# Tackling Stationary and Randomized Heston Models using Quantization

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# Stationary and Randomized Heston model

## The model

Dynamic of the asset price process $(S_t)_{t\geqslant 0}$ and its volatility $(v_t)_{t\geqslant 0}$ is given by

$$\begin{cases} dS_t = S_t\left(rdt + \sqrt{v_t}dW_t\right) \\ dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t}d\widetilde{W}_t \end{cases}$$

- $S_0 = s_0$ is the initial value of the process,
- $\kappa$ the mean reverting term,
- $\theta$ the long run average price variance,
- $\xi$ is the volatility of the volatility,
- $(W, \widetilde{W})$ is a standard correlated 2d Brownian motion with correlation $\rho$,

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- $(W, \widetilde{W})$ is a standard correlated 2d Brownian motion with correlation $\rho$,
- $v_0 = \mathcal{L}(\nu_0)$ **with** $\nu_0 \sim \Gamma(\alpha, \beta)$ **with** $\beta = (2\kappa)/\xi^2$ **and** $\alpha = \theta\beta$**.**

## History

Introduced by G. Pagès et F. Panloup in 2009 and later, studied by A. Jacquier and F. Shi in 2017.

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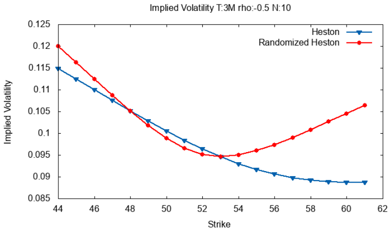
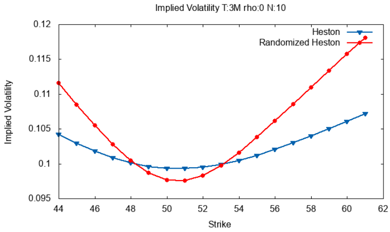
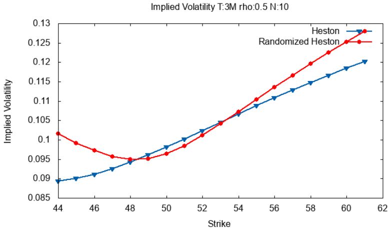



(a) $\rho = -0.5$




(b) $\rho = 0$




(c) $\rho = 0.5$

Figure: Implied volatility ($T = 0.25$) - 3M

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# Definitions

Let $\Gamma_N = \{x_1^N, \dots, x_N^N\} \subset \mathbb{R}$, a subset of size $N$, called $N$-quantizer, we define:

- The *Voronoï partition* of $\mathbb{R}$ induced by the $N$-quantizer:

$$C_i(\Gamma_N) = (x_{i-1/2}^N, x_{i+1/2}^N], \quad i \in [\![1, N-1]\!], \quad C_N(\Gamma_N) = (x_{N-1/2}, x_{N+1/2}^N).$$

  **Easily defined in dimension one.**

- The *Voronoï Quantization* of the random variable $X$:

$$\widehat{X}^{\Gamma_N} = \mathrm{Proj}_{\Gamma_N}(X) = \sum_{i=1}^{N} x_i^N \, \mathbb{1}_{X \in C_i(\Gamma_N)}$$

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- It is convenient to define the quadratic distortion function at level $N$:

$$\mathcal{Q}_{2,\mathrm{N}} : x = (x_1^N, \dots, x_N^N) \longmapsto \mathbb{E}\left[ \min_{i \in [\![1,N]\!]} |X - x_i^N|^2 \right] = \|X - \widehat{X}^N\|_2^2.$$

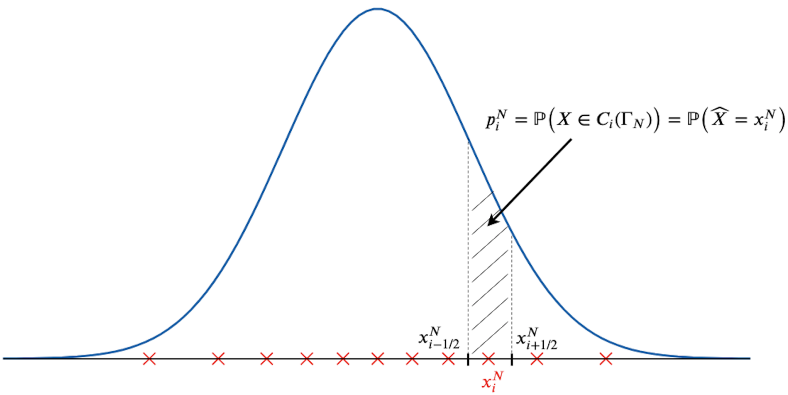


Figure: Gaussian Optimal Quantization

# How to build an Optimal Quantizer?

## 1. Differentiate the $\mathcal{Q}_{2,\mathrm{N}}$

The gradient is given by

$$\nabla \mathcal{Q}_{2,\mathrm{N}}(x_{\mathbf{1}:N}) = \left( \mathbb{E}\left[ \left( x_i^N - X \right) \mathbb{1}_{X \in (x^N_{i-\mathbf{1/2}}, x^N_{i+\mathbf{1/2}}]} \right] \right)_{i=1,\dots,N}$$

## 2. Solve the fixed point problem

Then, find $x_{1:N}$ that cancel the gradient

$$\nabla \mathcal{Q}_{2,\mathrm{N}}\left( x_{\mathbf{1}:N} \right) = 0 \iff x_i^N = \frac{\mathbb{E}\left[ X \, \mathbb{1}_{X \in (x^N_{i-\mathbf{1/2}}, x^N_{i+\mathbf{1/2}}]} \right]}{\mathbb{P}\left( X \in (x^N_{i-1/2}, x^N_{i+1/2}] \right)}, \qquad i = 1, \dots, N$$

$$\iff x_i^N = \frac{K_X\left(x^N_{i+1/2}\right) - K_X\left(x^N_{i-1/2}\right)}{F_X\left(x^N_{i+1/2}\right) - F_X\left(x^N_{i-1/2}\right)}, \qquad i = 1, \dots, N$$

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The price of the European option on the asset $S_T^{v_0}$ is given by

$$I_0 := \mathbb{E}\left[\, e^{-rT}\, \varphi(S_T^{v_0})\right].$$

By preconditioning we have

$$I_0 = \mathbb{E}\left[\, \mathbb{E}\left[\, e^{-rT}\, \varphi(S_T^{v_0}) \mid \sigma(v_0)\right]\right] = \mathbb{E}\left[ f(v_0)\right]$$

where $f(v)$ is the price of the European option in the Standard Heston model with deterministic initial conditions for a given initial volatility $v$.

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## Cubature formula

- Build an optimal quantizer of $v_0 \sim \Gamma(\alpha, \beta)$
- Approximate $I_0$ using the following quantization-based cubature formula

$$\widehat{I}_0^N := \mathbb{E}\left[f(\widehat{v}_0^N)\right] = \sum_{i=1}^{N} f(v_{0,i}^N)\,\mathbb{P}(\widehat{v}_0^N = v_{0,i}^N).$$

- Fast convergence of $\widehat{I}_0^N$ toward $I_0$, for smooth enough functions $f$

$$\lim_{N\to+\infty} N^{2}\left|\mathbb{E}\left[f(v_0)\right]-\mathbb{E}\left[f(\widehat{v}_0^N)\right]\right| < C_{v_0,f} < +\infty.$$

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- From European options (preconditioning by $v_0$ at time $t_0$) to path-dependent ones (need to precondition by $(v_k)_{k=0:n}$ at time $t_k$'s).

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# Litterature

## Recursive Quantization

- *Recursive marginal quantization of the Euler scheme of a diffusion process* by G. Pagès and A. Sagna. (2015)
- *Recursive Marginal Quantization of Higher-Order Schemes* by J. Kienitz, T. A. McWalter, E. Platen and R. Rudd. (2017)
- *Product Markovian quantization of an $R^d$-valued Euler scheme of a diffusion process with applications to finance* by L. Fiorin, G. Pagès and A. Sagna. (2018)

## Previous work on Heston model using Quantization

- *Pricing via Quantization in Stochastic Volatility Models* by G. Callegaro, L. Fiorin and M. Grasselli. (2016)
- *Fast Quantization of Stochastic Volatility Models* by J. Kienitz, T. A. McWalter, E. Platen and R. Rudd. (2017)
- *American quantized calibration in stochastic volatility* by G. Callegaro, L. Fiorin and M. Grasselli. (2018)
- And more...

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# Model transformation

$$
\begin{cases}
dS_t = S_t\left(rdt + \sqrt{v_t}dW_t\right) \\
dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t}d\widetilde{W}_t
\end{cases}
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**Remark**

$v_t$ is autonomous, hence $1d$ problem (again).

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**Remark**

$v_t$ is autonomous, hence $1d$ problem (again).

We will be working with $(X_t, Y_t)$ that are transformation of $(S_t, v_t)$ :

- For the volatility $\longrightarrow Y_t = e^{\kappa t}\, v_t$
- For the asset $\longrightarrow X_t = \log(S_t)$

# First, the volatility

## Milstein Scheme (preserving the positivity)

We consider the following *boosted* volatility process: let $Y_t = e^{\kappa t} v_t$, $t \in [0, T]$.

$$dY_t = e^{\kappa t} \kappa\theta dt + \xi e^{\frac{\kappa t}{2}} \sqrt{Y_t} d\widetilde{W}_t.$$

Now, if we look at the Milstein discretization scheme of $Y_t$

$$\bar{Y}_{t_{k+1}} = \mathcal{M}^{\Delta}_{\widetilde{b},\widetilde{\sigma}}\big(t_k, \bar{Y}_{t_k}, \widetilde{Z}_{k+1}\big)$$

where $\widetilde{Z}_{k+1} \sim \mathcal{N}(0,1)$ and

$$\mathcal{M}^{\Delta}_{\widetilde{b},\widetilde{\sigma}}(t,x,z) := x - \frac{\widetilde{\sigma}(t,x)}{2\widetilde{\sigma}'_x(t,x)} + \Delta\left(\widetilde{b}(t,x) - \frac{\widetilde{\sigma}\widetilde{\sigma}'_x(t,x)}{2}\right) + \frac{\widetilde{\sigma}\widetilde{\sigma}'_x(t,x)\Delta}{2}\left(z + \frac{1}{\sqrt{\Delta}\widetilde{\sigma}'_x(t,x)}\right)^2$$

with

$$\widetilde{b}(t,x) := e^{\kappa t} \kappa\theta, \qquad \widetilde{\sigma}(t,x) := \xi\sqrt{x}\, e^{\frac{\kappa t}{2}} \quad \text{and} \quad \widetilde{\sigma}'_x(t,x) := \frac{\xi e^{\frac{\kappa t}{2}}}{2\sqrt{x}}.$$

# Then, the log-asset

## Euler-Maruyama scheme

We consider the logarithm of the asset $X_t := \log(S_t)$, yielding

$$dX_t = \left(r - \frac{v_t}{2}\right) dt + \sqrt{v_t} dW_t.$$

Now, using an Euler-Maruyama scheme for the discretization of $X_t$, we have

$$\begin{cases} \bar{X}_{t_{k+1}} = \mathcal{E}^{\Delta}_{b,\sigma}(t_k, \bar{X}_{t_k}, \bar{Y}_{t_k}, Z_{k+1}) \\ \bar{Y}_{t_{k+1}} = \mathcal{M}^{\Delta}_{\tilde{b},\tilde{\sigma}}(t_k, \bar{Y}_{t_k}, \tilde{Z}_{k+1}) \end{cases}$$

where $Z_{k+1} \sim \mathcal{N}(0,1)$ and $\mathbb{C}\mathrm{orr}(Z_{k+1}, \tilde{Z}_{k+1}) = \rho$ and

$$\mathcal{E}^{\Delta}_{b,\sigma}(t,x,y,z) := x + b(t,x,y)\Delta + z\sigma(t,x,y)\sqrt{\Delta}$$

with

$$b(t,x,y) := r - \frac{\mathrm{e}^{-\kappa t} y}{2} \qquad \text{and} \qquad \sigma(t,x,y) := \sqrt{\mathrm{e}^{-\kappa t} y}.$$

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# First, the volatility

We build recursively the Markovian quantization tree $(\widehat{Y}_k)_{k \in [\![0,n]\!]}$ where $\widehat{Y}_{t_{k+1}}$ is the Voronoï quantization of $\widetilde{Y}_{k+1}$ defined by

$$\widetilde{Y}_{k+1} = \mathcal{M}^{\Delta}_{\widetilde{b},\widetilde{\sigma}}(t_k, \widehat{Y}_k, \widetilde{Z}_{k+1}), \qquad \widehat{Y}_{t_{k+1}} = \mathrm{Proj}_{\Gamma^Y_{N_2}}\left(\widetilde{Y}_{k+1}\right)$$

with $\Gamma^Y_{N_2} := \{y^k_1, \dots, y^k_{N_2}\}$ the optimal $N_2$-quantizer of $\widetilde{Y}_{k+1}$ and $\widetilde{Z}_{k+1} \sim \mathcal{N}(0,1)$.

# Then, the log-asset

Now, using the fact that $(Y_t)_t$ has already been quantized and the Euler-Maruyama scheme of $(X_t)_t$, we define the Markov quantized scheme

$$\widetilde{X}_{k+1} = \mathcal{E}^{\Delta}_{b,\sigma}\big(t_k, \widehat{X}_k, \widehat{Y}_k, Z_{k+1}\big), \qquad \widehat{X}_{k+1} = \mathrm{Proj}_{\Gamma^X_{N_1}}\left(\widetilde{X}_{k+1}\right)$$

with $\Gamma^X_{N_1} := \{x_1^k, \ldots, x_{N_1}^k\}$ the optimal $N_1$-quantizer of $\widetilde{X}_{k+1}$ and $Z_{k+1} \sim \mathcal{N}(0,1)$.

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# Bermudan Options

Its price, at time $t_0$, is given by

$$V_0 := \sup_{\tau \in \{t_1, \cdots, t_n\}} \mathbb{E}\left[ e^{-r\tau} \psi_\tau(X_\tau, Y_\tau) \mid \mathcal{F}_{t_0} \right].$$

Hence, we can define recursively the sequence of random variable $L^p$-integrable $(V_k)_{0 \leqslant k \leqslant n}$ :

$$\begin{cases} V_n = e^{-rt_n} \psi_n(X_n, Y_n), \\ V_k = \max\left( e^{-rt_k} \psi_k(X_k, Y_k), \mathbb{E}[V_{k+1} \mid \mathcal{F}_k] \right), \qquad 0 \leqslant k \leqslant n-1 \end{cases}$$

called *Backward Dynamical Programming Principle*.

# Bermudan Options

## Using the Product Recursive Quantizer

We approximate the *Backward Dynamical Programming Principle* by the following sequence involving the couple $(\widehat{X}_k, \widehat{Y}_k)_{0 \leqslant k \leqslant n}$:

$$\begin{cases} \widehat{V}_n = \mathrm{e}^{-rt_n} \, \psi_n(\widehat{X}_n, \widehat{Y}_n), \\ \widehat{V}_k = \max\Big( \mathrm{e}^{-rt_k} \, \psi_k(\widehat{X}_k, \widehat{Y}_k), \mathbb{E}\big[ \widehat{V}_{k+1} \mid (\widehat{X}_k, \widehat{Y}_k) \big] \Big), \qquad k = 1, \ldots, n-1 \end{cases}$$

# Bermudan Options

## Using the Product Recursive Quantizer

The last equation can be rewritten

$$\begin{cases} \widehat{v}_n(x_i^n, y_j^n) = \mathrm{e}^{-rt_n}\,\psi_n(x_i^n, y_j^n), \quad i = 1,\dots,N_1, \quad j = 1,\dots,N_2, \\ \widehat{v}_k(x_i^k, y_j^n) = \max\left(\mathrm{e}^{-rt_k}\,\psi_k(x_i^k, y_j^n), \sum_{l=1}^{N_1}\sum_{m=1}^{N_2} \pi^k_{(i,j),(l,m)}\,\widehat{v}_{k+1}(x_l^{k+1}, y_m^{k+1})\right), \\ \qquad k = 1,\dots,n-1, \quad i = 1,\dots,N_1, \quad j = 1,\dots,N_2. \end{cases}$$

with $\pi^k_{(i,j),(l,m)} := \mathbb{P}(\widehat{X}_{k+1} = x_l^{k+1}, \widehat{Y}_{k+1} = y_m^{k+1} \mid \widehat{X}_k = x_i^k, \widehat{Y}_k = y_j^k)$.
Finally, the approximation of the price of the bermudan option is given by

$$\mathbb{E}\left[\widehat{v}_k(x_0, \widehat{Y}_0)\right] = \sum_{i=1}^{N_{2,0}} p_i\,\widehat{v}_k(x_0, y_i^0)$$

with $p_i := \mathbb{P}(\widehat{Y}_0 = y_i^0)$ previously defined.

# Barrier Options

A Barrier option is a path-dependent financial product whose payoff at maturity date $T$ depends on the value of the process $X_T$ at date $T$ and its maximum or minimum over the period $[0, T]$. More precisely, we are interested by options with the following types of payoff $h$

$$h = f(X_T)\,\mathbb{1}_{\{\sup_{t\in[0,T]} X_t \in I\}} \qquad or \qquad h = f(X_T)\,\mathbb{1}_{\{\inf_{t\in[0,T]} X_t \in I\}}$$

where $I$ is an unbounded interval of $\mathbb{R}$, $T$ is the maturity date and $f$ can be any vanilla payoff function (Call, Put, Spread, Butterfly, ...).

# Barrier Options

## Using a representation formula

Now, using the representation formula based on the **conditional law of the Brownian Bridge** for the price of up-and-out options $\bar{P}_{UO}$ and down-and-out options $\bar{P}_{DO}$

$$\bar{P}_{UO} := \mathrm{e}^{-rT}\,\mathbb{E}\left[f(\bar{X}_T)\,\mathbb{1}_{\sup_{t\in[0,T]}\bar{X}_t\leqslant L}\right] = \mathrm{e}^{-rT}\,\mathbb{E}\left[f(\bar{X}_T)\prod_{k=0}^{n-1} G^k_{(\bar{X}_k,\overline{Y}_k),\bar{X}_{k+1}}(L)\right]$$

where $L$ is the barrier and

$$G^k_{(x,y),z}(u) = \left(1 - \mathrm{e}^{-2n\frac{(x-u)(z-u)}{T\sigma^2(t_k,x,y)}}\right)\mathbb{1}_{\{u\geqslant \max(x,z)\}}$$

**Equivalent formulas for other standard Barrier options.**

# Barrier Options

## Using the Product Recursive Quantizer

Finally, replacing $(\bar{X}_k, \bar{Y}_k)$ by $(\widehat{X}_k, \widehat{Y}_k)$ and using a recursive algorithm yield

$$\begin{cases} \widehat{V}_n = \mathrm{e}^{-rT} f(\widehat{X}_n), \\ \widehat{V}_k = \mathbb{E}\left[ g_k(\widehat{X}_k, \widehat{Y}_k, \widehat{X}_{k+1}) \widehat{V}_{k+1} \mid (\widehat{X}_k, \widehat{Y}_k) \right], \qquad 0 \leqslant k \leqslant n-1 \end{cases}$$

that can be rewritten

$$\begin{cases} \widehat{v}_n(x_i^n, y_j^n) = \mathrm{e}^{-rT} f(x_i^n), \quad i = 1, \dots, N_1, \quad j = 1, \dots, N_2 \\ \widehat{v}_k(x_i^k, y_j^n) = \displaystyle\sum_{l=1}^{N_1} \sum_{m=1}^{N_2} \pi_{(i,j),(l,m)}^k \widehat{v}_{k+1}(x_l^{k+1}, y_m^{k+1}) g_k(x_i^k, y_j^k, x_l^{k+1}), \\ \qquad\qquad k = 1, \dots, n-1, \quad i = 1, \dots, N_1, \quad j = 1, \dots, N_2 \end{cases}$$

with $\pi_{(i,j),(l,m)}^k := \mathbb{P}(\widehat{X}_{k+1} = x_l^{k+1}, \widehat{Y}_{k+1} = y_m^{k+1} \mid \widehat{X}_k = x_i^k, \widehat{Y}_k = y_j^k)$ and $g_k(x, y, z) = G_{(x,y),z}^k(L)$.

# Conclusion

## So far

- Introduced a model with steeper smile volatility surface for short maturities than Standard Heston model.
- Fast numerical solution for the pricing of European, Bermudan and Barrier options.

## And more..

- Asian Options
- Calibration

# Thank you for your attention!