$$\sup_{t\in[0,T]} |V(X, f, \Delta_n)_t - V'(f)^n| \le \Delta_n \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \left| f\left(\Delta_n^{-1/2}\Delta_i^n X\right) - f(\alpha_i^n)\right| \stackrel{\mathbb{P}}{\to} 0$$

This completes the proof of the theorem. □

What about a central limit theorem?

Preliminary condition

- Recall that we proved the convergence

$$V(X,f,\Delta_n)_t = \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} f\left(\Delta_n^{-1/2}\Delta_i^n X\right) \overset{u.c.p.}{\to} \int_0^t \rho_{\sigma_s}(f)ds$$

  and we now want to show the associated stable central limit theorem.

- In this situation we require a stronger condition on the volatility process σ. We assume that σ is also a continuous Itô semimartingale of the form

$$\sigma_t = \sigma_0 + \int_0^t a'_s ds + \int_0^t \sigma'_s dW_s + \int_0^t v'_s dV_s$$

  where V is an $(\mathcal{F}_t)$-Brownian motion independent of W. This condition is satisfied in most of financial models.

Stable central limit theorem

Theorem (Barndorff-Nielsen, Graversen, Jacod, P. & Shephard (06))

*Assume that the volatility process σ is a continuous Itô semimartingale as defined before. Furthermore, suppose that $f \in C^1(\mathbb{R})$ is an **even** function such that f, f' have at most polynomial growth. Then we obtain the functional stable convergence*

$$\Delta_n^{-1/2} \left(V(X, f, \Delta_n)_t - V(X, f)_t \right) \xrightarrow{d_{st}} \int_0^t \sqrt{\rho_{\sigma_s}(f^2) - \rho_{\sigma_s}(f)^2} \, dW_s'$$

where W' is a new Brownian motion independent of $\mathcal{F}$.

The result can be extended to certain non-differentiable functions. In particular, the theorem remains valid for the power variation class $f_p(x) = |x|^p$, $p > 0$. Furthermore, for $p = 2$ the semimartingale assumption on σ is not required!

Important example: Feasible estimation of the quadratic variation

Recall that $[X]_t = \int_0^t \sigma_s^2 ds$ and we deal with the function $f_2(x) = x^2$. We obtain the identities

$$\rho_x(f_2) = x^2 \qquad \text{and} \qquad \rho_x(f_2^2) = 3x^4$$

The stable central limit theorem implies that

$$\Delta_n^{-1/2} \left(\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Delta_i^n X)^2 - [X]_t \right) \stackrel{d_{st}}{\to} \sqrt{2} \int_0^t \sigma_s^2 dW_s' \sim \mathcal{MN}\left(0, 2\int_0^t \sigma_s^4 ds\right)$$

The law of large numbers gives us $V(X, f_4, \Delta_n)_t \stackrel{\mathbb{P}}{\to} 3\int_0^t \sigma_s^4 ds$. Hence, we obtain a feasible central limit theorem

$$\frac{\Delta_n^{-1/2} \left(\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Delta_i^n X)^2 - [X]_t \right)}{\sqrt{\frac{2}{3} V(X, f_4, \Delta_n)_t}} \stackrel{d}{\to} \mathcal{N}(0,1)$$

This result can be used to construct confidence regions.

Stable central limit theorem

Theorem (Barndorff-Nielsen, Graversen, Jacod, P. & Shephard (06))

Assume that the volatility process σ is a continuous Itô semimartingale as defined before. Furthermore, suppose that $f \in C^1(\mathbb{R})$ is an ***even*** *function such that f, f' have at most polynomial growth. Then we obtain the functional stable convergence*

$$\Delta_n^{-1/2} \left(V(X, f, \Delta_n)_t - V(X, f)_t \right) \stackrel{d_{st}}{\to} \int_0^t \sqrt{\rho_{\sigma_s}(f^2) - \rho_{\sigma_s}(f)^2} \, dW'_s$$

where W' is a new Brownian motion independent of $\mathcal{F}$.

Road map for the proof

The proof consists of three steps:

Step 1: Show that it is sufficient to prove the theorem in the setting where all processes $a, a', \sigma, \sigma', v'$ are bounded in (ω, t). This step is performed in a similar fashion as in the law of large numbers.

Step 2: Prove the theorem when the increments $\Delta_n^{-1/2} \Delta_i^n X$ are replaced by the approximation $\alpha_i^n = \Delta_n^{-1/2} \sigma_{(i-1)\Delta_n} \Delta_i^n W$.

Step 3: Show that the approximation in the previous step is asymptotically negligible.

Step 2: Central limit theorem for the approximation

Define the random variable

$$\beta_i^n := \Delta_n^{1/2}\left(f(\alpha_i^n) - \mathbb{E}[f(\alpha_i^n)|\mathcal{F}_{(i-1)\Delta_n}]\right) = \Delta_n^{1/2}\left(f(\alpha_i^n) - \rho_{\sigma_{(i-1)\Delta_n}}(f)\right)$$

We want to prove the functional stable convergence

$$U_t^n := \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \beta_i^n \xrightarrow{d_{st}} U_t := \int_0^t \sqrt{\rho_{\sigma_s}(f^2) - \rho_{\sigma_s}(f)^2}\, dW_s'$$

By construction $\mathbb{E}[\beta_i^n|\mathcal{F}_{(i-1)\Delta_n}] = 0$. On the other hand, we have

$$\mathbb{E}[(\beta_i^n)^2|\mathcal{F}_{(i-1)\Delta_n}] = \Delta_n\left(\rho_{\sigma_{(i-1)\Delta_n}}(f^2) - \rho_{\sigma_{(i-1)\Delta_n}}(f)^2\right)$$

which implies that

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[(\beta_i^n)^2|\mathcal{F}_{(i-1)\Delta_n}] \xrightarrow{\mathbb{P}} F_t = \int_0^t \left(\rho_{\sigma_s}(f^2) - \rho_{\sigma_s}(f)^2\right) ds$$

Recall again that $\alpha_i^n = \Delta_n^{-1/2}\sigma_{(i-1)\Delta_n}\Delta_i^n W$ and the function f is even. Hence, the function $x \mapsto xf(x)$ is odd, and we conclude

$$\mathbb{E}[\beta_i^n \Delta_i^n W | \mathcal{F}_{(i-1)\Delta_n}] = 0$$

We also have that

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[(\beta_i^n)^2 1_{\{|\beta_i^n|>\epsilon\}} | \mathcal{F}_{(i-1)\Delta_n}] \leq \epsilon^{-2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[(\beta_i^n)^4 | \mathcal{F}_{(i-1)\Delta_n}] = \epsilon^{-2} O_{\mathbb{P}}(\Delta_n)$$

Hence,

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[(\beta_i^n)^2 1_{\{|\beta_i^n|>\epsilon\}} | \mathcal{F}_{(i-1)\Delta_n}] \xrightarrow{\mathbb{P}} 0 \qquad \forall \epsilon > 0$$

Step 2: Cont.

Finally, let $N \in W_b^{\perp}$. We introduce the filtration $\widetilde{\mathcal{F}}_t = \mathcal{F}_{(i-1)\Delta_n + t}$ for $t \in [0, \Delta_n]$ and define

$$H_t := \mathbb{E}[f(\alpha_i^n)|\widetilde{\mathcal{F}}_t]$$

Note that H is a martingale and $\beta_i^n = \sqrt{\Delta_n}\ \Delta_0^n H$. By martingale representation theorem we obtain

$$H_t = H_0 + \int_{(i-1)\Delta_n}^{t} \eta_s \ dW_s$$

for some predictable process η. Now we deduce the identity

$$\mathbb{E}[\beta_i^n \Delta_i^n N|\mathcal{F}_{(i-1)\Delta_n}] = \sqrt{\Delta_n}\ \mathbb{E}[\Delta_0^n H \Delta_i^n N|\mathcal{F}_{(i-1)\Delta_n}]$$
$$= \sqrt{\Delta_n}\ \mathbb{E}[\int_{(i-1)\Delta_n}^{i\Delta_n} \eta_s d[W, N]_s|\mathcal{F}_{(i-1)\Delta_n}] = 0$$

Thus, it follows

$$U_t^n = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \beta_i^n \stackrel{d_{st}}{\to} U_t = \int_0^t \sqrt{\rho_{\sigma_s}(f^2) - \rho_{\sigma_s}(f)^2}\ dW_s'$$

□

Step 3: Negligibility of the approximation

It now remains to prove that the approximative statistic U_t^n is asymptotically equivalent to the original one. This is performed in several steps.

Step 3.1: Introduce the more "natural" statistic

$$\overline{U}_t^n := \Delta_n^{1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \Big(f(\Delta_n^{-1/2}\Delta_i^n X) - \mathbb{E}[f(\Delta_n^{-1/2}\Delta_i^n X)|\mathcal{F}_{(i-1)\Delta_n}] \Big)$$

and show that $\overline{U}^n - U^n \stackrel{u.c.p.}{\to} 0$.

Step 3.2: Prove that

$$\Delta_n^{1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} \left(\rho_{\sigma_s}(f) - \rho_{\sigma_{(i-1)\Delta_n}}(f)\right) ds \stackrel{u.c.p.}{\to} 0$$

Step 3.3: Finally, show that

$$\Delta_n^{1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[f(\Delta_n^{-1/2}\Delta_i^n X) - f(\alpha_i^n)|\mathcal{F}_{(i-1)\Delta_n}] \stackrel{u.c.p.}{\to} 0$$

Step 3: First arguments

To complete Step 3.1 we observe that $\overline{U}^n - U^n = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \gamma_i^n$ with $\mathbb{E}[\gamma_i^n | \mathcal{F}_{(i-1)\Delta_n}] = 0$. Thus, the proof of the statement $\overline{U}^n - U^n \stackrel{u.c.p.}{\to} 0$ is completed if we show that

$$\sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \mathbb{E}[(\gamma_i^n)^2 | \mathcal{F}_{(i-1)\Delta_n}] \stackrel{\mathbb{P}}{\to} 0$$

which is relatively straightforward. □

To show the convergence in Step 3.2 we use the mean value theorem

$$\Delta_n^{1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} \left(\rho_{\sigma_s}(f) - \rho_{\sigma_{(i-1)\Delta_n}}(f) \right) ds$$

$$\approx \Delta_n^{1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \rho'_{\sigma_{(i-1)\Delta_n}}(f) \int_{(i-1)\Delta_n}^{i\Delta_n} \left(\sigma_s - \sigma_{(i-1)\Delta_n} \right) ds$$

along with the semimartingale assumption on σ. □

Step 3: Last argument

It remains to prove the convergence in Step 3.3:

$$\Delta_n^{1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[f(\Delta_n^{-1/2}\Delta_i^n X) - f(\alpha_i^n)|\mathcal{F}_{(i-1)\Delta_n}] \stackrel{u.c.p.}{\to} 0$$

For this purpose we need the precise understanding of the approximation error under the semimartingale assumption on σ:

$$\Delta_n^{-1/2}\Delta_i^n X - \alpha_i^n \approx \Delta_n^{1/2} a_{(i-1)\Delta_n} + \Delta_n^{-1/2} \int_{(i-1)\Delta_n}^{i\Delta_n} \left(\sigma_s - \sigma_{(i-1)\Delta_n}\right) dW_s$$

$$\approx \Delta_n^{1/2} a_{(i-1)\Delta_n} + \Delta_n^{-1/2} \int_{(i-1)\Delta_n}^{i\Delta_n} \Big(\sigma'_{(i-1)\Delta_n}\left(W_s - W_{(i-1)\Delta_n}\right)$$

$$+v'_{(i-1)\Delta_n}\left(V_s - V_{(i-1)\Delta_n}\right)\Big) dW_s$$

Step 3: Cont.

The fact that the function $f \in C^1(\mathbb{R})$ is even, and hence f' is odd, turns out to be absolutely crucial. Indeed, we deduce that

$$\Delta_n^{1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[f(\Delta_n^{-1/2}\Delta_i^n X) - f(\alpha_i^n)|\mathcal{F}_{(i-1)\Delta_n}]$$

$$\approx \Delta_n^{1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[f'(\alpha_i^n)(\Delta_n^{-1/2}\Delta_i^n X - \alpha_i^n)|\mathcal{F}_{(i-1)\Delta_n}]$$

$$\approx \Delta_n^{1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[f'(\alpha_i^n)(\Delta_n^{1/2} a_{(i-1)\Delta_n} + \Delta_n^{-1/2} \int_{(i-1)\Delta_n}^{i\Delta_n}$$

$$(\sigma'_{(i-1)\Delta_n}(W_s - W_{(i-1)\Delta_n}) + v'_{(i-1)\Delta_n}(V_s - V_{(i-1)\Delta_n}))dW_s)|\mathcal{F}_{(i-1)\Delta_n}]$$

$$= 0$$

□

Final remarks

- The asymptotic theory for high frequency observations of semimartingales easily extends to the multivariate setting.

- Let X be a d-dimensional Itô semimartingale and $k \in \mathbb{N}$. For a measurable function $f : (\mathbb{R}^d)^k \to \mathbb{R}^m$, we consider the statistic

$$V(X, f, \Delta_n)_t = \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} f\left(\Delta_n^{-1/2}\Delta_i^n X, \ldots, \Delta_n^{-1/2}\Delta_{i+k-1}^n X\right)$$

  We can derive the complete asymptotic theory in this setting. In particular, it holds that

$$V(X, f, \Delta_n)_t \stackrel{u.c.p.}{\to} V(X, f)_t := \int_0^t \rho_{\sigma_s}(f)ds, \qquad \rho_x(f) = \mathbb{E}[f(xN)]$$

  and $N = (N_1, \ldots, N_k)$ with N_j i.i.d. $\sim \mathcal{N}_d(0, \mathrm{id})$.

- A particularly useful class are **multipower variations** that correspond to $d = 1$ and $f(x) = \prod_{j=1}^k |x_j|^{p_j}$.

Chapter 4.2: Limit theorems in the discontinuous case

The model and main statistics

- We consider a 1-dimensional Itô semimartingales of the form

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s + J_t \qquad t \geq 0$$

defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Here a and σ are assumed to be **cádllág** adapted processes, and J is a **compound Poisson process**, i.e.

$$J_t = \sum_{j=1}^{N_t} Z_j \qquad \text{where } N \sim \text{Po}(\lambda) \quad \text{and} \quad (Z_j)_{j\in\mathbb{N}} \text{ are i.i.d.}$$

We denote $\Delta X_s = \Delta J_s := J_s - J_{s-}$ and assume that $(N, Z) \perp\!\!\!\perp W$.

- In the discontinuous setting we only consider power variations of the form

$$V(X, p, \Delta_n)_t = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} |\Delta_i^n X|^p \qquad p > 0$$

Law of large numbers

Theorem (Jacod (08))

(i) *When* $p > 2$ *it holds that*

$$V(X, p, \Delta_n)_t \xrightarrow{\mathbb{P}} \sum_{s \in [0,t]} |\Delta X_s|^p$$

(ii) *When* $p = 2$ *it holds that*

$$V(X, 2, \Delta_n)_t \xrightarrow{\mathbb{P}} [X]_t = \int_0^t \sigma_s^2 ds + \sum_{s \in [0,t]} |\Delta X_s|^2$$

(iii) *When* $p < 2$ *it holds that*

$$\Delta_n^{1-p/2} V(X, p, \Delta_n)_t \xrightarrow{\mathbb{P}} m_p \int_0^t |\sigma_s|^p ds$$

Important application: Separation of volatility and jumps

In financial applications it is crucial to separate the continuous part of the quadratic variation from the discontinuous one. There are two classical methods in the literature:

Multipower variation: This approach has been proposed in Barndorff-Nielsen & Shephard (04). Their idea is based upon using multipower variations of the form

$$V(X, p_1, \ldots, p_k, \Delta_n)_t = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \prod_{j=0}^{k-1} |\Delta_{i+j}^n X|^{p_j}$$

with $\sum_{j=1}^{k} p_j = 2$ to obtain a **jump robust** measure of $\int_0^t \sigma_s^2 ds$.

Threshold estimator: The idea stems from Mancini (01). The jump robust estimator of $\int_0^t \sigma_s^2 ds$ is obtained via

$$TRV_t^n = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Delta_i^n X)^2 1_{\{|\Delta_i^n X| \le \Delta_n^{\nu}\}} \qquad \nu \in (0, 1/2)$$

Theorem (Barndorff-Nielsen & Shephard (04))

Assume that $k > 1$ and $\sum_{j=1}^{k} p_j = 2$. Then it holds that

$$V(X, p_1, \ldots, p_k, \Delta_n)_t \xrightarrow{\mathbb{P}} \left(\prod_{j=1}^{k} m_{p_j} \right) \cdot \int_0^t \sigma_s^2 ds$$

If moreover $\max_j(p_j) < 1$, then we obtain the stable convergence

$$\Delta_n^{-1/2} \left(V(X, p_1, \ldots, p_k, \Delta_n)_t - \left(\prod_{j=1}^{k} m_{p_j} \right) \cdot \int_0^t \sigma_s^2 ds \right)$$

$$\xrightarrow{d_{st}} \sqrt{m_{p_1,\ldots,p_k}} \int_0^t \sigma_s^2 dW'_s$$

where W' is a new Brownian motion independent of $\mathcal{F}$ and $m_{p_1,\ldots,p_k}$ is a certain constant.

Sketch of the proof

Since J is a compound Poisson process, there is at most one jump in the interval $[i\Delta_n, (i+k)\Delta_n]$ with high probability. Define the set

$$\Omega^n_{\text{jump}} := \{i \le \lfloor t/\Delta_n \rfloor : \text{ there is a jump in } [i\Delta_n, (i+k)\Delta_n]\}$$

Applying the limit theory for continuous Itô semimartingales we conclude that

$$\sum_{i \notin \Omega^n_{\text{jump}}} \prod_{j=0}^{k-1} |\Delta^n_{i+j} X|^{p_j} \xrightarrow{\mathbb{P}} \left(\prod_{j=1}^{k} m_{p_j} \right) \cdot \int_0^t \sigma_s^2 ds$$

On the other hand, we have that

$$\sum_{i \in \Omega^n_{\text{jump}}} \prod_{j=0}^{k-1} |\Delta^n_{i+j} X|^{p_j} = O_{\mathbb{P}}(\Delta_n^{\max_j (p_j)/2})$$

The latter implies the assertion of the theorem. □

Limit theory for the threshold estimator

Theorem (Mancini (01))

It holds that

$$\Delta_n^{-1/2}\left(TRV_t^n - \int_0^t \sigma_s^2 ds\right) \xrightarrow{d_{st}} \sqrt{2}\int_0^t \sigma_s^2 dW_s'$$

where W' is a new Brownian motion independent of $\mathcal{F}$. In other words, the jump part J does not affect the central limit theorem.

For this result to hold it is essential that J has **finite variation**. In the setting of infinite variation jumps the statement fails to hold.

Sketch of the proof

Recall the definition

$$TRV_t^n = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Delta_i^n X)^2 1_{\{|\Delta_i^n X| \le \Delta_n^\nu\}} \qquad \nu \in (0, 1/2)$$

Let us denote by

$$X_t^c = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s \qquad t \ge 0$$

the continuous part of X. By the law of iterated logarithm for the Brownian motion, we deduce for any $\nu \in (0, 1/2)$:

$$\mathbb{P}(|\Delta_i^n X^c| < \Delta_n^\nu) = 1 \qquad \text{and} \qquad \mathbb{P}(|\Delta_i^n J| < \Delta_n^\nu) \to 0$$

when J has jumps on $[(i-1)\Delta_n, i\Delta_n)$. Hence, the threshold asymptotically eliminates the jumps and we obtain the same stable limit theorem as in the continuous case. □

What about limit theorems when jumps are dominating?

Some notations

- We again consider the power variation case. Similar to the law of large numbers, the weak limit theory strongly depends on the considered power $p > 0$.

- We write $f_p(x) = |x|^p$ and introduce random variables

  $$(U_m^+)_{m\in\mathbb{N}},\ (U_m^-)_{m\in\mathbb{N}} \text{ i.i.d. } \mathcal{N}(0,1) \quad \text{and} \quad (\kappa_m)_{m\in\mathbb{N}} \text{ i.i.d. } \mathcal{U}(0,1)$$

  that are defined on the extended space $(\Omega', \mathcal{F}', \mathbb{P}')$ and independent of $\mathcal{F}$.

- The stochastic processes W', U^+, U^- and κ are mutually independent.

Central limit theorem

Theorem (Jacod (08))

Let $(T_m)_{m\geq 1}$ denote the jump times of X and define the process

$$L(p)_t = \sum_{m:\ T_m\in[0,t]} f_p'(\Delta X_{T_m})\left(\sqrt{\kappa_m}\sigma_{T_m-}U_m^- + \sqrt{1-\kappa_m}\,\sigma_{T_m}U_m^+\right)$$

(i) When $p>3$ it holds that

$$\Delta_n^{-1/2}\Big(V(X,p,\Delta_n)_t - \sum_{s\in[0,t]}|\Delta X_s|^p\Big) \xrightarrow{d_{st}} L(p)_t$$

(ii) When $p=2$ it holds that

$$\Delta_n^{-1/2}\left(V(X,2,\Delta_n)_t - [X]_t\right) \xrightarrow{d_{st}} L(p)_t + \sqrt{2}\int_0^t \sigma_s^2 dW_s'$$

(iii) When $p<1$ and σ is a continuous Itô semimartingale

$$\Delta_n^{-1/2}\left(\Delta_n^{1-p/2}V(X,p,\Delta_n)_t - V(X,f_p)_t\right) \xrightarrow{d_{st}} \sqrt{m_{2p}-m_p^2}\int_0^t |\sigma_s|^p dW_s'$$

A statistical remark

When the processes σ and J have **no common jumps**, the process $L(p)$ can be simplified to

$$L(p)_t = \sum_{m:\ T_m \in [0,t]} f_p'(\Delta X_{T_m}) \sigma_{T_m} U_m^+$$

Since $(U_m^+)_{m \in \mathbb{N}}$ are i.i.d. $\mathcal{N}(0,1)$-distributed and independent of $\mathcal{F}$, we readily deduce that

$$L(p)_t \sim \mathcal{MN}\left(0, \sum_{m:\ T_m \in [0,t]} \left(f_p'(\Delta X_{T_m}) \sigma_{T_m}\right)^2\right)$$

The condition variance can be estimated via the threshold approach:

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (f_p')^2(\Delta_i^n X)\, \widehat{\sigma}^2_{(i-1)\Delta_n} 1_{\{|\Delta_i^n X| > \Delta_n^{\nu}\}} \qquad \nu \in (0, 1/2)$$

The latter can be used for statistical applications.

Central limit theorem

Theorem (Jacod (08))

Let $(T_m)_{m\geq 1}$ denote the jump times of X and define the process

$$L(p)_t = \sum_{m:\ T_m\in[0,t]} f_p'(\Delta X_{T_m})\left(\sqrt{\kappa_m}\sigma_{T_m-}U_m^- + \sqrt{1-\kappa_m}\,\sigma_{T_m}U_m^+\right)$$

(i) When $p>3$ it holds that

$$\Delta_n^{-1/2}\left(V(X,p,\Delta_n)_t - \sum_{s\in[0,t]}|\Delta X_s|^p\right) \xrightarrow{d_{st}} L(p)_t$$

(ii) When $p=2$ it holds that

$$\Delta_n^{-1/2}\left(V(X,2,\Delta_n)_t - [X]_t\right) \xrightarrow{d_{st}} L(p)_t + \sqrt{2}\int_0^t \sigma_s^2 dW_s'$$

(iii) When $p<1$ and σ is a continuous Itô semimartingale

$$\Delta_n^{-1/2}\left(\Delta_n^{1-p/2}V(X,p,\Delta_n)_t - V(X,f_p)_t\right) \xrightarrow{d_{st}} \sqrt{m_{2p}-m_p^2}\int_0^t |\sigma_s|^p dW_s'$$

Sketch of the proof

We already showed how to deduce part (iii), hence we concentrate on the argument behind part (i). Since J is a compound Poisson process, it is easy to prove that

$$\Delta_n^{-1/2}\Big(V(J,p,\Delta_n)_t - \sum_{s\in[0,t]} |\Delta X_s|^p\Big) \stackrel{\mathbb{P}}{\to} 0$$

In the next step we observe that $|f_p'(x)| = p|x|^{p-1}$ and $p-1>2$ since we assumed that $p>3$. Consequently, we obtain the asymptotic equivalence

$$\Delta_n^{-1/2}\Big(V(X,p,\Delta_n)_t - V(J,p,\Delta_n)_t\Big) \approx \Delta_n^{-1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} f'(\Delta_i^n J)\Delta_i^n X^c$$

Recall our basic approximation

$$\Delta_i^n X^c \approx \int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s dW_s$$

Cont.

Let us now consider an arbitrary jump time $T_m \in [0, t]$ of J and let i_m denote the random index such that $T_m \in [(i_m - 1)\Delta_n, i_m\Delta_n)$. It obviously holds that

$$\Delta_{i_m}^n J \stackrel{a.s.}{\to} \Delta J_{T_m}$$

On the other hand, since σ is cádlág, we have that

$$\Delta_n^{-1/2} \int_{(i_m-1)\Delta_n}^{i_m\Delta_n} \sigma_s dW_s = \Delta_n^{-1/2} \left(\int_{(i_m-1)\Delta_n}^{T_m} \sigma_s dW_s + \int_{T_m}^{i_m\Delta_n} \sigma_s dW_s \right)$$
$$\approx \Delta_n^{-1/2} \left(\sigma_{T_m-}(W_{T_m} - W_{(i_m-1)\Delta_n}) + \sigma_{T_m}(W_{i_m\Delta_n} - W_{T_m}) \right)$$

Finally, note the identity

$$\Delta_n^{-1} \left(T_m - (i_m - 1)\Delta_n \right) = \{ T_m/\Delta_n \} \stackrel{d_{st}}{\to} \kappa_m \sim \mathcal{U}(0, 1)$$

Cont.

Now it remains to put everything together. Due to mutual independence of the involved processes we conclude that

$$\Delta_n^{-1/2}\Big(V(X,p,\Delta_n)_t - V(J,p,\Delta_n)_t\Big)$$

$$\approx \Delta_n^{-1/2}\sum_m f'(\Delta_{i_m}^n J)\left(\sigma_{T_m-}(W_{T_m} - W_{(i_m-1)\Delta_n}) + \sigma_{T_m}(W_{i_m\Delta_n} - W_{T_m})\right)$$

$$\xrightarrow{d_{st}} \sum_{m:\ T_m\in[0,t]} f_p'(\Delta X_{T_m})\left(\sqrt{\kappa_m}\sigma_{T_m-}U_m^- + \sqrt{1-\kappa_m}\,\sigma_{T_m}U_m^+\right)$$

This completes the proof of part (i). □

Key application

Testing for jumps in price processes based on high frequency data

Statistical formulation of the problem

- One of the classical questions in financial applications is whether the **unobserved** path $(X_t(\omega))_{t\in[0,T]}$ has jumps or not. Let us define the set

$$\Omega_0(T) := \{\omega \in \Omega : \ J_T = 0\}$$

and note that $\mathbb{P}(\Omega_0(T)) \in (0,1)$. We set $\Omega_1(T) = \Omega \setminus \Omega_0(T)$.

- In the literature there are several approaches to test for

$$H_0 : \ \omega \in \Omega_0(T) \qquad \text{vs.} \qquad H_1 : \ \omega \in \Omega_1(T)$$

based on high frequency data $X_0, X_{\Delta_n}, \ldots, X_{\lfloor T/\Delta_n \rfloor \Delta_n}$.

- Appropriate statistical formulation: Define by $\mathbb{P}_\theta$ the probability measure $\mathbb{P}$ restricted to $\Omega_\theta(T)$ with $\theta = 0, 1$. The formal statistical test:

$$H_0 : \ \theta = 0 \qquad \text{vs.} \qquad H_1 : \ \theta = 1$$

- In Barndorff-Nielsen & Shephard (06) the authors propose to use **bipower variation** to test for jumps in X. Their idea is based on the following observation:

$$\sum_{i=1}^{\lfloor T/\Delta_n \rfloor} |\Delta_i^n X||\Delta_{i+1}^n X| \xrightarrow{\mathbb{P}} m_1^2 \int_0^T \sigma_s^2 ds$$

In particular, this statistic is robust to the presence of jumps.

- They introduced the test statistic

$$S(T)_n := \Delta_n^{-1/2} \left(\sum_{i=1}^{\lfloor T/\Delta_n \rfloor} (\Delta_i^n X)^2 - m_1^{-2} \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} |\Delta_i^n X||\Delta_{i+1}^n X| \right)$$

It is easy to see that $S(T)_n \to \infty$ under $\mathbb{P}_1$, while

$$S(T)_n \xrightarrow{d_{st}} \mathcal{MN}(0, s^2) \qquad \text{under } \mathbb{P}_0$$

The power variation based test for jumps

- In Aït-Sahalia & Jacod (09) the authors propose to use the 4th power variaion to test for jumps in X. Their idea is based upon computing power variation at two different frequencies. In particular, it holds that

$$R(T)_n := \frac{V(X,4,2\Delta_n)_T}{V(X,4,\Delta_n)_T} \xrightarrow{\mathbb{P}} \begin{cases} 2: & \theta = 0 \\ 1: & \theta = 1 \end{cases}$$

which follows from our asymptotic theory.

- The formal testing procedure is obtained by an application of stable limit theorems presented in this chapter.

Chapter 5: Optimal estimation of volatility functionals

Central questions

Are the estimators presented in the previous chapter **optimal**?

- In the classical test theory the model parameters are **deterministic** objects. There exist numerous approaches to access the optimality of estimators: Cramer-Rao bounds, maximum likelihood theory, minimax approach, Le Cam theory, etc.
- However, in the high frequency setting the objects of interests are often **random**. Examples include quadratic variation, realised jumps, supremum/infimum of a process, local times, occupation time measures etc.
- In this framework very little is known about how to construct optimal estimates.

Motivating examples

- We have already obtained an estimator of the quadratic variation in the continuous setting:

$$\Delta_n^{-1/2}\left(\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}(\Delta_i^n X)^2-\int_0^t \sigma_s^2 ds\right)\xrightarrow{d_{st}}\mathcal{MN}\left(0,2\int_0^t \sigma_s^4 ds\right)$$

Is this estimator efficient? The answer is: **Yes**!

- We also know that

$$\Delta_n^{-1/2}\left(\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}(\Delta_i^n X)^4-3\int_0^t \sigma_s^4 ds\right)\xrightarrow{d_{st}}\mathcal{MN}\left(0,96\int_0^t \sigma_s^8 ds\right)$$

Is this estimator efficient? The answer is: **No**! But it is rate optimal.

A warning

In certain (unrealistic) situation there may exist estimators with a faster rate of convergence than $\Delta_n^{-1/2}$! Let us consider a stochastic differential equation of the form

$$X_t = X_0 + \int_0^t a(X_s)ds + \int_0^t \sigma(X_s)dW_s$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ is a **known** function. In this setting we have that

$$[X]_t = \int_0^t \sigma^2(X_s)ds$$

We can estimate $[X]_t$ by the Riemann sum. Indeed, one can prove that

$$\Delta_n^{-1} \left(\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \sigma^2(X_{i\Delta_n}) - [X]_t \right)$$

converges stably in law to a mixed normal limit.

A simple example

Consider a simple model

$$X_t = \sigma W_t, \qquad \sigma > 0$$

in which case $(\Delta_n^{-1/2} \Delta_i^n X)_{i \in \mathbb{N}}$ are i.i.d. $\mathcal{N}(0, \sigma^2)$-distributed. In this setting we can compute the maximum likelihood estimator

$$\widehat{\sigma}^2_{\mathrm{MLE}} = \sum_{i=1}^{\lfloor 1/\Delta_n \rfloor} (\Delta_i^n X)^2$$

which is efficient for σ^2. What about estimation of σ^4? The asymptotically efficient estimator of σ^4 is given by $(\widehat{\sigma}^2_{\mathrm{MLE}})^2$. We obtain via δ-method

$$\Delta_n^{-1/2} \left((\widehat{\sigma}^2_{\mathrm{MLE}})^2 - \sigma^4 \right) \xrightarrow{d} \mathcal{N}(0, 8\sigma^8)$$

Extension to continuous Itô semimartingales

Let us now consider a continuous Itô semimartingale of the form

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s$$

Assume that the object of interest is given by

$$\Phi(\phi)_t = \int_0^t \phi(\sigma_s^2) ds$$

for a smooth function $\phi : \mathbb{R} \to \mathbb{R}$. Following the ideas of the parametric case, it is natural to propose the statistic

$$\Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \phi\left(\widehat{\sigma}^2_{(i-1)\Delta_n}\right)$$

as an estimator of $\Phi(\phi)_t$. Indeed, a bias corrected version of this statistic has been investigated in Jacod & Rosenbaum (13).

An optimality result for functionals $\Phi(\phi)_t$

We now present an optimality result from Clement, Delattre & Gloter (13). They consider a stochastic differential equation of the type

$$X_t = X_0 + \int_0^t a(X_s)ds + \int_0^t \sigma(X_s, b_s)dW_s$$

where b is a continuous Itô semimartingale of the form

$$b_t = b_0 + \int_0^t a'_s ds + \int_0^t v'_s dV_s$$

and V is a Brownian motion **independent** of W. In this setting we have that

$$\Phi(\phi)_t = \int_0^t \phi(\sigma^2(X_s, b_s))ds$$

To present the main result we need to introduce the notion of **local asymptotic mixed normality** (LAMN).

Definition of the local asymptotic mixed normality

Definition (LAMN, 1-dimensional case)

Assume that the data is generated from the probability measure $\mathbb{P}_\theta^n$ with $\theta \in \Theta \subset \mathbb{R}$. We say that the family $\{\mathbb{P}_\theta^n\}$ satisfies the LAMN property at θ_0 if there exist measurable random variables $N_{\theta_0}^n$ and $I_{\theta_0}^n$ such that the following decomposition holds:

$$\log \frac{\mathbb{P}^n_{\theta_0+h/\sqrt{n}}}{\mathbb{P}^n_{\theta_0}} = hN^n_{\theta_0} - \frac{1}{2}h^2 I^n_{\theta_0} + o_{\mathbb{P}^n_{\theta_0}}(1)$$

Furthermore, it must hold that

$$\left(I^n_{\theta_0}, N^n_{\theta_0}\right) \xrightarrow{d} (I_{\theta_0}, \mathcal{MN}(0, I_{\theta_0})) \qquad \text{wrt } \mathbb{P}^n_{\theta_0}$$

The definition of LAMN can be extended to the infinite dimensional setting. This version is used in Clement, Delattre & Gloter (13).

Main result

In the following we denote by

$$\mathbb{P}_\theta^n, \qquad \theta \in C([0,t])$$

the law of the data $(X_{i\Delta_n})_{0\le i\le \lfloor t/\Delta_n\rfloor}$ conditionally on $b=\theta$.

Theorem (Clement, Delattre & Gloter (13))

Set $\Phi(\phi)_t = \int_0^t \phi(\sigma^2(X_s,b_s))ds$ and let $\Phi(\phi)_t^n$ be an estimator of $\Phi(\phi)_t$ such that

$$\Delta_n^{-1/2}\left(\Phi(\phi)_t^n - \Phi(\phi)_t\right) \xrightarrow{d} Z$$

Under certain regularity conditions on the model the LAMN property for $\{\mathbb{P}_\theta^n\}$ holds and Z has the form

$$Z = \sqrt{2\int_0^t (\phi')^2(\sigma^2(X_s,b_s))\sigma^4(X_s,b_s)ds} \times N + R$$

where $N \sim \mathcal{N}(0,1)$ and independent of R. In other words, the first term in Z gives an optimal lower bound, which is achieved by the estimator proposed in Jacod & Rosenbaum (13).

Optimality result for estimation of jumps

Let us now consider a stochastic differential equation of the type

$$dX_t = a(X_t)dt + \sigma(X_t)dW_t + dJ_t$$

where J is a jump process with **finite** number of jumps on compact intervals.

Theorem (Clement, Delattre & Gloter (14))

Let us denote by (T_k) the jump times of J and assume that an estimator Φ_t^n satisfies the convergence

$$\Delta_n^{-1/2}\left(\Phi_t^n - (\Delta J_{T_m})_{m:T_m \leq t}\right) \xrightarrow{d} Z$$

Under certain regularity conditions on a, σ and the jump process J it necessarily holds that

$$Z = \left(\sqrt{\kappa_m}\sigma(X_{T_m-})U_m^- + \sqrt{1-\kappa_m}\,\sigma(X_{T_m})U_m^+\right)_{m:T_m \leq t} + R$$

where R is independent of $(U_m^-, U_m^+)_{m\in\mathbb{N}}$.

Chapter 6: New optimality results for supremum, local times and occupation time measure

- The results of Clement, Delattre & Gloter (13) only cover estimation problems for volatility functionals. In this chapter we will rather focus on the following random objects:

$$\overline{X}_t := \sup_{s\in[0,t]} X_s$$

$$l_t(x) := \lim_{\epsilon\downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{(-\epsilon,\epsilon)}(X_s - x)ds$$

$$L_t(x) := \int_0^t 1_{(x,\infty)}(X_s)ds$$

which is the supremum, local time and occupation time measure of the process X, respectively.

- We are interested in **optimal estimation** of these objects given high frequency data $(X_{i\Delta_n})_{0\leq i\leq \lfloor t/\Delta_n\rfloor}$.

A remark on optimality

We will see that many naive estimators are rate optimal, but not efficient! In fact, efficient estimators are easy to introduce.

Let $Q = \Phi((X_s)_{s\in[0,t]})$ be a random variable of interest. An optimal estimator of Q is given as

(i) in L^2-sense: $\mathbb{E}[Q|\ (X_{i\Delta_n})_{0\le i\le \lfloor t/\Delta_n\rfloor}]$

(ii) in L^1-sense: $\text{median}[Q|\ (X_{i\Delta_n})_{0\le i\le \lfloor t/\Delta_n\rfloor}]$

We will investigate the asymptotic theory for these type of estimates in the setting of supremum, local time and occupation time measure of the process X, where X is a Brownian motion, stable Lévy process or a continuous diffusion process.

Chapter 6.1: The case of supremum

- It is rather simple to propose the following estimate for the supremum

$$M_n := \max_{i=1,\dots,\lfloor 1/\Delta_n \rfloor} X_{i\Delta_n} \xrightarrow{\mathbb{P}} \overline{X}_1$$

where the consistency holds for all Lévy processes L.

- The asymptotic theory for the maximum has been studied in several papers including Asmussen, Glynn & Pitman (95) (Brownian motion) and Ivanovs (18) (general Lévy processes).

- Since $M_n < \overline{X}_1$, the estimator M_n is downward biased and there were several attempts to correct the bias.

A result on zooming-in at supremum

The following result from the theory of Lévy processes will be extremely useful for our asymptotic theory.

Theorem (Ivanovs (18))

Let X be an α-stable Lévy process with $\alpha \in (0,2]$. Denote by τ the time of the supremum of X on the interval $[0,1]$. Then we obtain the functional stable convergence

$$(Z_t^n)_{t\in\mathbb{R}} := \left(\Delta_n^{-1/\alpha}\left(X_{\tau+t\Delta_n} - X_\tau\right)\right)_{t\in\mathbb{R}} \stackrel{d_{st}}{\to} \left(\widehat{X}_t\right)_{t\in\mathbb{R}}$$

*where $\widehat{X}$ is the so called **Lévy process conditioned to stay negative**, which is independent of $\mathcal{F}$. When X is a Brownian motion, we deduce the identity*

$$\widehat{X}_t = -\|B_t\|$$

where B is a 3-dimensional Brownian motion.

Application to estimation of the supremum

The previous result has the following consequence.

Theorem (Ivanovs (18))

Let X be an α-stable Lévy process with $\alpha \in (0,2]$. Then it holds that

$$\Delta_n^{-1/\alpha}\left(M_n - \overline{X}_1\right) \xrightarrow{d} \max_{j\in\mathbb{Z}}(\widehat{X}_{j+U})$$

where $U \sim \mathcal{U}(0,1)$ is independent of $\widehat{X}$ and $\mathcal{F}$.

Proof: Note that

$$\Delta_n^{-1/\alpha}\left(X_{i\Delta_n} - X_\tau\right) = Z^n_{i-\tau/\Delta_n} = Z^n_{i-\lfloor\tau/\Delta_n\rfloor-\{\tau/\Delta_n\}}$$

Recall that $\{\tau/\Delta_n\} \xrightarrow{d_{st}} U \sim \mathcal{U}(0,1)$. Since $Z^n \xrightarrow{d_{st}} \widehat{X}$ and the functional $\max_{j\in\mathbb{Z}}(\cdot)$ is **shift invariant**, we conclude that

$$\Delta_n^{-1/\alpha}\left(M_n - \overline{X}_1\right) \xrightarrow{d} \max_{j\in\mathbb{Z}}(\widehat{X}_{j+U})$$

□

The basis of our approach is the computation of the conditional probability

$$H_n(x) := \mathbb{P}\left(\overline{X}_1 \le x |\ (X_{i\Delta_n})_{0\le i\le \lfloor 1/\Delta_n \rfloor}\right) \qquad x > 0.$$

Due to Markov and self-similarity property of X, we easily see that

$$H_n(x) = \prod_{i=1}^{n} F\left(\Delta_n^{-1/2}(x - X_{\frac{i-1}{n}}), \Delta_n^{-1/2}\Delta_i^n X\right)$$

where $F(x,y) = \mathbb{P}\left(\overline{X}_1 \le x |\ X_1 = y\right) = 1 - \exp(-2x(x-y))$. After rescaling we deduce the stable convergence

$$H_n\left(\Delta_n^{1/2}x + M_n\right) = \prod_{i\in\mathbb{Z}} F\left(x + \Delta_n^{-1/2}(M_n - X_{(i-1)\Delta_n}), \Delta_n^{-1/2}\Delta_i^n X\right)$$

$$\xrightarrow{d_{st}} G(x) := \prod_{i\in\mathbb{Z}} F\left(x + \max_{j\in\mathbb{Z}} \widehat{X}_{j+U} - \widehat{X}_{i+U}, \widehat{X}_{i+1+U} - \widehat{X}_{i+U}\right).$$

Asymptotic theory for the optimal estimators: Brownian case

Theorem (Ivanovs & P. (19))

Define the estimates

$$T_n^{(1)} := median\left[\overline{X}_1 \middle| \, (X_{i\Delta_n})_i\right], \qquad T_n^{(2)} := \mathbb{E}\left[\overline{X}_1 \middle| \, (X_{i\Delta_n})_i\right].$$

(i) *It holds that*

$$\Delta_n^{-1/2}\left(T_n^{(1)} - \overline{X}_1\right) \xrightarrow{d} \max_{j\in\mathbb{Z}}(\widehat{X}_{j+U}) + G^{-1}(1/2).$$

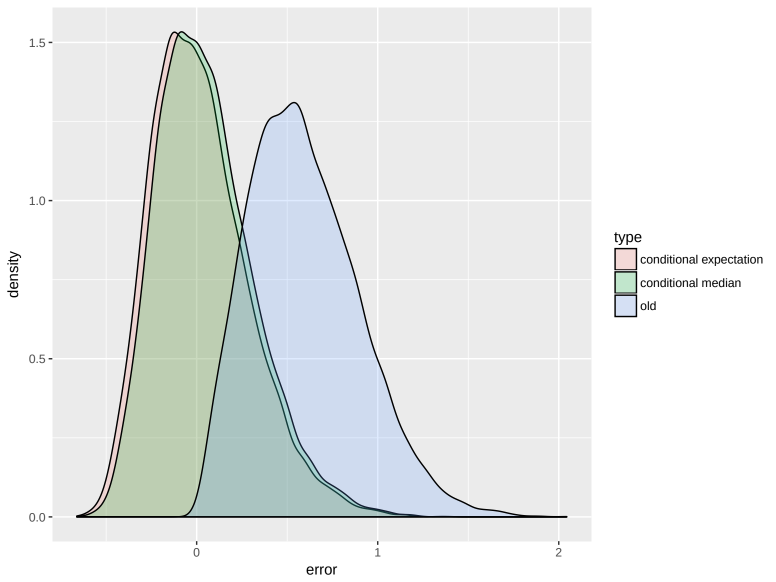
(ii) *Furthermore,*

$$\Delta_n^{-1/2}\left(T_n^{(2)} - \overline{X}_1\right) \xrightarrow{d} \max_{j\in\mathbb{Z}}(\widehat{X}_{j+U}) + \int_0^\infty (1 - G(y))dy.$$

In particular, we have that

$$\frac{MSE(M_n)}{MSE(T_n^{(2)})} \approx 6.25 \; !$$

Simulation of asymptotic distributions



Theorem (Ivanovs & P. (19))

Let X be an α-stable Lévy motion with $\alpha \in (0,2)$.

(i) *Define $T_n^{(1)} = median[\overline{X}_1 |\ (X_{i\Delta_n})_i]$. Then we obtain*

$$\Delta_n^{-1/\alpha}\left(T_n^{(1)} - \overline{X}_1\right) \xrightarrow{d} \max_{j\in\mathbb{Z}}(\widehat{X}_{j+U}) + G^{-1}(1/2).$$

and the estimator is L^1-optimal for $\alpha \in (1,2)$.

(ii) *Define $T_n^{(2)} = \mathbb{E}[\overline{X}_1 |\ (X_{i\Delta_n})_i]$ for $\alpha \in (1,2)$. Then it holds that*

$$\Delta_n^{-1/\alpha}\left(T_n^{(2)} - \overline{X}_1\right) \xrightarrow{d} \max_{j\in\mathbb{Z}}(\widehat{X}_{j+U}) + \int_0^\infty (1 - G(y))dy.$$

Chapter 6.2: The case of local times

Naive estimators

- In this chapter we assume that X is a Brownian motion. Recall the definition of local time:

$$l_t(x) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{(-\epsilon,\epsilon)}(X_s - x) ds$$

where $x \in \mathbb{R}$.

- A straightforward estimator of $l_t(x)$ is given as

$$l_t^n(x) := a_n \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} g\left(a_n(X_{i\Delta_n} - x)\right) \xrightarrow{\mathbb{P}} l_t(x)$$

where g is a kernel satisfying $\int_{\mathbb{R}} g(x)dx = 1$, and $a_n \to \infty$ with $a_n \Delta_n \to 0$.

- We will focus on a more general class of statistics:

$$V(h,x)_t^n := a_n \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} g\left(a_n(X_{i\Delta_n} - x), \Delta_n^{-1/2} \Delta_i^n X\right)$$

Theorem (Borodin (86), Jacod (98))

Assume that $a_n = \Delta_n^{-1/2}$ and h satisfies the condition $|h(y,z)| \leq h_1(y)\exp(\lambda|z|)$ for some $\lambda > 0$ and $\int_{\mathbb{R}} |y|^p h_1(y)dy < \infty$ for some $p > 3$. Then it holds that

$$V(h,x)_t^n \stackrel{u.c.p.}{\rightarrow} c_h l_t(x)$$

where $c_h = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(y,z)\varphi(z)dz\right) dy$ and φ denotes the density of the standard normal distribution. Furthermore, we obtain the stable convergence

$$\Delta_n^{-1/4}\left(V(h,x)_t^n - c_h l_t(x)\right) \stackrel{d_{st}}{\rightarrow} \mathcal{MN}(0, v_h l_t(x))$$

for a certain constant $v_h > 0$.

An interesting example is the **number of crossings at level 0** which corresponds to $x = 0$ and $h(y,z) = 1_{(-\infty,0)}(y(y+z))$.

L^2-optimal estimator of the local time

As we mentioned earlier, the L^2-optimal estimator of the local time is given by

$$\widehat{l}_t^n(x) = \mathbb{E}\left[l_t(x) \middle| \, (X_{i\Delta_n})_{1 \le i \le \lfloor t/\Delta_n \rfloor} \right]$$

The following distributional identity connects the law of local times to the law of the supremum:

$$(l_t(0), |X_t|)_{t \in \mathbb{R}} = \left(\overline{X}_t, \overline{X}_t - X_t \right)_{t \in \mathbb{R}}$$

Applying the Markov and self-similarity property of the Brownian motion we deduce that

$$\widehat{l}_t^n(x) = V(h_0, x)_t^n \qquad \text{with} \qquad a_n = \Delta_n^{-1/2}$$

and

$$h_0(y,z) = 2|y| e^{z^2/2} \int_0^1 s^{-3/2} e^{-y^2/(2s)} \overline{\Phi}\left(\frac{|y+z|}{\sqrt{1-s}} \right) ds$$

Here $\overline{\Phi}$ denotes the tail distribution of the standard normal law.

Asymptotic theory for $V(h, x)^n$

Theorem (Ivanovs & P. (19))

We obtain the stable convergence

$$\Delta_n^{-1/4} \left(V(h_0, x)_t^n - c_{h_0} l_t(x) \right) \xrightarrow{d_{st}} \mathcal{MN}(0, v_{h_0} l_t(x))$$

We conjecture that this result can be extended to continuous stochastic differential equations.

Chapter 6.3: The case of the occupation time measure

- In this chapter we consider a Brownian motion X. The object of interest is the occupation time measure

$$L_t(x) = \int_0^t 1_{(x,\infty)}(X_s)ds$$

which turns out to be easier to treat than the previous two cases.

- We will again compute the conditional mean estimator

$$L_t^n(x) := \mathbb{E}\left[L_t(x)|\ (X_{i\Delta_n})_{1\leq i\leq \lfloor t/\Delta_n\rfloor}\right]$$

Define $L_{i-1}^i(x) = \int_{(i-1)\Delta_n}^{i\Delta_n} 1_{(x,\infty)}(X_s)ds$ and observe the identity

$$\mathbb{E}\left[L_{i-1}^i(x)|X_{(i-1)\Delta_n}, \Delta_n^{-1/2}\Delta_i^n X\right]$$
$$= \Delta_n \int_0^1 \overline{\Phi}_{t(1-t)}\left(\Delta_n^{-1/2}(x - X_{(i-1)\Delta_n} - t\Delta_i^n X)\right) dt$$

where $\overline{\Phi}_t$ is the tail of $\mathcal{N}(0,t)$.

Using again the Markov property of the Brownian motion we obtain the formula

$$L_t^n(x) = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[L_{i-1}^{i}(x) \middle| \, (X_{i\Delta_n})_{1\le i\le \lfloor t/\Delta_n \rfloor} \right]$$

$$= \Delta_n \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} f\left(\Delta_n^{-1/2}(x - X_{(i-1)\Delta_n}), \Delta_n^{-1/2} \Delta_i^n X \right)$$

with

$$f(y,z) = \int_0^1 \overline{\Phi}_{t(1-t)}\,(y - tz)\,dt$$

We remark that the function f does not satisfy the assumptions of the previous chapter! In particular, $L_t^n(x)$ does not convergence to the local time.

Asymptotic theory for $L_t^n(x)$

Theorem (Ivanovs & P. (19))

We obtain the stable convergence

$$\Delta_n^{-3/4}\left(L_t^n(x) - \int_0^t 1_{(x,\infty)}(X_s)ds\right) \xrightarrow{d_{st}} \mathcal{MN}(0, v_f l_t(x))$$

where $v_f > 0$ is a certain constant. We conjecture that a similar result holds for symmetric α-stable Lévy processes with $\alpha \in (1,2)$:

$$\Delta_n^{-1/2-1/\alpha}\left(L_t^n(x) - L_t(x)\right) \xrightarrow{d_{st}} \mathcal{MN}(0, v_\alpha l_t(x))$$

The rate optimality of the rate $\Delta_n^{-3/4}$ has been shown in Ngo & Ogawa (11) in the setting of continuous diffusion models.

Sketch of proof: We apply once again Jacod's stable limit theorem. Observe that

$$\Delta_n^{-3/4}\left(L_t^n(x)-\int_0^t 1_{(x,\infty)}(X_s)ds\right)=\sum_{i=1}^{\lfloor t/\Delta_n\rfloor} Z_{in}$$

with

$$Z_{in}:=\Delta_n^{-3/4}\left(\Delta_n f\left(\Delta_n^{-1/2}(x-X_{(i-1)\Delta_n}),\Delta_n^{-1/2}\Delta_i^n X\right)-L_{i-1}^i(x)\right)$$

By construction we have that $\mathbb{E}[Z_{in}|\mathcal{F}_{(i-1)\Delta_n}]=0$. Furthermore, it holds that

$$\mathbb{E}[Z_{in}^2|\mathcal{F}_{(i-1)\Delta_n}]=\Delta_n^{1/2}f_1(\Delta_n^{-1/2}(x-X_{(i-1)\Delta_n}))$$

$$\mathbb{E}[Z_{in}\Delta_i^n X|\mathcal{F}_{(i-1)\Delta_n}]=\Delta_n^{3/4}f_2(\Delta_n^{-1/2}(x-X_{(i-1)\Delta_n}))$$

where $f_1,f_2\in L^1(\mathbb{R})$.

Cont.

From the asymptotic theory for local times we thus conclude that

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[Z_{in}^2 | \mathcal{F}_{(i-1)\Delta_n}] = \Delta_n^{1/2} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} f_1(\Delta_n^{-1/2}(x - X_{(i-1)\Delta_n})) \stackrel{\mathbb{P}}{\to} \|f_1\|_{L^1(\mathbb{R})} l_t(x)$$

and

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}[Z_{in} \Delta_i^n X | \mathcal{F}_{(i-1)\Delta_n}] = \Delta_n^{3/4} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} f_2(\Delta_n^{-1/2}(x - X_{(i-1)\Delta_n})) \stackrel{\mathbb{P}}{\to} 0$$

The other conditions of Jacod's theorem are straightforward to prove. Hence, we obtain the stable convergence

$$\Delta_n^{-3/4} \left(L_t^n(x) - \int_0^t 1_{(x,\infty)}(X_s) ds \right) \stackrel{d_{st}}{\to} \mathcal{MN}\left(0, \|f_1\|_{L^1(\mathbb{R})} l_t(x)\right)$$

□

End of the course

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Thank you very much for your attention!