High Frequency Data - Part II

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- Spot volatility estimation
- Multiple spot volatility: a uniformity result
- Spot drift estimation and drift bursts
- An economic model for immediacy: Grossman and Miller (1988)
- Application: flash crashes in stock markets

Three techniques:

- Localization of squared returns

$$\widehat{\sigma}_t^2 = \frac{1}{h_n} \sum_{i=1}^{n} K\left(\frac{t_{i-1} - t}{h_n}\right) \left(\Delta_i^n X\right)^2$$

- Localization of bipower variation

$$\widehat{\sigma}_t^2 = \frac{1}{h_n} \sum_{i=1}^{n-1} K\left(\frac{t_{i-1} - t}{h_n}\right) \left|\Delta_i^n X\right| \left|\Delta_{i+1}^n X\right|$$

- Localization of truncated square returns

$$\widehat{\sigma}_t^2 = \frac{1}{h_n} \sum_{i=1}^{n-1} K\left(\frac{t_{i-1} - t}{h_n}\right) \left(\Delta_i^n X\right)^2 I_{\{|\Delta_i^n X| < \theta_n\}}$$

Main model:

$$dX_t = \mu_t dt + \sigma_t dW_t + dJ_t, \tag{0.1}$$

we need extra assumptions on the coefficients to work.

- X is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathcal{P})$, X_0 is $\mathcal{F}_0$-measurable,
- $\mu = (\mu_t)_{t\geqslant 0}$ is a locally bounded and predictable drift
- $\sigma = (\sigma_t)_{t\geqslant 0}$ is an adapted, càdlàg and strictly positive (almost surely) volatility
- $W = (W_t)_{t\geqslant 0}$ is a standard Brownian motion
- $J = (J_t)_{t\geqslant 0}$ is a pure-jump process.

The bandwidths h_n is a sequence of positive real numbers, such that, as $n \to \infty$, $h_n \to 0$, $nh_n \to \infty$.

The kernel $K : \mathbb{R} \to \mathbb{R}_+$ is any function with the properties:

(K0) $K(x) = 0$ for $x > 0$ (left-sided kernel);

(K1) K is bounded and differentiable with bounded first derivative;

(K2) $\int_{-\infty}^{0} K(x)\mathrm{d}x = 1$ and $K_2 = \int_{-\infty}^{0} K^2(x)\mathrm{d}x < \infty$;

(K3) It holds that for every positive sequence $g_n \to \infty$, $\int_{-\infty}^{-g_n} K(x)\mathrm{d}x \leqslant C g_n^{-B}$ for some $B > 0$ and $C > 0$ (i.e., K has a fast vanishing tail);

(K4) $m_K(\alpha) = \int_{-\infty}^{0} K(x)|x|^{\alpha}\mathrm{d}x < \infty$, for all $\alpha > -1$;
$m'_K(\alpha) = \int_{-\infty}^{0} K^2(x)|x|^{\alpha}\mathrm{d}x < \infty$, for all $\alpha > -1$.

The jump process J_t is of the form:

$$J_t = \int_0^t \int_{\mathbb{R}} \delta(s,x) I_{\{|\delta(s,x)| \leqslant 1\}} \left(\nu(\mathrm{d}s,\mathrm{d}x) - \tilde{\nu}(\mathrm{d}s,\mathrm{d}x)\right) + \int_0^t \int_{\mathbb{R}} \delta(s,x) I_{\{|\delta(s,x)| > 1\}} \nu(\mathrm{d}s,\mathrm{d}x), \tag{0.2}$$

where ν is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$, $\tilde{\nu}(\mathrm{d}s,\mathrm{d}x) = \lambda(\mathrm{d}x)\mathrm{d}s$ a compensator, and λ is a σ-finite measure on $\mathbb{R}$, while $\delta : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is predictable and such that there exists a sequence $(\tau_n)_{n \geqslant 1}$ of $\mathcal{F}_t$-stopping times with $\tau_n \to \infty$ and, for each n, a deterministic and nonnegative Γ_n with $\min(|\delta(t,x)|, 1) \leqslant \Gamma_n(x)$ and $\int_{\mathbb{R}} \Gamma_n^2(x)\lambda(\mathrm{d}x) < \infty$ for all (t,x) and $n \geqslant 1$.

Fix $t \in (0, T]$ and let $B_\epsilon(t) = [t - \epsilon, t]$ with $\epsilon > 0$ fixed. We assume there exists a $\Gamma > 0$ and a sequence of $\mathcal{F}_t$-stopping times $\tau_m \to \infty$ and constants $C_t^{(m)}$ such that for all m, $(\omega, s) \in \Omega \times B_\epsilon(t) \cap [0, \tau_m(\omega)[$, and $u \in B_\epsilon(t)$,

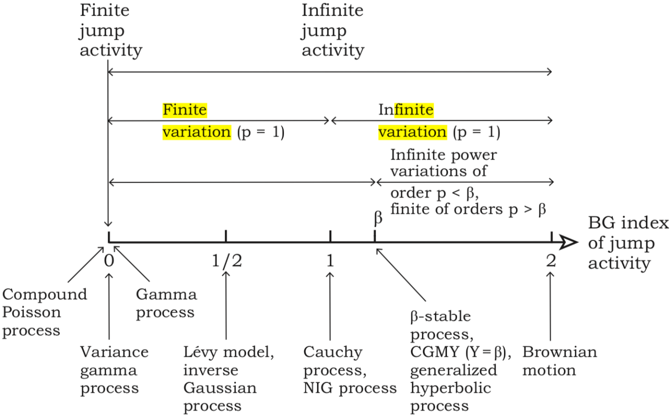
$$E_{u \wedge s}\left[|\mu_u - \mu_s|^2 + |\sigma_u - \sigma_s|^2\right] \leqslant C_t^{(m)} |u - s|^\Gamma, \tag{0.3}$$

where $E_t[\cdot] = E[\cdot|\mathcal{F}_t]$.

The localization procedure (see, e.g., Jacod and Protter (2012), Section 4.4.1) implies that we can and shall assume μ_t, σ_t, and $\delta(t, x)$ are bounded (as (ω, t, x) vary within $\Omega \times [0, T] \times \mathbb{R}$) and that $|\delta(t, x)| \leqslant \overline{\Gamma}(x)$, where $\overline{\Gamma}(x)$ is bounded and such that $\int_{\mathbb{R}} \overline{\Gamma}(x)^2 \lambda(\mathrm{d}x) < \infty$.

The jump activity index (denoted BG, sinces it coincides with the Blumenthal-Getoor index for pure jump Lévy processes) is defined as

$$BG = \inf_p \left\{ p \geqslant 0 : \sum_{s \leqslant t} |\Delta X_s|^p < \infty \right\}.$$



Theorem

Consider the simplest estimator:

$$\widehat{\sigma}_t^2 = \frac{1}{h_n} \sum_{i=1}^{n} K\left(\frac{t_{i-1} - t}{h_n}\right) (\Delta_i^n X)^2 .$$

For every fixed $t \in (0, T]$, *as* $n \to \infty$ *and* $h_n \to 0$ *such that* $nh_n \to \infty$, *it holds that* $\hat{\sigma}_t^2 \xrightarrow{p} \sigma_{t-}^2$.

This result is somewhat surprising. Let's see why it holds.

The proof is the combination of Mancini, Mattiussi and Renò (2015) with Theorem 9.3.2 in Jacod and Protter (2012).

Assume $J = 0$ (and $\mu = 0$).

Write:

$$\sigma_{t-}^2 = \frac{1}{h_n} \int_0^T K\left(\frac{s-t}{h_n}\right) \sigma_s^2 ds + R_n(t)$$

Then,

$$\widehat{\sigma}_t^2 - \sigma_{t-}^2 = \frac{1}{h_n} \sum_{i=1}^{n} K\left(\frac{t_{i-1}-t}{h_n}\right) \left(\left(\int_{t_{i-1}}^{t_i} \sigma_s^2 dW_s \right)^2 - \int_{t_{i-1}}^{t_i} \sigma_s^2 ds \right) + o_p(1)$$

and we invoke a LLN.

(The consistency result also holds with $\mu \neq 0$.)

Now assume $J \neq 0$.

We compensate the large jump term and write

$$X'_t = \int_0^t \mu^*_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s,$$

where $\mu^*_t = \mu_t + \int_{\mathbb{R}} \delta(t,x) I_{\{|\delta(t,x)|>1\}} \lambda(\mathrm{d}x)$ is bounded, and

$$X''_t = X_t - X'_t = \int_{\mathbb{R}} \delta(s,x) \left(\nu(\mathrm{d}s, \mathrm{d}x) - \tilde{\nu}(\mathrm{d}s, \mathrm{d}x)\right).$$

We already know that:

$$\frac{1}{h_n} \sum_{i=1}^{n} K\left(\frac{t_{i-1} - t}{h_n}\right) \left(\Delta_i^n X'\right)^2 \xrightarrow{p} \sigma^2_{t-}.$$

We need to show that:

$$R_n^{\hat{\sigma}} = \frac{1}{h_n} \sum_{i=1}^{n} K\left(\frac{t_{i-1} - t}{h_n}\right) \left(\left(\Delta_i^n X\right)^2 - \left(\Delta_i^n X'\right)^2\right) \xrightarrow{p} 0.$$

For $\kappa \in (0,1)$, write $\Delta_i^n X = \Delta_i^n X' + \Delta_i^n X_1''(\kappa) + \Delta_i^n X_2''(\kappa)$, where

$$\Delta_i^n X_1''(\kappa) = \int_{\mathbb{R}} \delta(s,x) I_{\{\overline{\Gamma}(x) \leqslant \kappa\}} \left(\nu(\mathrm{d}s, \mathrm{d}x) - \tilde{\nu}(\mathrm{d}s, \mathrm{d}x)\right)$$

$$\Delta_i^n X_2''(\kappa) = \int_{\mathbb{R}} \delta(s,x) I_{\{\overline{\Gamma}(x) > \kappa\}} \left(\nu(\mathrm{d}s, \mathrm{d}x) - \tilde{\nu}(\mathrm{d}s, \mathrm{d}x)\right).$$

The following inequality holds:

$$\left|(a+b+c)^2 - a^2\right| \leqslant \epsilon a^2 + \frac{C}{\epsilon}(b^2 + c^2)$$

for a fixed constant C valid for all $\epsilon \in]0,1]$.

Thus

$$\begin{aligned} |R_n^{\hat{\sigma}}| &\leqslant \frac{1}{h_n} \sum_{i=1}^n K\left(\frac{t_{i-1} - t}{h_n}\right) \left| \left(\Delta_i^n X\right)^2 - \left(\Delta_i^n X'\right)^2 \right| \\ &\leqslant \frac{1}{h_n} \sum_{i=1}^n K\left(\frac{t_{i-1} - t}{h_n}\right) \left(\epsilon \left(\Delta_i^n X'(\kappa)\right)^2 + \frac{C}{\epsilon}\left(\left(\Delta_i^n X_1''(\kappa)\right)^2 + \left(\Delta_i^n X_2''(\kappa)\right)^2\right)\right), \end{aligned}$$

We now split the probability space in two parts.

We define $\Omega_n(\psi, \kappa) \subseteq \Omega$ as the set of the events in which the Poisson process $\nu([0, t] \times \{x : \overline{\Gamma}(x) > \kappa\})$ has no jumps in the interval $(t - h_n^{\psi}, t]$, for $0 < \psi < 1$ and $0 < \kappa < 1$.

Note that $\Omega_n(\psi, \kappa) \to \Omega$, as $n \to \infty$ since $h_n^{\psi} \to 0$.

Conditioning on this (local) set where "large" jumps are absent, we just need to take care of the large jumps in the kernel tail:

$$
\begin{aligned}
& E\left[\frac{1}{h_n} \sum_{i=1}^{n} K\left(\frac{t_{i-1} - t}{h_n}\right) \frac{C}{\epsilon} \left(\Delta_i^n X_2''(\kappa)\right)^2 \mid \Omega_n(\psi, \kappa) \right] \\
&= \frac{1}{h_n} \sum_{t_{i-1} \leqslant t - h_n^{\psi}} K\left(\frac{t_{i-1} - t}{h_n}\right) \frac{C}{\epsilon} E\left[\left(\Delta_i^n X_2''(\kappa)\right)^2 \right] \\
&\leqslant \frac{1}{h_n} \sum_{t_{i-1} \leqslant t - h_n^{\psi}} K\left(\frac{t_{i-1} - t}{h_n}\right) \frac{C}{\epsilon} \Delta_{i,n} \left(\sim \int_{-\infty}^{h_n^{\psi - 1}} K(s) ds \right) \\
&\leqslant \frac{C}{\epsilon} h_n^{B(1-\psi)}
\end{aligned}
$$

For the "small" jumps we have:

$$E\Big[\left(\Delta_i^n X_1''(\kappa)\right)^p\Big] \leqslant C\Delta_{i,n}\int_{\left\{x:\overline{\Gamma}(x)\leqslant\kappa\right\}}\overline{\Gamma}(x)^p\lambda(\mathrm{d}x),$$

for all $p \geqslant 1$. We use $p = 2$ here.

So that, using again the convergence of the no-jump part:

$$E\Big[|R_n^{\hat{\sigma}}| \mid \Omega_n(\psi,\kappa)\Big] \leqslant C\epsilon + \frac{C}{\epsilon}\left(\int_{\left\{x:\overline{\Gamma}(x)\leqslant\kappa\right\}}\overline{\Gamma}(x)^2\lambda(\mathrm{d}x) + h_n^{B(1-\psi)}\right).$$

The conclusion is now classical. Write:

$$\begin{aligned}\mathcal{P}\left(|R_n^{\hat{\sigma}}| > c\right) =& \mathcal{P}\left(|R_n^{\hat{\sigma}}| > c \mid \Omega_n^{\complement}(\psi,\kappa)\right) + \mathcal{P}\left(|R_n^{\hat{\sigma}}| > c \mid \Omega_n(\psi,\kappa)\right)\\ \leqslant& \mathcal{P}\left(\Omega_n^{\complement}(\psi,\kappa)\right) + E\Big[|R_n^{\hat{\sigma}}| \mid \Omega_n(\psi,\kappa)\Big]/c\end{aligned}$$

(by Markov's inequality).

We proved that, or all $\epsilon, \kappa \in (0,1)$,

$$\limsup_{n\to\infty} \mathcal{P}\left(|R_n^{\hat{\sigma}}| > c\right) \leqslant \frac{1}{c}\left(C\epsilon + \frac{C}{\epsilon}\left(\int_{\left\{x:\overline{\Gamma}(x)\leqslant\kappa\right\}} \overline{\Gamma}(x)^2\lambda(\mathrm{d}x) + h_n^{B(1-\psi)}\right)\right).$$

Setting $\epsilon = \left(\int_{\left\{x:\overline{\Gamma}(x)\leqslant\kappa\right\}} \overline{\Gamma}(x)^2\lambda(\mathrm{d}x) + h_n^{B(1-\psi)}\right)^{\psi'}$ with $0 < \psi' < 1$ and noticing that $\left(\int_{\left\{x:\overline{\Gamma}(x)\leqslant\kappa\right\}} \overline{\Gamma}(x)^2\lambda(\mathrm{d}x) + h_n^{B(1-\psi)}\right) \to 0$ as $\kappa \to 0$ and $n \to \infty$, we deduce that $\mathcal{P}\left(|R_n^{\hat{\sigma}}| > c\right) \to 0$. ▫

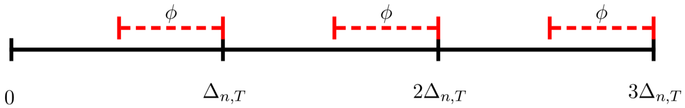
- Can we thus think that, under certain conditions, volatility can be observed and plugged in, e.g., a regression?
- Not really.
- We need indeed a uniformity result.
- We will now provide such a result using the truncated estimator.

We consider the price/variance system:

$$\mathrm{d}\log p_t = \mu(\sigma_t^2)\mathrm{d}t + \sigma_t \mathrm{d}W_t^r + \mathrm{d}J_t^r, \quad (0.4)$$

$$\mathrm{d}f(\sigma_t^2) = m_{f(\sigma^2)}(\sigma_t^2)\mathrm{d}t + \Lambda_{f(\sigma^2)}(\sigma_t^2)\mathrm{d}W_t^\sigma + \mathrm{d}J_t^\sigma, \quad (0.5)$$

where $\{\mathrm{d}W_t^r, \mathrm{d}W_t^\sigma\} = \{\rho(\sigma_t^2)dW_t^1 + \sqrt{1-\rho^2(\sigma_t^2)}dW_t^2, dW_t^1\}$ with $-1 \leqslant \rho(.) \leqslant 1$, $\left\{W_t^1, W_t^2\right\}$ are independent, standard Brownian motions, $\{J_t^r, J_t^\sigma\}$ are Poisson jump processes independent of each other and independent of $\left\{W_t^1, W_t^2\right\}$ with intensities $\lambda^r(.)$ and $\lambda^{f(\sigma^2)}(.)$ respectively, $f(.)$ is a monotonically non-decreasing transformation of variance, and $\mu(.)$, $m_{f(\sigma^2)}(.)$, $\Lambda_{f(\sigma^2)}(.)$, and $\rho(.)$ are generic functions satisfying suitable smoothness conditions.



We assume $k+1$ intra-period price observations (on dashed lines of total length $\phi < \Delta_{n,T}$) for each price observation sampled at $i\Delta_{n,T}$ with $i = 1, ..., n$. In the graph, we consider 3 periods ($n = 3$). Spot variance is estimated for every time $i\Delta_{n,T}$ using $k+1$ intra-period observations over ϕ.

Localized (in time) threshold realized variance estimator (Mancini, 2009), namely

$$\widehat{\sigma}^2_{i\Delta_{n,T}} = \frac{1}{\phi}\sum_{j=1}^{k} r_{i,j}^2 \ \mathbf{1}_{\{r_{i,j}^2 \leqslant \vartheta\}}, \tag{0.6}$$

Theorem

(Bandi and Renò, Econometric Theory, 2018) Assume $\widehat{\sigma}^2_{i\Delta_{n,T}}$ *is given by Eq. (0.6). If, as* $\vartheta \to 0$, *we have* $\frac{1}{\vartheta}\frac{\phi}{k}\log\left(\frac{k}{\phi}\right) \to 0$, *and when the driving functions and* σ_t^2 *are uniformly bounded, we have*

$$\max_{1\leqslant i\leqslant n}\left|\widehat{\sigma}^2_{i\Delta_{n,T}} - \sigma^2_{i\Delta_{n,T}}\right| = O_p\left(\sqrt{\frac{\log n \log nk}{k}} + \phi^{1/2}\sqrt{\log n} + n\phi\right).$$

We assume (among other things):

1. There exists a function $F(\cdot)$ satisfying the property $\frac{\mathrm{E}[\sup_{s\leqslant T}\left|f^*(\sigma_s^2)\right|]}{\mathrm{E}[\sup_{s\leqslant T}|F(\sigma_s^2)|]} = \frac{\mathrm{E}[\mathcal{M}(f^*(\sigma_s^2))]}{\mathrm{E}[\mathcal{M}(F(\sigma_s^2))]} \leqslant 1$, where $\mathcal{M}(g(\sigma_s^2))$ is the maximal process of the generic variance transformation $g(\sigma_s^2)$ and $f^*(\sigma^2)$ is a function of the variance state σ^2

If the functions $\mu(.), m_{f(\sigma^2)}(.), \ldots$ and the process σ_t^2 are not uniformly bounded, then the generic function $f^*(\sigma_t^2)$ will, of course, not be uniformly bounded. Thus, the condition

$$\max_{1\leqslant i\leqslant n}\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left|f^*(\sigma_s^2)\right| \mathrm{d}s \overset{p}{\sim} \Delta_{n,T}(\to 0), \tag{0.7}$$

which is a routine approximation in this literature, may not be valid as $T\to\infty$, in general. However, $\max\limits_{1\leqslant i\leqslant n}\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left|f^*(\sigma^2)\right| \mathrm{d}s \overset{p}{\sim} \mathrm{E}\left(\mathcal{M}(f^*(\sigma_s^2))\right)\Delta_{n,T}$ will be valid with $\mathcal{M}(f^*(\sigma_s^2)) = \sup\limits_{s\leqslant T}\left|f^*(\sigma_s^2)\right|$ defining the maximal process of $f^*(\sigma_s^2)$.

We write $\max_{1\leqslant i\leqslant n}\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left|f^*(\sigma_s^2)\right| \mathrm{d}s \overset{p}{\sim} \mathrm{E}\left(\mathcal{M}(f^*(\sigma_s^2))\right)\Delta_{n,T} \leqslant \mathrm{E}\left(\mathcal{M}(F(\sigma_s^2))\right)\Delta_{n,T} := \Delta^*_{n,T}$

- Brownian motion: (Doob's inequality)

$$\mathrm{E}\left(\sup_{s\leqslant\tau}|W_s|\right)\leqslant\sqrt{2}\sqrt{\mathrm{E}\left(\tau\right)},$$

- Bessel process of dimension δ starting at zero:

$$\mathrm{E}^{\frac{1}{p}}\left(\left(\sup_{s\leqslant\tau}X_s\right)^p\right)\leqslant\sqrt{\delta}\left(\frac{4-p}{2-p}\right)^{1/p}\mathrm{E}\left(\tau^{p/2}\right)$$

  for all $0<p<2$

- The square-root process $\mathrm{d}X_t=(a+bX_t)\mathrm{d}t+c\sqrt{X_t}\mathrm{d}W_t$ with $a,c>0$ and $b<0$

$$\mathrm{E}^{\frac{1}{p}}\left(\left(\sup_{s\leqslant\tau}X_s\right)^p\right)\leqslant\gamma_p\frac{c^2 2^{\frac{2a}{c^2}}}{|b|}\mathrm{E}\left(\log^p\left(1+\frac{a|b|}{c^2}\tau\right)\right),$$

Write the price process as $\log p = \log \widetilde{p} + J$, where $\widetilde{p}$ is the continuous component and J is the jump component. Recall $\phi < \Delta_{n,T}$.

Notation: write $\widetilde{p}_j = \widetilde{p}_{t_j}$, where
$t_j = \left(\lfloor \frac{j-1}{k+1} \rfloor + 1 \right) \Delta_{n,T} - \phi + \left(\frac{j-1}{k} - \lfloor \frac{j-1}{k+1} \rfloor \left(1 + \frac{1}{k}\right) \right) \phi$, with $j = 1, \ldots, n(k+1)$,
where $\lfloor . \rfloor$ is the "floor" function.

Notation: rhe symbol $\mathbf{1}_{i,j}$ signifies $\mathbf{1}_{\left\{ i-1 < \frac{j}{k+1} \leqslant i \right\}}$.

Simplification: use $f(\sigma^2) = \sigma^2$.

Rewrite:

$$\widetilde{\sigma}^2_{i\Delta_{n,T}} := \frac{1}{\phi} \sum_{j=1}^{n(k+1)-1} \mathbf{1}_{i,j} \left(\log \widetilde{p}_{j+1} - \log \widetilde{p}_j \right)^2,$$

Decomposition, using Ito's Lemma:

$$\max_{1\leqslant i\leqslant n}\left|\widehat{\sigma}^2_{i\Delta_{n,T}}-\sigma^2_{i\Delta_{n,T}}\right| \leqslant \max_{1\leqslant i\leqslant n}\left|\widetilde{\sigma}^2_{i\Delta_{n,T}}-\sigma^2_{i\Delta_{n,T}}\right| + \max_{1\leqslant i\leqslant n}\left|\widehat{\sigma}^2_{i\Delta_{n,T}}-\widetilde{\sigma}^2_{i\Delta_{n,T}}\right|$$

$$\leqslant \underbrace{\max_{1\leqslant i\leqslant n}\left|\frac{2}{\phi}\sum_{j=1}^{n(k+1)-1}\mathbf{1}_{i,j}\int_{t_j}^{t_{j+1}}\left(\int_{t_j}^{s}\mu(\sigma_v^2)\mathrm{d}v\right)\mu(\sigma_s^2)\mathrm{d}s\right|}_{V_{1,T,k,\phi}}$$

$$+\underbrace{\max_{1\leqslant i\leqslant n}\left|\frac{2}{\phi}\sum_{j=1}^{n(k+1)-1}\mathbf{1}_{i,j}\int_{t_j}^{t_{j+1}}\left(\int_{t_j}^{s}\sigma_v\mathrm{d}W_v^r\right)\mu(\sigma_s^2)\mathrm{d}s\right|}_{V_{2,n,T,k,\phi}}$$

$$+\underbrace{\max_{1\leqslant i\leqslant n}\left|\frac{2}{\phi}\sum_{j=1}^{n(k+1)-1}\mathbf{1}_{i,j}\int_{t_j}^{t_{j+1}}\left(\int_{t_j}^{s}\sigma_v\mathrm{d}W_v^r\right)\sigma_s\mathrm{d}W_s^r\right|}_{V_{3,n,T,k}}$$

$$+\underbrace{\max_{1\leqslant i\leqslant n}\left|\frac{2}{\phi}\sum_{j=1}^{n(k+1)-1}\mathbf{1}_{i,j}\int_{t_j}^{t_{j+1}}\left(\int_{t_j}^{s}\mu(\sigma_v^2)\mathrm{d}v\right)\sigma_s\mathrm{d}W_s^r\right|}_{V_{4,n,T,k,\phi}}$$

$$+\underbrace{\max_{1\leqslant i\leqslant n}\left|\frac{1}{\phi}\int_{i\Delta_{n,T}-\phi}^{i\Delta_{n,T}}\sigma_s^2\mathrm{d}s-\sigma^2_{i\Delta_{n,T}}\right|}_{B_{n,T,\phi}}+\underbrace{\max_{1\leqslant i\leqslant n}\left|\widehat{\sigma}^2_{i\Delta_{n,T}}-\widetilde{\sigma}^2_{i\Delta_{n,T}}\right|}_{J_{n,T,k,\phi}}.$$

We start with the bias term $B_{n,T,\phi}$. Write

$$
\begin{aligned}
&\max_{1\leqslant i\leqslant n}\left|\frac{1}{\phi}\int_{i\Delta_{n,T}-\phi}^{i\Delta_{n,T}}\sigma_s^2\mathrm{d}s-\sigma_{i\Delta_{n,T}}^2\right|=\max_{1\leqslant i\leqslant n}\left|\frac{1}{\phi}\int_{i\Delta_{n,T}-\phi}^{i\Delta_{n,T}}\left(\sigma_s^2-\sigma_{i\Delta_{n,T}}^2\right)\mathrm{d}s\right|\\
&\leqslant\max_{1\leqslant i\leqslant n}\sup_{i\Delta_{n,T}-\phi\leqslant s\leqslant i\Delta_{n,T}}\left|\sigma_s^2-\sigma_{i\Delta_{n,T}}^2\right|\\
&\leqslant\max_{1\leqslant i\leqslant n}\left(\sup_{i\Delta_{n,T}-\phi\leqslant s\leqslant i\Delta_{n,T}}\left|\int_s^{i\Delta_{n,T}}m(\sigma_u^2)\mathrm{d}u\right|+\sup_{i\Delta_{n,T}-\phi\leqslant s\leqslant i\Delta_{n,T}}\left|\int_s^{i\Delta_{n,T}}\Lambda(\sigma_u^2)\mathrm{d}W_u^\sigma\right|\right.\\
&\left.+\sup_{i\Delta_{n,T}-\phi\leqslant s\leqslant i\Delta_{n,T}}\left|\int_s^{i\Delta_{n,T}}\int_\xi\xi^\sigma v_\sigma(\mathrm{d}u,\mathrm{d}\xi^\sigma)\right|\right)\\
&\leqslant\max_{1\leqslant i\leqslant n}\left(\left|\int_{i\Delta_{n,T}-\phi}^{i\Delta_{n,T}}m(\sigma_u^2)\mathrm{d}u\right|+\left|\int_{i\Delta_{n,T}-\phi}^{i\Delta_{n,T}}\Lambda(\sigma_u^2)\mathrm{d}W_u^\sigma\right|+\left|\int_{i\Delta_{n,T}-\phi}^{i\Delta_{n,T}}\int_\xi\xi^\sigma v_\sigma(\mathrm{d}u,\mathrm{d}\xi^\sigma)\right|\right)\\
&+\max_{1\leqslant i\leqslant n}\left(\sup_{i\Delta_{n,T}-\phi\leqslant s\leqslant i\Delta_{n,T}}\left|\int_{i\Delta_{n,T}-\phi}^{s}m(\sigma_u^2)\mathrm{d}u\right|+\sup_{i\Delta_{n,T}-\phi\leqslant s\leqslant i\Delta_{n,T}}\left|\int_{i\Delta_{n,T}-\phi}^{s}\Lambda(\sigma_u^2)\mathrm{d}W_u^\sigma\right|\right.\\
&\left.+\sup_{i\Delta_{n,T}-\phi\leqslant s\leqslant i\Delta_{n,T}}\left|\int_{i\Delta_{n,T}-\phi}^{s}\int_\xi\xi^\sigma v_\sigma(\mathrm{d}u,\mathrm{d}\xi^\sigma)\right|\right).
\end{aligned}
$$

We only consider the last three terms since the first three are dominated asymptotically.

First term.

$$\max_{1\leqslant i\leqslant n}\sup_{i\Delta_{n,T}-\phi\leqslant s\leqslant i\Delta_{n,T}}\left|\int_{i\Delta_{n,T}-\phi}^{s} m(\sigma_u^2)\mathrm{d}u\right| \leqslant \sup_{s\leqslant T}\left|m(\sigma_s^2)\right|\phi.$$

For the second term, we use a "classical" trick and and (after Boole's inequality) exponential inequality.

Write:

$$\Pr(A) = \Pr(A\cap B) + \Pr(A\cap \overline{B}) \leqslant \Pr(A\cap B) + P(\overline{B}),$$

so that

$$\Pr\left(\max_{1\leqslant i\leqslant n}\sup_{i\Delta_{n,T}-\phi\leqslant s\leqslant i\Delta_{n,T}}\left|\int_{i\Delta_{n,T}-\phi}^{s}\Lambda(\sigma_u^2)\mathrm{d}W_u^{\sigma}\right|\geqslant C_{n,T,\phi}\right)$$
$$\leqslant \Pr\left(\max_{1\leqslant i\leqslant n}\sup_{i\Delta_{n,T}-\phi\leqslant s\leqslant i\Delta_{n,T}}\left|\int_{i\Delta_{n,T}-\phi}^{s}\Lambda(\sigma_u^2)\mathrm{d}W_u^{\sigma}\right|\geqslant C_{n,T,\phi}, \max_{1\leqslant i\leqslant n}\int_{i\Delta_{n,T}-\phi}^{i\Delta_{n,T}}\Lambda^2(\sigma_u^2)\mathrm{d}u\leqslant \beta_{T,\phi}\right)$$
$$+\Pr\left(\max_{1\leqslant i\leqslant n}\int_{i\Delta_{n,T}-\phi}^{i\Delta_{n,T}}\Lambda^2(\sigma_u^2)\mathrm{d}u > \beta_{T,\phi}\right).$$

For the first term, using Boole and a suitable exponential inequality (van Zanten, 2015):

$$
\begin{aligned}
&\Pr\left(\max_{1\leqslant i\leqslant n}\sup_{i\Delta_{n,T}-\phi\leqslant s\leqslant i\Delta_{n,T}}\left|\int_{i\Delta_{n,T}-\phi}^{s}\Lambda(\sigma_u^2)\mathrm{d}W_u^{\sigma}\right|\geqslant C_{n,T,\phi},\ \max_{1\leqslant i\leqslant n}\int_{i\Delta_{n,T}-\phi}^{i\Delta_{n,T}}\Lambda^2(\sigma_u^2)\mathrm{d}u\leqslant\beta_{T,\phi}\right)\\
&\leqslant\sum_{i=1}^{n}\Pr\left(\sup_{i\Delta_{n,T}-\phi\leqslant s\leqslant i\Delta_{n,T}}\left|\int_{i\Delta_{n,T}-\phi}^{s}\Lambda(\sigma_u^2)\mathrm{d}W_u^{\sigma}\right|\geqslant C_{n,T,\phi},\ \max_{1\leqslant i'\leqslant n}\int_{i'\Delta_{n,T}-\phi}^{i'\Delta_{n,T}}\Lambda^2(\sigma_u^2)\mathrm{d}u\leqslant\beta_{T,\phi}\right)\\
&\leqslant\sum_{i=1}^{n}\Pr\left(\sup_{i\Delta_{n,T}-\phi\leqslant s\leqslant i\Delta_{n,T}}\left|\int_{i\Delta_{n,T}-\phi}^{s}\Lambda(\sigma_u^2)\mathrm{d}W_u^{\sigma}\right|\geqslant C_{n,T,\phi},\ \int_{i\Delta_{n,T}-\phi}^{i\Delta_{n,T}}\Lambda^2(\sigma_u^2)\mathrm{d}u\leqslant\beta_{T,\phi}\right)\\
&\leqslant 2n\exp\left\{-\frac{C_{n,T,\phi}^2}{2\beta_{T,\phi}}\right\},
\end{aligned}
$$

while for the second term, by Markov's inequality, we also have

$$
\Pr\left(\max_{1\leqslant i\leqslant n}\int_{i\Delta_{n,T}-\phi}^{i\Delta_{n,T}}\Lambda^2(\sigma_u^2)\mathrm{d}u>\beta_{T,\phi}\right)\leqslant\frac{\mathrm{E}\left[\max_{1\leqslant i\leqslant n}\int_{i\Delta_{n,T}-\phi}^{i\Delta_{n,T}}\Lambda^2(\sigma_u^2)\mathrm{d}u\right]}{\beta_{T,\phi}}\leqslant\mathrm{E}\left[\mathcal{M}(\Lambda^2)\right]\frac{\phi}{\beta_{T,\phi}}.
$$

Now, setting

$$\beta_{T,\phi} = c\mathrm{E}\left[\mathcal{M}(\Lambda^2)\right]\phi$$

and

$$C_{n,T,\phi} = c\left(\mathrm{E}\left[\mathcal{M}(\Lambda^2)\right]\right)^{1/2}\sqrt{\phi\log n} = cC^*_{n,T,\phi},$$

we obtain

$$\Pr\left(\frac{1}{C^*_{n,T,\phi}}\max_{1\leqslant i\leqslant n}\sup_{i\Delta_{n,T}-\phi\leqslant s\leqslant i\Delta_{n,T}}\left|\int_{i\Delta_{n,T}-\phi}^{s}\Lambda(\sigma_u^2)\mathrm{d}W_u^\sigma\right|\geqslant c\right)\leqslant 2n^{1-c/2}+\frac{1}{c},$$

which gives $\max_{1\leqslant i\leqslant n}\sup_{i\Delta_{n,T}-\phi\leqslant s\leqslant i\Delta_{n,T}}\left|\int_{i\Delta_{n,T}-\phi}^{s}\Lambda(\sigma_u^2)\mathrm{d}W_u^\sigma\right| = O_p\left(C^*_{n,T,\phi}\right) = O_p\left(\mathcal{M}^{*1/2}(\Lambda^2)\sqrt{\phi\log n}\right)$, which is the order of the second term of the bias.

Now we deal with the jump term of the bias (the third term).

$$\Pr\left(\frac{\max_{1\leqslant i\leqslant n}\sup_{i\Delta_{n,T}-\phi\leqslant s\leqslant i\Delta_{n,T}}\left|\int_{i\Delta_{n,T}-\phi}^{s}\int_{\xi}\xi^{\sigma}v_{\sigma}(\mathrm{d}u,\mathrm{d}\xi^{\sigma})\right|}{n\phi}\geqslant c\right)$$

$$\leqslant\sum_{i=1}^{n}\Pr\left(\frac{\sup_{i\Delta_{n,T}-\phi\leqslant s\leqslant i\Delta_{n,T}}\left|\int_{i\Delta_{n,T}-\phi}^{s}\int_{\xi}\xi^{\sigma}v_{\sigma}(\mathrm{d}u,\mathrm{d}\xi^{\sigma})\right|}{n\phi}\geqslant c\right)$$

$$\leqslant\sum_{i=1}^{n}\Pr\left(\frac{\int_{i\Delta_{n,T}-\phi}^{i\Delta_{n,T}}\int_{\xi}|\xi^{\sigma}|\,v_{\sigma}(\mathrm{d}u,\mathrm{d}\xi^{\sigma})}{n\phi}\geqslant c\right)$$

$$\leqslant\sum_{i=1}^{n}\frac{\mathrm{E}\left[\int_{i\Delta_{n,T}-\phi}^{i\Delta_{n,T}}\int_{\xi}|\xi^{\sigma}|\,v_{\sigma}(\mathrm{d}u,\mathrm{d}\xi^{\sigma})\right]}{cn\phi}\leqslant\frac{\mathrm{E}\left[|\xi^{\sigma}|\right]\overline{\lambda}^{\sigma^2}}{c},$$

where $\overline{\lambda}^{\sigma^2}$ is the upper bound on the intensity of the variance jumps, i.e., $\lambda^{\sigma^2}(.)$. This proves that

$$\max_{1\leqslant i\leqslant n}\sup_{i\Delta_{n,T}-\phi\leqslant s\leqslant i\Delta_{n,T}}\left|\int_{i\Delta_{n,T}-\phi}^{s}\int_{\xi}\xi^{\sigma}v_{\sigma}(\mathrm{d}u,\mathrm{d}\xi^{\sigma})\right|=O_p(n\phi).$$

For the variance term, we expect the dominating order to come from $V_{3,n,T,k}$.

We have

$$V_{3,n,T,k} \leqslant \max_{1\leqslant i\leqslant n} \sup_{i\Delta_{n,T}-\phi\leqslant\tau\leqslant i\Delta_{n,T}} \left| \underbrace{\frac{2}{\phi} \sum_{j=1}^{n(k+1)-1} \mathbf{1}_{i,j} \mathbf{1}_{\{t_j<\tau\}} \int_{t_j}^{t_{j+1}\wedge\tau} \left(\int_{t_j}^{s} \sigma_v \mathrm{d} W_v^r \right) \sigma_s \mathrm{d} W_s^r}_{V^i_{3,n,T,k}(\tau)} \right|.$$

As before, write

$$\Pr\left(V_{3,n,T,k} \geqslant C_{n,T,k}\right) \leqslant \Pr\left(\max_{1\leqslant i\leqslant n} \sup_{i\Delta_{n,T}-\phi\leqslant\tau\leqslant i\Delta_{n,T}} \left| V^i_{3,n,T,k}(\tau) \right| \geqslant C_{n,T,k} \right)$$

$$\leqslant \Pr\left(\max_{1\leqslant i\leqslant n} \sup_{i\Delta_{n,T}-\phi\leqslant\tau\leqslant i\Delta_{n,T}} \left| V^i_{3,n,T,k}(\tau) \right| \geqslant C_{n,T,k}, \max_{1\leqslant i\leqslant n} [V^i_{3,n,T,k}(i\Delta_{n,T})] \leqslant \beta_{n,T,k,\phi} \right)$$

$$+ \Pr\left(\max_{1\leqslant i\leqslant n} [V^i_{3,n,T,k}(i\Delta_{n,T})] > \beta_{n,T,k,\phi} \right)$$

$$\leqslant \sum_{i=1}^{n} \Pr\left(\sup_{i\Delta_{n,T}-\phi\leqslant\tau\leqslant i\Delta_{n,T}} \left| V^i_{3,n,T,k}(\tau) \right| \geqslant C_{n,T,k}, [V^i_{3,n,T,k}(i\Delta_{n,T})] \leqslant \beta_{n,T,k,\phi} \right)$$

$$+ \Pr\left(\max_{1\leqslant i\leqslant n} [V^i_{3,n,T,k}(i\Delta_{n,T})] > \beta_{n,T,k,\phi} \right),$$

To the first term, we apply, once more, the exponential inequality:

$$\Pr\left(\sup_{i\Delta_{n,T}-\phi\leqslant\tau\leqslant i\Delta_{n,T}}\left|V^i_{3,n,T,k}(\tau)\right|\geqslant C_{n,T,k},[V^i_{3,n,T,k}(i\Delta_{n,T})]\leqslant\beta_{n,T,k,\phi}\right)\leqslant 2\exp\left\{-\frac{C^2_{n,T,k}}{2\beta_{n,T,k,\phi}}\right\}$$

For the second term, we note that each quantity $V^i_{3,n,T,k}(i\Delta_{n,T})$ is a martingale whose quadratic variation satisfies

$$\begin{aligned}
\max_{1\leqslant i\leqslant n}[V^i_{3,n,T,k}(i\Delta_{n,T})] &= \frac{4}{\phi^2}\max_{1\leqslant i\leqslant n}\sum_{j=1}^{n(k+1)-1}\mathbf{1}_{i,j}\left\{\int_{t_j}^{t_{j+1}}\left(\int_{t_j}^{s}\sigma_v \mathrm{d}W^r_v\right)^2\sigma^2_s\mathrm{d}s\right\}\\
&\leqslant \frac{4}{\phi^2}\mathcal{M}(\sigma^2)\max_{1\leqslant i\leqslant n}\sum_{j=1}^{n(k+1)-1}\mathbf{1}_{i,j}\left\{\int_{t_j}^{t_{j+1}}\left(\sup_{t_j\leqslant s\leqslant t_j+\phi/k}\left|\int_{t_j}^{s}\sigma_v \mathrm{d}W^r_v\right|\right)^2\mathrm{d}s\right\}\\
&\leqslant \frac{4}{\phi^2}\mathcal{M}(\sigma^2)\frac{\phi}{k}\max_{1\leqslant i\leqslant n}\sum_{j=1}^{n(k+1)-1}\mathbf{1}_{i,j}\left(\sup_{t_j\leqslant s\leqslant t_j+\phi/k}\left|\int_{t_j}^{s}\sigma_v \mathrm{d}W^r_v\right|\right)^2\\
&\leqslant \frac{4}{\phi^2}\mathcal{M}(\sigma^2)\frac{\phi}{k}k\max_{1\leqslant i\leqslant n}\left(\max_{i-1\leqslant j/(k+1)\leqslant i}\sup_{t_j\leqslant s\leqslant t_j+\phi/k}\left|\int_{t_j}^{s}\sigma_v \mathrm{d}W^r_v\right|\right)^2\\
&= \frac{4}{\phi}\mathcal{M}(\sigma^2)\left(\max_{1\leqslant j\leqslant (k+1)n}\sup_{t_j\leqslant s\leqslant t_j+\phi/k}\left|\int_{t_j}^{s}\sigma_v \mathrm{d}W^r_v\right|\right)^2,
\end{aligned}$$

so that

$$
\begin{aligned}
&\Pr\left(\max_{1\leqslant i\leqslant n}[V^i_{3,n,T,k}(i\Delta_{n,T})] > \beta_{n,T,k,\phi}\right)\\
&\leqslant \Pr\left(\frac{4}{\phi}\mathcal{M}(\sigma^2)\left(\max_{1\leqslant j\leqslant (k+1)n}\sup_{t_j\leqslant s\leqslant t_j+\phi/k}\left|\int_{t_j}^{s}\sigma_v \mathrm{d}W_v^r\right|\right)^2 > \beta_{n,T,k,\phi}\right)\\
=&\Pr\left(\max_{1\leqslant j\leqslant (k+1)n}\sup_{t_j\leqslant s\leqslant t_j+\phi/k}\left|\int_{t_j}^{s}\sigma_v \mathrm{d}W_v^r\right| > \beta^{1/2}_{n,T,k,\phi}\left(\frac{4}{\phi}\mathcal{M}(\sigma^2)\right)^{-1/2}\right)\\
\leqslant&\Pr\left(\max_{1\leqslant j\leqslant (k+1)n}\sup_{t_j\leqslant s\leqslant t_j+\phi/k}\left|\int_{t_j}^{s}\sigma_v \mathrm{d}W_v^r\right| > \beta^{1/2}_{n,T,k,\phi}\left(\frac{4}{\phi}\mathcal{M}(\sigma^2)\right)^{-1/2},\ \mathcal{M}(\sigma^2)\leqslant M_T\right)\\
&+\Pr\left(\mathcal{M}(\sigma^2) > M_T\right)\\
\leqslant&\Pr\left(\max_{1\leqslant j\leqslant (k+1)n}\sup_{t_j\leqslant s\leqslant t_j+\phi/k}\left|\int_{t_j}^{s}\sigma_v \mathrm{d}W_v^r\right| > \beta^{1/2}_{n,T,k,\phi}\left(\frac{4}{\phi}M_T\right)^{-1/2}\right) + \frac{\mathrm{E}\left[\mathcal{M}(\sigma^2)\right]}{M_T},
\end{aligned}
$$

and using again the exponential inequality:

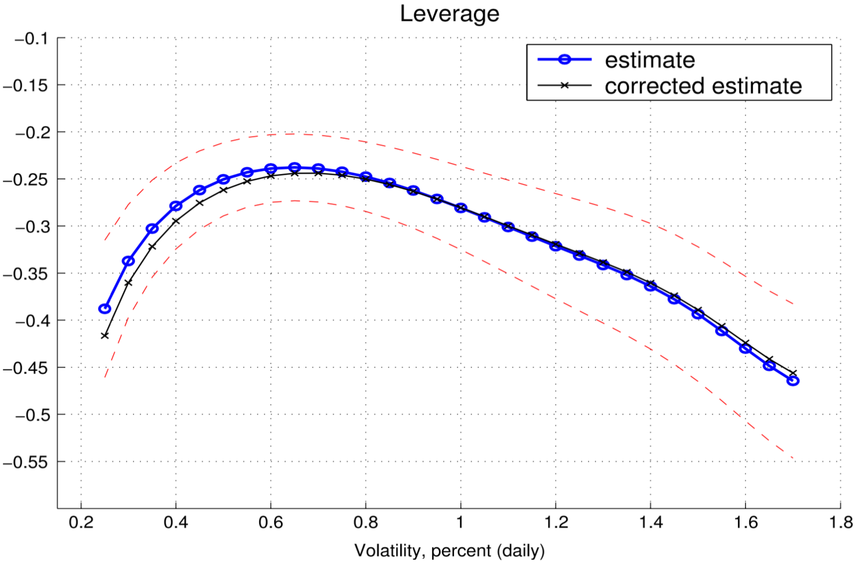
$$\begin{aligned}&\Pr\left(\max_{1\leqslant i\leqslant n}[V^i_{3,n,T,k}(i\Delta_{n,T})]>\beta_{n,T,k,\phi}\right)\leqslant 2n(k+1)\exp\left\{-\frac{\beta_{n,T,k,\phi}\phi}{4M_T\gamma_{T,k,\phi}}\right\}\\&+\Pr\left(\max_{1\leqslant j\leqslant (k+1)n}\int_{t_j}^{t_{j+1}}\sigma_s^2 ds>\gamma_{T,k,\phi}\right)+\frac{\mathrm{E}\left[\mathcal{M}(\sigma^2)\right]}{M_T}\\ \leqslant & 2n(k+1)\exp\left\{-\frac{\beta_{n,T,k,\phi}\phi}{4M_T\gamma_{T,k,\phi}}\right\}+\frac{\mathrm{E}[\mathcal{M}(\sigma^2)]\frac{\phi}{k}}{\gamma_{T,k,\phi}}+\frac{\mathrm{E}\left[\mathcal{M}(\sigma^2)\right]}{M_T},\end{aligned}$$

for $\gamma_{T,k,\phi}>0$.

Now, set $M_T=\sqrt{c}\mathcal{M}*(\sigma^2)$, $\gamma_{T,k,\phi}=\sqrt{c}\mathcal{M}*(\sigma^2)\frac{\phi}{k}$, $\beta_{n,T,k,\phi}=\gamma_{T,k,\phi}M_T\frac{\log n(k+1)}{\phi}$ and $C_{n,T,k}=c\mathcal{M}*(\sigma^2)\sqrt{\frac{\log n\log n(k+1)}{k}}$ to achieve

$$\Pr\left(V_{3,n,T,k}>C_{n,T,k}\right)\leqslant 2n^{1-c/2}+2(n(k+1))^{1-c/4}+\frac{2}{\sqrt{c}},$$

which proves that $V_{3,n,T,k}=O_p\left(\mathcal{M}*(\sigma^2)\sqrt{\frac{\log n\log n(k+1)}{k}}\right)=O_p\left(\mathcal{M}*(\sigma^2)\sqrt{\frac{\log n\log nk}{k}}\right)$. □


Leverage
estimate
corrected estimate
−0.1
−0.15
−0.2
−0.25
−0.3
−0.35
−0.4
−0.45
−0.5
−0.55
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
Volatility, percent (daily)

- Natural estimator:

$$\widehat{\mu}_t = \frac{1}{h_n} \sum_{i=1}^{n} K\left(\frac{t_{i-1} - t}{h_n}\right) \Delta_i^n X$$

- However we have the following problem:

$$\sqrt{h_n} \left(\hat{\mu}_t^n - \mu_{t-}\right) \xrightarrow{d} N\left(0, K_2 \sigma_{t-}^2\right),$$

- Assume the usual model:

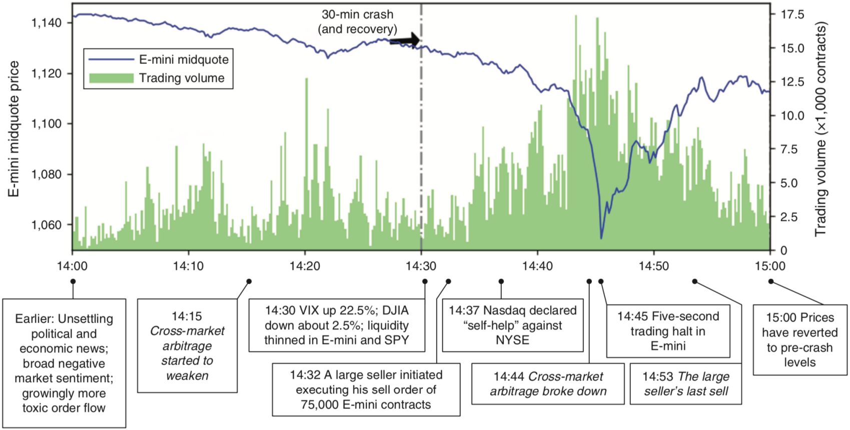
$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t + \mathrm{d}J_t,$$

- Then, around any point τ:

$$\int_{\tau-\overline{\Delta}}^{\tau+\overline{\Delta}} |\mu_s| \mathrm{d}s = O_p\left(\overline{\Delta}\right) \qquad \text{and} \qquad \int_{\tau-\overline{\Delta}}^{\tau+\overline{\Delta}} \sigma_s \mathrm{d}W_s = O_p\left(\sqrt{\overline{\Delta}}\right).$$

- So, no chances for the drift?

Figure 1. (Color online) Timeline of Main Events



Notes. This figure illustrates the sequence of events on the afternoon of May 6, 2010. The graph combines different sources: media ("US Shares Plunge Amid Fears Over Debt," *Financial Times*, May 7, 2010), regulators' reports (CFTC and SEC 2010a, b), and academic studies (Kirilenko et al. 2017, Easley et al. 2012). Earlier in the day, the market experienced "unsettling political and economic news," such as "European debt crisis" and "broad negative market sentiment" (CFTC and SEC 2010a). Easley et al. (2012) show that order flow had grown steadily more toxic (rising VPIN) in the course of the day.

- The drift can prevail in an alternative model where, in the neighborhood of τ_{db}, it is allowed to diverge in such a way that:

$$\int_{\tau_{\mathrm{db}}-\overline{\Delta}}^{\tau_{\mathrm{db}}+\overline{\Delta}} |\mu_s| \mathrm{d}s = O_p\left(\overline{\Delta}^{\gamma_\mu}\right), \tag{0.8}$$

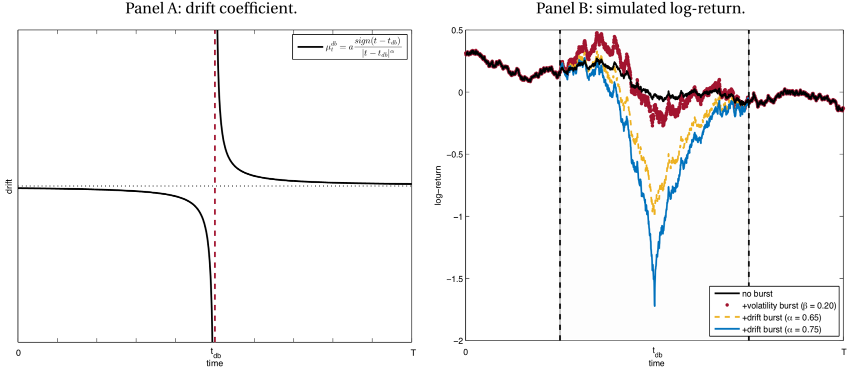
with $0 < \gamma_\mu < 1/2$.

- A simple example of an exploding drift leading to a drift burst is:

$$\mu_t^{\mathrm{db}} = \begin{cases} a_1 \left(\tau_{\mathrm{db}} - t\right)^{-\alpha} & t < \tau_{\mathrm{db}} \\ a_2 \left(t - \tau_{\mathrm{db}}\right)^{-\alpha} & t > \tau_{\mathrm{db}} \end{cases}. \tag{0.9}$$

with $1/2 < \alpha < 1$ and a_1, a_2 constants.

Figure 2: Illustration of a log-price with a drift burst.



Note. In Panel A, the drift coefficient is shown against time, while Panel B shows the evolution of a simulated log-price with a burst in: (i) nothing, (ii) volatility, and (iii) drift and volatility. The latter are based on Eq. (5) and (8) with $-a_1 = a_2 = 3$, $b = 0.15$, $\alpha = 0.65$ or 0.75 and $\beta = 0.2$.

- Set:

$$\hat{\sigma}_t^n = \left(\frac{1}{h'_n} \sum_{i=1}^{n} K\left(\frac{t_{i-1} - t}{h'_n} \right) (\Delta_i^n X)^2 \right)^{1/2}, \quad \text{for } t \in (0, T], \tag{0.10}$$

- Define the following t-statistic:

$$T_t^n = \sqrt{\frac{h_n}{K_2}} \frac{\hat{\mu}_t^n}{\hat{\sigma}_t^n}. \tag{0.11}$$

T_t^n has an intuitive interpretation with the indicator kernel. In that case, it is the ratio of the drift part to the volatility part of the log-return over the interval $[t - h_n, t]$. As $h_n, h'_n \to 0$, this is valid with any kernel satisfying the stated assumptions.

- In the model, an outside customer with a trading imbalance i cannot match his need with other customers immediately, so that he uses market makers. Market makers ask a compensation for it, and this causes the reversal.
- Two assets: a risk-free asset with zero interest rate and a risky asset with price P_t.
- Three times:

  - Time 1: the outside customer trades with market makers

  - Time 2: market makers trade with a new customer (and the outside customer)

  - Time 3: terminal condition.
- Each agent maximizes the expected utility of her terminal wealth:

$$u(W) = -e^{-\gamma W}$$

- The problem of the seller:

$$\max_{x_1, x_2, B_1, B_2} E\left[u(W_3)\right]$$

where x_t, B_t are asset and cash holdings at time t, subject to:

$$W_3 = B_2 + x_3 P_3$$

$$P_2 x_2 + B_2 = P_2 x_1 + B_1$$

$$P_1 x_1 + B_2 = P_1 i + W_0$$

- We solve the problem by backward induction. At time $t = 2$:

$$W_3 = W_2 - P_2 i + (P_3 - P_2) y_2 + P_3 i$$

where $y_t = x_t - i$

- The solution in $t = 2$ for the outside customer, assuming normally distributed prices, is:

$$y_2^C = \frac{E_2[P_3] - P_2}{\gamma Var_2[P_3]} - i$$

- Now assume there are M market makers with zero inventory. They solve the same problem but with $i = 0$. Their aggregate order in $t = 2$ is thus:

$$My_2^{MM} = M\frac{E_2[P_3] - P_2}{\gamma Var_2[P_3]}$$

- At time $t = 2$ there is also a new customer. By construction, her imbalance is $-i$. Thus her demand is:

$$y_2^N = \frac{E_2[P_3] - P_2}{\gamma Var_2[P_3]} + i$$

- Market clearing requires:

$$y_2^N + My_2^{MM} + y_2^C = 0$$

which implies:

$$P_2 = E_2[P_3]$$

and

$$x_2^C = 0.$$

- Now at time $t = 1$. Similar reasoning leads to:

$$y_1^C = \frac{E_1[P_3] - P_1}{\gamma Var_1[E_2[P_3]]} - i$$

$$y_1^{MM} = \frac{E_1[P_3] - P_1}{\gamma Var_1[E_2[P_3]]}$$

- Market clearing now requires:

$$y_1^C + My_1^{MM} = 0$$

- Solving for P_1:

$$P_1 = \underbrace{E_1[P_3]}_{\text{efficient term}} - \underbrace{\frac{i\gamma}{1+M} Var_1[E_2[P_3]]}_{\text{frictional term}}$$

Theorem (The t-statistic under the null)

Assume that X is a semimartingale as defined above, and given assumptions, as $n \to \infty$, it holds that:

$$T_t^n \xrightarrow{d} N(0,1). \tag{0.12}$$

We decompose T_t^n into:

$$T_t^n = \underbrace{\sqrt{\frac{h_n}{\mathbf{K}_2}} \frac{\left(\hat{\mu}_t^n - \mu_{t-}^*\right)}{\hat{\sigma}_t^n}}_{T_1} + \underbrace{\sqrt{\frac{h_n}{\mathbf{K}_2}} \frac{\mu_{t-}^*}{\hat{\sigma}_t^n}}_{T_2},$$

where $\mu_t^* = \mu_t + \int_{\mathbb{R}} \delta(t,x) I_{\{|\delta(t,x)|>1\}} \lambda(\mathrm{d}x)$ is the compensated drift.

This, together with the boundedness of μ_t^*, σ_t and $\delta(t,x)$, yields the following:

$$|T_2| \leqslant C \frac{\sqrt{h_n}}{\hat{\sigma}_t^n} = O_p\left(\sqrt{h_n}\right).$$

For every fixed $t \in (0, T]$ as $n \to \infty$ and $h_n \to 0$, it holds that:

$$A_n = \frac{1}{h_n} \sum_{i=1}^{n} K\left(\frac{t_{i-1} - t}{h_n}\right) \int_{t_{i-1}}^{t_i} \mu_s \mathrm{d}s - \int_0^T \frac{1}{h_n} K\left(\frac{s-t}{h_n}\right) \mu_s \mathrm{d}s = O_p\left(\frac{1}{nh_n}\right),$$

$$B_n = \frac{1}{\Delta_n h_n} \sum_{i=1}^{n} K\left(\frac{t_{i-1} - t}{h_n}\right) \left(\int_{t_{i-1}}^{t_i} \mu_s \mathrm{d}s\right)^2 - \int_0^T \frac{1}{h_n} K\left(\frac{s-t}{h_n}\right) \mu_s^2 \mathrm{d}s = O_p\left(\frac{1}{nh_n}\right)$$

(This also applies if μ_t is replaced by σ_t.)

Lemma 2

For every fixed $t \in (0, T]$ as $n \to \infty$ and $h_n \to 0$, it holds that:

$$B_n = \int_0^T \frac{1}{h_n} K\left(\frac{s-t}{h_n}\right) \mu_s \mathrm{d}s - \mu_{t-} = O_p\left(h_n^{\Gamma/2} + h_n^B\right).$$

(This also applies if μ_t is replaced by σ_t.)

Write

$$\mu_{t-} = \mu_{t-} \int_{-\infty}^0 K(x)\mathrm{d}x = \mu_{t-}\left(\int_{-\infty}^{-t/h_n} K(x)\mathrm{d}x + \int_{-t/h_n}^0 K(x)\mathrm{d}x\right),$$

so we can write:

$$|B_n| = \left|\int_0^T \frac{1}{h_n} K\left(\frac{s-t}{h_n}\right)(\mu_s - \mu_{t-})\mathrm{d}s + \mu_{t-}\int_{-\infty}^{-t/h_n} K(x)\mathrm{d}x\right| \leqslant \int_0^T \frac{1}{h_n} K\left(\frac{s-t}{h_n}\right)|\mu_s - \mu_{t-}|\mathrm{d}s + Ch_n^B,$$

where (K3) is applied. Then, by Jensen's inequality and the Lipschitz condition on coefficients:

$$E_{s\wedge t}\left[|\mu_s - \mu_{t-}|\right] \leqslant C|s-t|^{\Gamma/2}.$$

Together with (K4) and a change of variable, this implies that:

$$E\left[\int_0^T \frac{1}{h_n} K\left(\frac{s-t}{h_n}\right)|\mu_s - \mu_{t-}|\mathrm{d}s\right] \leqslant \int_0^T \frac{1}{h_n} K\left(\frac{s-t}{h_n}\right)|s-t|^{\Gamma/2}\mathrm{d}s = \int_{-t/h_n}^0 K(x)|x|^{\Gamma/2} h_n^{\Gamma/2}\mathrm{d}x \leqslant Ch_n^{\Gamma/2}.$$

This concludes the proof.

Now, using the Lemmas we can write:

$$\hat{\mu}_t^n - \mu_{t-}^* = \frac{1}{h_n}\sum_{i=1}^{n} K\left(\frac{t_{i-1}-t}{h_n}\right)\Delta_i^n X - \frac{1}{h_n}\sum_{i=1}^{n} K\left(\frac{t_{i-1}-t}{h_n}\right)\int_{t_{i-1}}^{t_i}\mu_s^* \mathrm{d}s + O_p\left(\frac{1}{nh_n} + h_n^{\Gamma/2} + h_n^B\right).$$

Hence,

$$\sqrt{h_n}\left(\hat{\mu}_t^n - \mu_{t-}^*\right) = \underbrace{\frac{1}{\sqrt{h_n}}\sum_{i=1}^{n} K\left(\frac{t_{i-1}-t}{h_n}\right)\int_{t_{i-1}}^{t_i}\sigma_s \mathrm{d}W_s}_{G_n}$$

$$+ \underbrace{\frac{1}{\sqrt{h_n}}\sum_{i=1}^{n} K\left(\frac{t_{i-1}-t}{h_n}\right)\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}}\delta(s,x)\left(\nu(\mathrm{d}s,\mathrm{d}x) - \tilde{\nu}(\mathrm{d}s,\mathrm{d}x)\right)}_{G_n'}$$

$$+ O_p\left(\frac{\sqrt{h_n}}{nh_n} + h_n^{\Gamma/2+1/2} + h_n^{B+1/2}\right).$$

To deal with the term G_n', write:

$$\int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} \delta(s,x) \left(\nu(\mathrm{d}s,\mathrm{d}x) - \tilde{\nu}(\mathrm{d}s,\mathrm{d}x)\right) = \Delta_i^n X_1''(\kappa) + \Delta_i^n X_2''(\kappa)$$

with, as before,

$$\Delta_i^n X_1''(\kappa) = \int_{\mathbb{R}} \delta(s,x) I_{\{\overline{\Gamma}(x) \leqslant \kappa\}} \left(\nu(\mathrm{d}s,\mathrm{d}x) - \tilde{\nu}(\mathrm{d}s,\mathrm{d}x)\right)$$

$$\Delta_i^n X_2''(\kappa) = \int_{\mathbb{R}} \delta(s,x) I_{\{\overline{\Gamma}(x) > \kappa\}} \left(\nu(\mathrm{d}s,\mathrm{d}x) - \tilde{\nu}(\mathrm{d}s,\mathrm{d}x)\right).$$

and define $\Omega_n(\psi,\kappa) \subseteq \Omega$ as the set of the events in which the Poisson process $\nu([0,t] \times \left\{x : \overline{\Gamma}(x) > \kappa\right\})$ has no jumps in the interval $(t - h_n^{\psi}, t]$, for $0 < \psi < 1$ and $0 < \kappa < 1$.

Conditioning on this (local) set where "large" jumps are absent, we just need to take care of the large jumps outside the set, which are in the kernel tail:

$$
\begin{aligned}
&E\Big[|G'_{n,2}| \mid \Omega_n(\kappa,\psi)\Big] \leqslant E\left[\frac{1}{\sqrt{h_n}}\sum_{i=1}^{n} K\left(\frac{t_{i-1}-t}{h_n}\right)|\Delta_i^n X_2''(\kappa)| \mid \Omega_n(\psi,\kappa)\right] \\
&= \frac{1}{\sqrt{h_n}}\sum_{t_{i-1}\leqslant t-h_n^{\psi}} K\left(\frac{t_{i-1}-t}{h_n}\right) E\Big[|\Delta_i^n X_2''(\kappa)|\Big] \\
&\leqslant \frac{1}{\sqrt{h_n}}\sum_{t_{i-1}\leqslant t-h_n^{\psi}} K\left(\frac{t_{i-1}-t}{h_n}\right) C\Delta_{i,n}\int_{\{x:\overline{\Gamma}(x)>\kappa\}}\overline{\Gamma}(x)\lambda(\mathrm{d}x) \leqslant C\sqrt{h_n}h_n^{B(1-\psi)},
\end{aligned}
$$

Now, since $\mathcal{P}\left(|G'_{n,2}|>c\right) \leqslant \mathcal{P}\left(\Omega_n^{\complement}(\kappa,\psi)\right) + E\Big[|G'_{n,2}| \mid \Omega_n(\kappa,\psi)\Big]/c$, it again follows that

$$
\limsup_{n\to\infty} \mathcal{P}\left(|G'_{n,2}|>c\right) \leqslant \frac{1}{c}C\sqrt{h_n}h_n^{B(1-\psi)},
$$

so that $G'_{n,2} \xrightarrow{p} 0$ as $n\to\infty$.

For the small jumps:

$$E\Big[|G'_{n,1}|\Big] \leqslant \frac{1}{\sqrt{h_n}} \sum_{i=1}^{n} K\left(\frac{t_{i-1}-t}{h_n}\right) E\Big[|\Delta_i^n X_1''(\kappa)|\Big]$$

$$\leqslant \frac{C}{\sqrt{h_n}} \sum_{i=1}^{n} K\left(\frac{t_{i-1}-1}{h_n}\right) \Delta_{i,n} \int_{\left\{x:\overline{\Gamma}(x)\leqslant\kappa\right\}} \overline{\Gamma}(x)\lambda(\mathrm{d}x).$$

The bound converges to $C\mathbf{K}_2 \int_{\left\{x:\overline{\Gamma}(x)\leqslant\kappa\right\}} \overline{\Gamma}(x)\lambda(\mathrm{d}x)$, which can be made arbitrarily small (with jumps of finite variation) by letting $\kappa \to 0$. We conclude that $G'_{n,1} \xrightarrow{p} 0$ and, therefore, $G'_n = o_p(1)$.

The main term is thus G_n, which we write as $G_n = \sum_{i=1}^{n} \Delta_i^n u$ with

$$\Delta_i^n u = \frac{1}{\sqrt{h_n}} K\left(\frac{t_{i-1} - t}{h_n}\right) \int_{t_{i-1}}^{t_i} \sigma_s \mathrm{d}W_s.$$

The aim is to prove that G_n – and hence $\sqrt{h_n}\left(\hat{\mu}_t^n - \mu_{t-}^*\right)$ – converges stably in law to $N\left(0, \mathbf{K}_2 \sigma_{t-}^2\right)$. We exploit again Jacod's theorem, which lists four sufficient conditions for this to hold :

$$\sum_{i=1}^{n} E_{t_{i-1}}\left[\Delta_i^n u\right] \xrightarrow{p} 0, \tag{0.13}$$

$$\sum_{i=1}^{n} E_{t_{i-1}}\left[(\Delta_i^n u)^2\right] \xrightarrow{p} \mathbf{K}_2 \sigma_{t-}^2, \tag{0.14}$$

$$\sum_{i=1}^{n} E_{t_{i-1}}\left[(\Delta_i^n u)^4\right] \xrightarrow{p} 0, \tag{0.15}$$

$$\sum_{i=1}^{n} E_{t_{i-1}}\left[\Delta_i^n u \Delta_i^n Z\right] \xrightarrow{p} 0, \tag{0.16}$$

where either $Z_t = W_t$ or $Z_t = W_t'$ with W_t' being orthogonal to W_t. The condition in Eq. (0.13) is immediate.

From Itō's Lemma, we deduce that:

$$\left(\int_{t_{i-1}}^{t_i} \sigma_s \mathrm{d}W_s\right)^2 = \int_{t_{i-1}}^{t_i} \sigma_s^2 \mathrm{d}s + 2\int_{t_{i-1}}^{t_i} \sigma_s \left(\int_{t_{i-1}}^{s} \sigma_u \mathrm{d}W_u\right) \mathrm{d}W_s,$$

so that

$$\sum_{i=1}^{n} E_{t_{i-1}}\left[(\Delta_i^n u)^2\right] = \sum_{i=1}^{n} \frac{1}{h_n} K^2\left(\frac{t_{i-1}-t}{h_n}\right) E_{t_{i-1}}\left[\left(\int_{t_{i-1}}^{t_i} \sigma_s \mathrm{d}W_s\right)^2\right]$$

$$= \sum_{i=1}^{n} \frac{1}{h_n} K^2\left(\frac{t_{i-1}-t}{h_n}\right) E_{t_{i-1}}\left[\int_{t_{i-1}}^{t_i} \sigma_s^2 \mathrm{d}s\right]$$

$$= \sum_{i=1}^{n} \frac{1}{h_n} K^2\left(\frac{t_{i-1}-t}{h_n}\right) \left(\sigma_{t_{i-1}}^2 \Delta_{i,n} + E_{t_{i-1}}\left[\int_{t_{i-1}}^{t_i} \left(\sigma_s^2 - \sigma_{t_{i-1}}^2\right) \mathrm{d}s\right]\right).$$

The first term converges to $K_2 \sigma_{t-}^2$.

The second term is negligible by continuity:

$$\sum_{i=1}^{n} \frac{1}{h_n} K^2 \left(\frac{t_{i-1} - t}{h_n} \right) E_{t_{i-1}} \left[\int_{t_{i-1}}^{t_i} \left(\sigma_s^2 - \sigma_{t_{i-1}}^2 \right) \mathrm{d}s \right]$$

$$\leqslant \sum_{i=1}^{n} \frac{1}{h_n} K^2 \left(\frac{t_{i-1} - t}{h_n} \right) \Delta_{i,n} \Delta_{i,n}^{\Gamma} = O_p \left(\Delta_n^{\Gamma} \right).$$

To deal with the third condition we use the Burkholder-Davis-Gundy inequality:

$$E_{t_{i-1}} \left[\left(\int_{t_{i-1}}^{t_i} \sigma_s \mathrm{d} W_s \right)^4 \right] \leqslant C E_{t_{i-1}} \left[\left(\int_{t_{i-1}}^{t_i} \sigma_s^2 \mathrm{d}s \right)^2 \right] \leqslant C (\Delta_{i,n})^2,$$

which leads to

$$\sum_{i=1}^{n} E_{t_{i-1}} \left[(\Delta_i^n u)^4 \right] = \sum_{i=1}^{n} \frac{1}{h_n^2} K^4 \left(\frac{t_{i-1} - t}{h_n} \right) E_{t_{i-1}} \left[\left(\int_{t_{i-1}}^{t_i} \sigma_s \mathrm{d} W_s \right)^4 \right]$$

$$\leqslant C \sum_{i=1}^{n} \frac{1}{h_n^2} K^4 \left(\frac{t_{i-1} - t}{h_n} \right) (\Delta_{i,n})^2 = O \left(\frac{\Delta_n}{h_n} \right).$$

To deal with the fourth condition, we first set $Z_t = W_t$. Then, using the Cauchy-Schwartz inequality:

$$E_{t_{i-1}}\left[\Delta_i^n W \int_{t_{i-1}}^{t_i} \sigma_s \mathrm{d}W_s\right] \leqslant \sqrt{E_{t_{i-1}}\left[(\Delta_i^n W)^2\right]}\sqrt{E_{t_{i-1}}\left[\left(\int_{t_{i-1}}^{t_i} \sigma_s \mathrm{d}W_s\right)^2\right]}$$

$$= \sqrt{\Delta_{i,n}}\sqrt{E_{t_{i-1}}\left[\int_{t_{i-1}}^{t_i} \sigma_s^2 \mathrm{d}s\right]} = O_p\left(\Delta_n\right),$$

and therefore

$$\sum_{i=1}^{n} E_{t_{i-1}}\left[\Delta_i^n u \Delta_i^n W\right] \leqslant C\frac{1}{\sqrt{h_n}}\sum_{i=1}^{n} K\left(\frac{t_{i-1}-t}{h_n}\right)\Delta_{i,n} \to 0.$$

If $Z_t = W'_t$, the process $W'_t \int_0^t \sigma_s \mathrm{d}W_s$ is a martingale by orthogonality, so that:

$$E_{t_{i-1}}\left[\Delta_i^n W' \int_{t_{i-1}}^{t_i} \sigma_s \mathrm{d}W_s\right] = 0.$$

Theorem (The t-statistic under the alternative)

Assume that $\tilde{X}$ is of the form:

$$d\tilde{X}_t = dX_t + \frac{c_{1,t}}{(\tau_{\text{db}} - t)^{\alpha}} dt + \frac{c_{2,t}}{(\tau_{\text{db}} - t)^{\beta}} dW_t,$$

where $\tau_{\text{db}} > 0$, dX_t *is the model (with bounded coefficients) in Eq.* (35) *for which the conditions of Theorem 3 hold,* $c_{1,t}$, $c_{2,t}$ *are adapted stochastic processes which satisfy the same conditions of* μ_t, σ_t, *and* α, β *are constants such that* $0 \leqslant \beta < 1/2$ *and* $0 < \alpha < 1$. *Then, as* $n \to \infty$, *we have:*

$$T^n_{\tau_{\text{db}}} \begin{cases} \overset{a.s.}{\to} \infty & \text{if } \alpha - \beta > 1/2 \\ \overset{d}{\to} c_{K,\beta} N(0,1) + d_{K,\beta,c_1,c_2} & \text{if } \alpha - \beta = 1/2 \\ \overset{d}{\to} c_{K,\beta} N(0,1) & \text{if } \alpha - \beta < 1/2 \end{cases}$$

where

$$c_{K,\beta} = \left(\frac{\int_{\mathbb{R}} K^2(x)|x|^{-2\beta} dx}{K_2 \int_{\mathbb{R}} K(x)|x|^{-2\beta} dx} \right)^{\frac{1}{2}} \text{ and } d_{K,\beta,c_1,c_2} = \frac{c_{1,1}}{c_{2,1}} \frac{\int_{\mathbb{R}} K^2(x)|x|^{-\beta-1/2} dx}{\left(K_2 \int_{\mathbb{R}} K(x)|x|^{-2\beta} dx\right)^{1/2}}.$$

Without loss of generality, we set $\tau_{\text{db}} = 1$. We write $\tilde{X}_t = X_t + D_t + V_t$, where $D_t = \int_0^t c_{1,s}(1-s)^{-\alpha}\mathrm{d}s$ and $V_t = \int_0^t c_{2,s}(1-s)^{-\beta}\mathrm{d}W_s$.

Look at the term $\hat{\mu}_t^n$. In Theorem 3, we already showed that $\frac{1}{h_n}\sum_{i=1}^n K\left(\frac{t_{i-1}-1}{h_n}\right)\Delta_i^n X = O_p\left(\frac{1}{\sqrt{h_n}}\right)$. Now,

$$
\begin{aligned}
A_n =& \frac{1}{h_n}\sum_{i=1}^n K\left(\frac{t_{i-1}-1}{h_n}\right)\int_{t_{i-1}}^{t_i} c_{1,s}(1-s)^{-\alpha}\mathrm{d}s \\
=& \frac{1}{h_n}\sum_{i=1}^n K\left(\frac{t_{i-1}-1}{h_n}\right)\Delta_{i,n} c_{1,\xi_{t_{i-1},t_i}}(1-\xi_{t_{i-1},t_i})^{-\alpha},
\end{aligned}
$$

where $t_{i-1} \leqslant \xi_{t_{i-1},t_i} \leqslant t_i$.The last term is, following a change of variable and Riemann summation, asymptotically equivalent to $c_{1,1}h_n^{-\alpha}m_K(-\alpha)$, where $m_K(-\alpha)$ is a constant.

For the second term $B_n = \frac{1}{h_n} \sum_{i=1}^n K\left(\frac{t_{i-1}-1}{h_n}\right) \int_{t_{i-1}}^{t_i} c_{2,s}(1-s)^{-\beta} \mathrm{d}W_s$, we easily show that:

$$h_n^{1/2+\beta} B_n \xrightarrow{d} N(0, m'_K(-2\beta)c_{2,1}^2),$$

where the above convergence is stable in law.

Thus, $\hat{\mu}_t^n$ is dominated by A_n when $\alpha - \beta > 1/2$, by B_n when $\alpha - \beta < 1/2$, whereas both terms are needed when $\alpha - \beta = 1/2$.

We turn to the denominator and set
$(\hat{\sigma}_t^n)^2 = \frac{1}{h_n}\sum_{i=1}^n K\left(\frac{t_{i-1}-1}{h_n}\right)\left[(\Delta_i^n D)^2 + (\Delta_i^n V)^2\right] + R_n'$, where

$$R_n' = \frac{1}{h_n}\sum_{i=1}^n K\left(\frac{t_{i-1}-1}{h_n}\right)\left((\Delta_i^n X + \Delta_i^n D + \Delta_i^n V)^2 - (\Delta_i^n D)^2 - (\Delta_i^n V)^2\right),$$

which is negligible since, using the fact that for all $\epsilon > 0$ and a, b and c real: $(a+b+c)^2 - a^2 - b^2 \leqslant \epsilon(a^2+b^2) + \frac{1+\epsilon}{\epsilon}c^2$, can be bounded as

$$\begin{aligned} |R_n'| &\leqslant \frac{1}{h_n}\sum_{i=1}^n K\left(\frac{t_{i-1}-1}{h_n}\right)\left(\epsilon\left((\Delta_i^n D)^2 + (\Delta_i^n V)^2\right) + \frac{1+\epsilon}{\epsilon}(\Delta_i^n X)^2\right) \\ &= \epsilon\frac{1}{h_n}\sum_{i=1}^n K\left(\frac{t_{i-1}-1}{h_n}\right)\left((\Delta_i^n D)^2 + (\Delta_i^n V)^2\right) + \frac{1+\epsilon}{\epsilon}O_p(1), \end{aligned}$$

so setting $\epsilon = h_n^\beta$, we can make $R_n' \sim h_n^{-\beta}$ (which will be negligible).

Write:

$$
\begin{aligned}
A_n' &= \frac{1}{h_n}\sum_{i=1}^{n} K\left(\frac{t_{i-1}-1}{h_n}\right)(\Delta_i^n D)^2 \\
&= \frac{1}{h_n}\sum_{i=1}^{n} K\left(\frac{t_{i-1}-1}{h_n}\right)\left(c_{1,t_{i-1}}^2 + o_p(1)\right)\left(\int_{t_{i-1}}^{t_i}(1-s)^{-\alpha}\mathrm{d}s\right)^2 \\
&\overset{p}{\sim} \frac{1}{h_n}\sum_{i=1}^{n} K\left(\frac{t_{i-1}-1}{h_n}\right) c_{1,t_{i-1}}^2 (1-t_{i-1})^{-2\alpha}\Delta_{i,n}^2,
\end{aligned}
$$

Let's simplify this a little bit: K is the indicator function, $c_1 = 1$, equally spaced observations. We have:

$$
A_n' \overset{p}{\sim} \frac{1}{n^{2-2\alpha}h_n}\sum_{i=1}^{[h_n/\Delta_n]} i^{-2\alpha}
$$

Thus, when $\alpha > 1/2$, we have $A_n' \overset{p}{\sim} \frac{1}{n^{2-2\alpha}h_n}$.

Next, setting $\Theta_n = \frac{1}{h_n}\sum_{i=1}^n K\left(\frac{t_{i-1}-1}{h_n}\right)(\Delta_i^n V)^2$ and using Itō's Lemma:

$$\left(\int_{t_{i-1}}^{t_i} c_{2,s}(1-s)^{-\beta}\mathrm{d}W_s\right)^2 = 2\int_{t_{i-1}}^{t_i}\left(\int_{t_{i-1}}^{s} c_{2,u}(1-u)^{-\beta}\mathrm{d}W_u\right)c_{2,s}(1-s)^{-\beta}\mathrm{d}W_s + \int_{t_{i-1}}^{t_i} c_{2,s}^2(1-s)^{-2\beta}\mathrm{d}s,$$

and we split $\Theta_n = \Theta_{1,n} + \Theta_{2,n}$ accordingly. $\Theta_{2,n}$ is asymptotically equivalent to $c_{2,1}^2 h_n^{-2\beta} m_K(-2\beta)$.We write $\Theta_{1,n} = \sum_{i=1}^n E_{t_{i-1}}\left[\Delta u_i'\right]$, where

$$\Delta u_i' = \frac{1}{h_n}K\left(\frac{t_{i-1}-1}{h_n}\right)2\int_{t_{i-1}}^{t_i} c_{2,s}(1-s)^{-\beta}\left(\int_{t_{i-1}}^{s} c_{2,u}(1-u)^{-\beta}\mathrm{d}W_u\right)\mathrm{d}W_s,$$

so $\Theta_{1,n}$ is a sum of martingale differences. Now, using a Taylor expansion on $c_{2,s}$,

$$\begin{aligned}
\sum_{i=1}^n E_{t_{i-1}}\left[\left(\Delta u_i'\right)^2\right] &= \frac{1}{h_n^2}\sum_{i=1}^n K^2\left(\frac{t_{i-1}-1}{h_n}\right)(c_{2,t_{i-1}}^4 + o_p(1))E_{t_{i-1}}\left[\left(2\int_{t_{i-1}}^{t_i}(1-s)^{-\beta}\left(\int_{t_{i-1}}^{s}(1-u)^{-\beta}\mathrm{d}W_u\right)\mathrm{d}W_s\right)^2\right.\\
&= \frac{4}{h_n^2}\sum_{i=1}^n K^2\left(\frac{t_{i-1}-1}{h_n}\right)(c_{2,t_{i-1}}^4 + o_p(1))\int_{t_{i-1}}^{t_i}(1-s)^{-2\beta}E_{t_{i-1}}\left[\left(\int_{t_{i-1}}^{s}(1-u)^{-\beta}\mathrm{d}W_u\right)^2\right]\mathrm{d}s\\
&= \frac{4}{h_n^2}\sum_{i=1}^n K^2\left(\frac{t_{i-1}-1}{h_n}\right)(c_{2,t_{i-1}}^4 + o_p(1))\int_{t_{i-1}}^{t_i}(1-s)^{-2\beta}\left(\int_{t_{i-1}}^{s}(1-u)^{-2\beta}\mathrm{d}u\right)\mathrm{d}s\\
&= \frac{2}{h_n^2}\sum_{i=1}^n K^2\left(\frac{t_{i-1}-1}{h_n}\right)(c_{2,t_{i-1}}^4 + o_p(1))\left((1-t_{i-1})^{-4\beta}\Delta_{i,n}^2 + O(\Delta_{i,n}^3)\right),
\end{aligned}$$

Thus, $\Theta_{1,n}$ is at most of order $\frac{1}{n^{1-2\beta} h_n}$ and is always dominated by $\Theta_{2,n}$.

Comparing $\Theta_{2,n} \sim h_n^{-2\beta}$ and $A'_n \sim \frac{1}{n^{2-2\alpha} h_n}$, both can dominate.

However, when $\alpha - \beta > 1/2$, if $\Theta_{2,n}$ dominates,

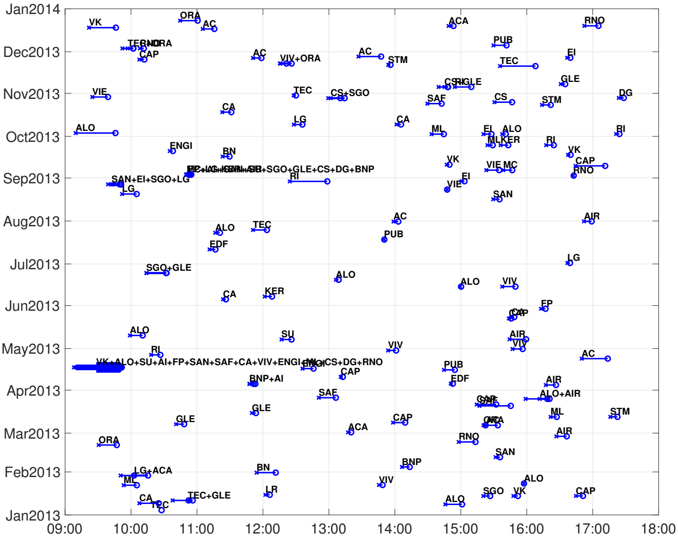
$$T \sim \frac{h_n^{1/2} h_n^{-\alpha}}{h_n^{-\beta}} \sim h_n^{-(\alpha-\beta-1/2)} \to \infty$$

If A'_n dominates,

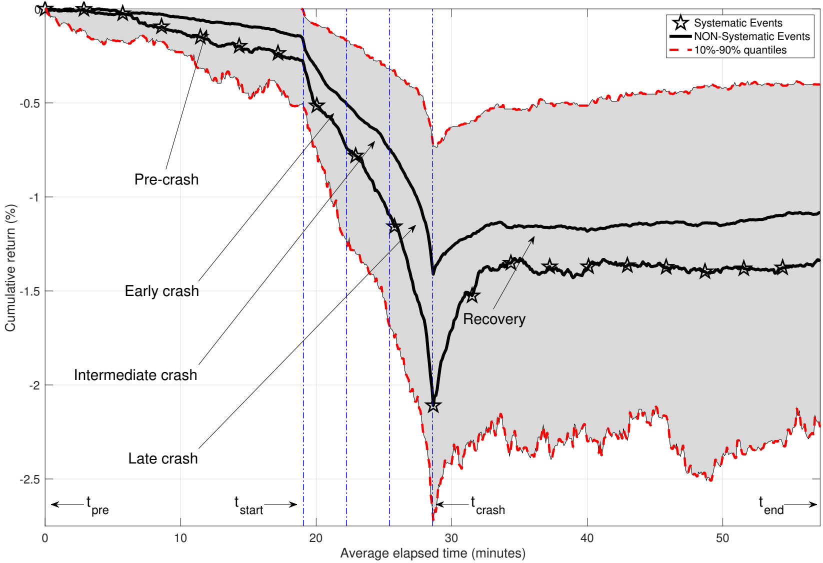
$$T \sim \frac{h_n^{1/2} h_n^{-\alpha}}{\frac{1}{n^{1-\alpha} h_n^{1/2}}} \sim (nh_n)^{1-\alpha} \to \infty$$

and the T-statistics explodes anyway.

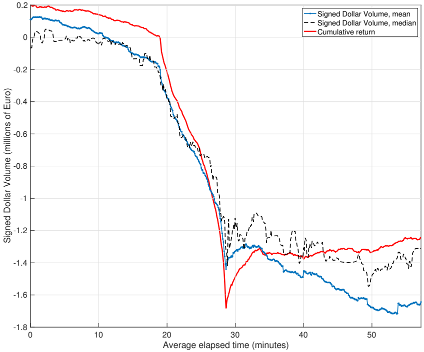
Flash Crashes over time and day



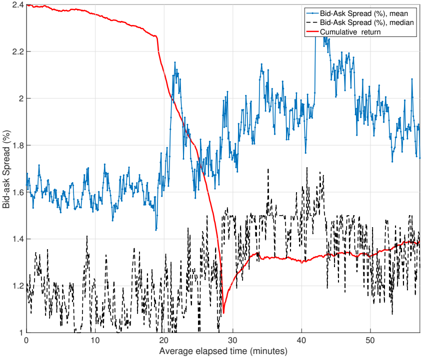
We identify 148 flash crash events in our sample. Average duration of a flash crash is 9.5 minutes, an average price drop during a flash crash is −1.35%. Systematic flash crashes: April 17 (14 stocks involved) and September 3 (13 stocks involved)


Systematic Events
NON-Systematic Events
10%-90% quantiles
Pre-crash
Early crash
Intermediate crash
Late crash
Recovery
t pre
t start
t crash
t end
Cumulative return (%)
Average elapsed time (minutes)
0
-0.5
-1
-1.5
-2
-2.5
0
10
20
30
40
50

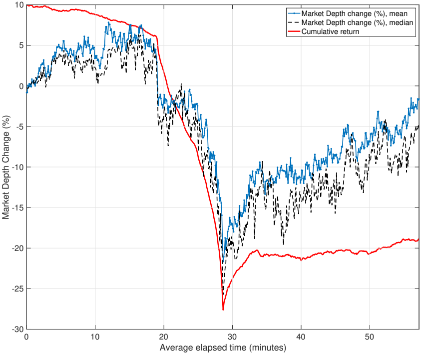
Panel A: Signed volume.



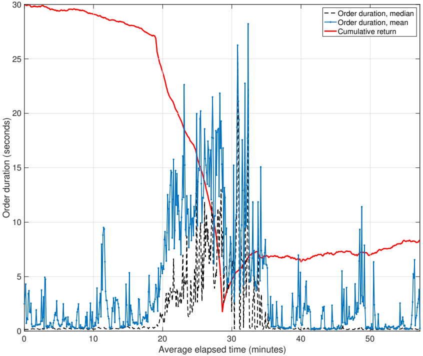
Panel B: Bid-ask spread.



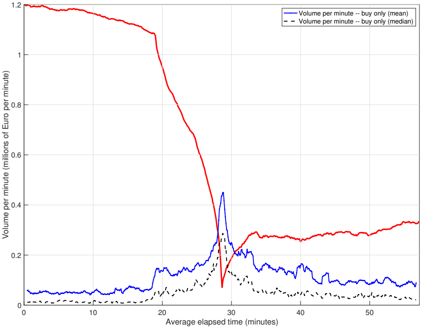
Panel C: Market depth.



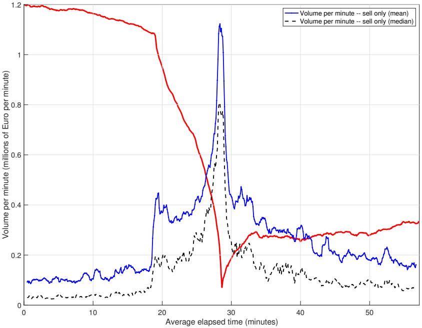
Panel D: Executed order age.

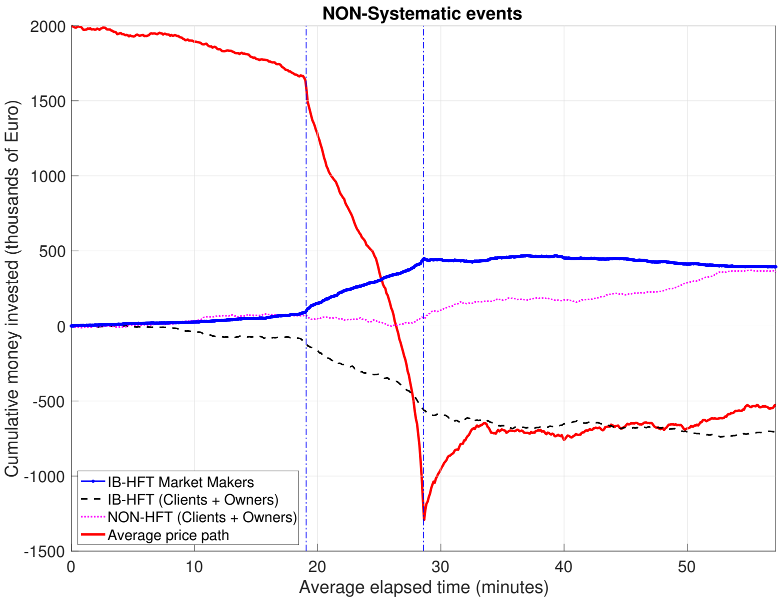


Panel A: buyer-initiated volume

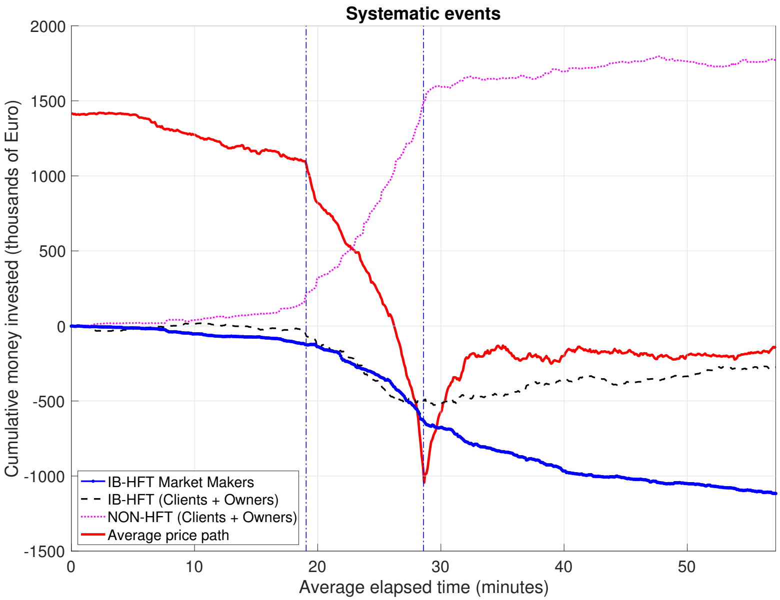


Panel B: seller-initiated volume




NON-Systematic events
2000
1500
1000
500
0
-500
-1000
-1500
Cumulative money invested (thousands of Euro)
0
10
20
30
40
50
Average elapsed time (minutes)
IB-HFT Market Makers
IB-HFT (Clients + Owners)
NON-HFT (Clients + Owners)
Average price path

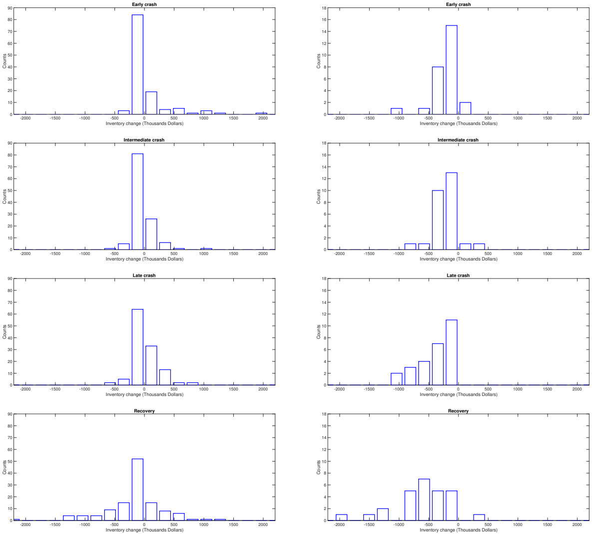
Aggregate signed inventory: systematic events

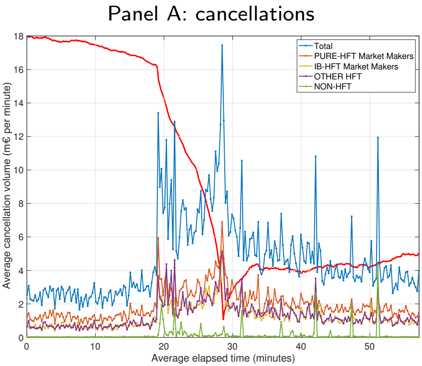


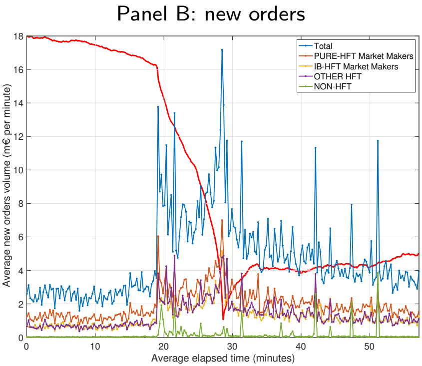
Cumulative net trading imbalances per minute of IB-HFT MM

Column A: NON-systematic flash crashes

Column B: Systematic flash crashes.

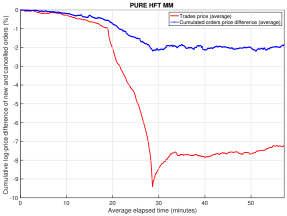



Panel A: cancellations
Total
PURE-HFT Market Makers
IB-HFT Market Makers
OTHER HFT
NON-HFT
Average cancellation volume (m€ per minute)
Average elapsed time (minutes)
0
2
4
6
8
10
12
14
16
18
0
10
20
30
40
50

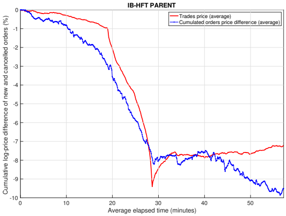

Panel B: new orders
Total
PURE-HFT Market Makers
IB-HFT Market Makers
OTHER HFT
NON-HFT
Average new orders volume (m€ per minute)
Average elapsed time (minutes)
0
2
4
6
8
10
12
14
16
18
0
10
20
30
40
50

Average price evolution of orders (through cancellations and new orders) for the most active trader categories (non-systematic).

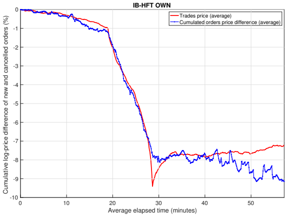
Panel A



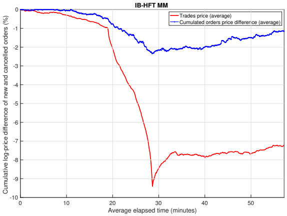
Panel B



Panel C



Panel D



Net monetary profit (€) during flash crashes.

	Systematic	Non-Systematic
PURE-HFT CLIENT	24.36	11.07
	(25.94)	(124.21)
PURE-HFT MM	−377.78	−164.09
	(541.03)	(444.22)
PURE-HFT OWN	−60.17	−27.52
	(108.69)	(312.32)
IB-HFT CLIENT	−1024.37	241.62
	(889.47)	(774.40)
IB-HFT MM	−75.61	−2799.72***
	(959.95)	(706.58)
IB-HFT OWN	5396.13**	2239.13*
	(2713.46)	(1204.43)
IB-HFT PARENT	−208.34	−423.07*
	(484.79)	(238.88)
Non-HFT CLIENT	−3164.53	−438.11
	(2250.18)	(996.79)
Non-HFT OWN	−765.22	1375.07**
	(1457.17)	(690.90)

- Cross-sectional analysis of trade imbalances (net and based on aggressive and passive trades separately) of different trading categories
- The analysis of quoting activity of different trading categories
- Testing whether HFTs change their trading behaviour during flash crashes in a model for inventory changes of different trader groups (as in Kirilenko, Kyle, Samadi, and Tuzun, 2017).
    - In contrast to the results of Kirilenko et al. (2017) we show that HFTs change their trading behaviour during flash crashes.
- Comparison with Extreme Price Movements (Broogard et al, 2018)
    - EPM methods allows to detect from 18.92% to 26.35% of flash crashes in our sample.

- HFT, in particular IB-HFT, do play a significant role in causing flash crashes via both trading and **quote revision**
- IB-HFT Owners start the crash with informed selling; IB-HFT Clients follow to profit opportunistically; IB-HFT Market Makers also start selling to back-run (or to predate on the price)

The crash

The joint behaviour of IB-HFT and HFT Market Makers in a market that became already illiquid **creates a transitory crash.**

- **Market Makers (i):** main liquidity providers at the beginning of the crash, only when they happen in single stocks, but they do not help with the recovery
- **Market Makers (ii):** When crashes affect several stocks, they sell along all phases of crash

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