Multiple curve modeling with CBI processes

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The multiple curve paradigm

Pre-crisis environment

The interest rate market:

The multiple curve paradigm

Pre-crisis environment

The interest rate market:

- A variety of reference rates: OIS, Ibor, Eonia...
  New ones forthcoming by 2021 (On Ibor transition, see Mercurio (2018));
- Pre-crisis environment: textbook situation, one single curve, the reference rates are linked by simple no-arbitrage relations.

The multiple curve paradigm

Post-crisis environment

The post-crisis interest rate market:

The multiple curve paradigm

Post-crisis environment

The post-crisis interest rate market:

- Strong increase of liquidity and credit risk in interbank transactions:
  - Ibor rates get riskier;
  - No more no-arbitrage relations between reference rates.
- Emergence of spreads between Ibor and OIS rates.

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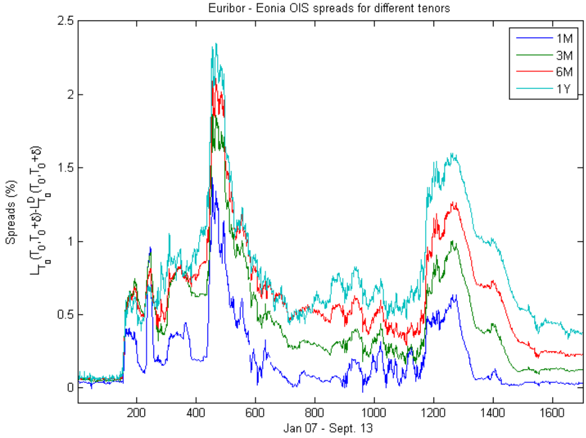
$$\Downarrow$$

Multiple yield curves

such that each curve represents a specific tenor (length of the loan) of the market (see Grbac & Runggaldier (2015))

(EUR)Ibor - OIS spreads from 01/2007 to 09/2013

Source: European Central Bank



Empirical analysis of the Ibor-OIS spreads

Typical market scenario for the spreads of the market

- Generally positive;
- Increasing with respect to the tenor;
- Volatility clustering;
- Common upward jumps along with strong dependence between the spreads of different tenors.

In this talk

The modeling approach

Continuous-state branching processes with immigration (CBI) for our main modeling quantities:

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Continuous-state branching processes with immigration (CBI) for our main modeling quantities:

- Spot multiplicative spreads between Ibor and OIS rates;
- The OIS short rate.

Straight upsides:

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Straight upsides:

- Satisfies the empirical features of spreads;
- Fits the initially observed term structure;
- Allows for efficient pricing of fixed income derivatives.

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- Allows for efficient pricing of fixed income derivatives.

$\Longrightarrow$ Calibration of the present model to market data.

Ibor and OIS rates

Consider the set of tenors of the market: $\{\delta_1, \ldots, \delta_m\}$ with $\delta_i \leq \delta_{i+1}$

- Ibor rate at t for $[t, t+\delta_i]$: $L(t, t+\delta_i)$ with δ_i being any tenor of the above set.
- OIS rate defined as the fair market swap rate of an Overnight Indexed Swap (OIS), providing the following:
  - The term structure of OIS zero-coupon bonds $T \mapsto B_t(T)$;
  - The spot simply compounded OIS rate at t for $[t, t+\delta]$:

$$L^{OIS}(t, t+\delta_i) = \frac{1}{\delta_i}\left(\frac{1}{B_t(t+\delta_i)} - 1\right); \qquad (1)$$

  - The OIS short rate denoted by $(r_t)_{t\geq 0}$.

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In the post-crisis market: $L(t, t+\delta_i) \neq L^{OIS}(t, t+\delta_i)$.

Multiplicative spreads

Spot multiplicative spread at t for the tenor δ_i:

$$S^{\delta_i}(t) = \frac{1 + \delta_i L(t, t + \delta_i)}{1 + \delta_i L^{OIS}(t, t + \delta_i)}. \tag{2}$$

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- Directly inferred from market quotes;
- Time-t market expectation of the interbank risk over $[t, t + \delta_i]$;
- Typical market behavior:
  - $S^{\delta_i}(t) \geq 1$;
  - $\delta_i \leq \delta_j \implies S^{\delta_i}(t) \leq S^{\delta_j}(t)$.

Such an approach is initially due to Henrard (2014), further studied in Cuchiero et al. (2016, 2018), Eberlein et al. (2018) and others.

The flow of CBI processes

Definition

Let $(\Omega, \mathbb{F}, \mathbb{Q})$ be a usual filtered probability space supporting:

- a white noise W on $(0, +\infty)^2$ of intensity $dsdu$;
- a Poisson time-space random measure N on $(0, +\infty)^3$ of intensity $ds\pi(dz)du$,

where π is a tempered alpha-stable measure:

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where π is a tempered alpha-stable measure:

$$\pi(dz) = -\frac{1}{\Gamma(\alpha)\cos(\pi\alpha/2)} \frac{e^{-\theta z}}{z^{1+\alpha}} \mathbb{1}_{z>0} dz \tag{3}$$

with $\alpha \in (1, 2)$ and $\theta > \eta$.

The flow of CBI processes

Definition

For each tenor δ_i, let $(Y_t^{\delta_i})_{t\geq 0}$ be the unique solution of the following:

$$Y_t^{\delta_i} = Y_0^{\delta_i} + \int_0^t (a(\delta_i) - bY_s^{\delta_i})ds + \sigma \int_0^t \int_0^{Y_s^{\delta_i}} W(ds, du)$$

$$+\eta \int_0^t \int_0^{+\infty} \int_0^{Y_{s-}^{\delta_i}} z\tilde{N}(ds, dz, du), \qquad (4)$$

where $b, \sigma, \eta \geq 0$ and $a : \{\delta_1, \ldots, \delta_m\} \to \mathbb{R}_+$ with $a(\delta_i) \leq a(\delta_{i+1})$. $\{Y^{\delta_i}, 1 \leq i \leq m\}$ is a flow of CBI processes (see Dawson & Li (2012)).

CBI-driven multi-curve model

Definition

Martingale modeling approach under $\mathbb{Q}$ in the spirit of the affine short rate multi-curve model (see Cuchiero et al. (2018)).
Given a flow of CBI processes $Y_t = \{Y_t^{\delta_i}, 1 \leq i \leq m\}$:

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- The spot multiplicative spread for each δ_i:

$$\log S^{\delta_i}(t) = c_i(t) + Y_t^{\delta_i}. \tag{6}$$

CBI-driven multi-curve model

Propreties

- The functions l and c_i allow for an exact fit to the initially observed term structure;
- Spreads satisfy the typical market behavior by construction;
- The processes $\{Y^{\delta_i}, 1 \leq i \leq m\}$ are generated by the same sources of randomness W and N:

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  $\Longrightarrow$ Common upward jumps and strong dependence between spreads;

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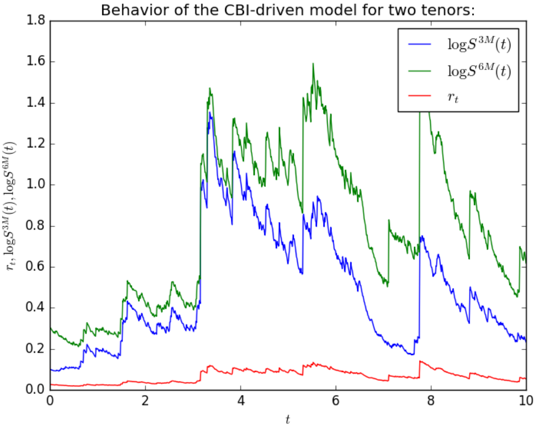
- Spreads satisfy the typical market behavior by construction;
- The processes $\{Y^{\delta_i}, 1 \leq i \leq m\}$ are generated by the same sources of randomness W and N:
  $\Longrightarrow$ Common upward jumps and strong dependence between spreads;
- Mutually exciting behavior between spreads:
  The higher $S^{\delta_i}(t)$ is, the greater the probability of upward jumps for all spreads with tenor $\delta_j \geq \delta_i$ will be.

The spot multiplicative spreads and the OIS short rate

Sample paths



The affine feature of the CBI process

Mathematical meaning

CBI processes are affine processes (see Duffie et al. (2003)):
For each δ_i, $\forall t, p \geq 0$:

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-pY_t^{\delta_i}\right)\right] = \exp\left(-y_0^{\delta_i} v(t,p) - a(\delta_i)\int_0^t v(s,p)ds\right), \tag{7}$$

where v is the unique solution of the following ODE:

$$\frac{\partial v}{\partial t}(t,p) = -\psi(v(t,p)), \quad v(0,p) = p, \tag{8}$$

where ψ is the branching mechanism of the flow:

$$\psi(x) = bx + \frac{1}{2}\sigma^2 x^2 + \frac{\theta^\alpha + x\eta\theta^{\alpha-1}\alpha - (x\eta+\theta)^\alpha}{\cos(\pi\alpha/2)}. \tag{9}$$

The affine feature of the CBI process

Consequences

- Existence of exponential moments of Y^{δ_i} (see Keller-Ressel & Mayerhofer (2015)):

$$b \geq \sigma^2 \frac{\theta}{2\eta} + \frac{\eta(1-\alpha)\theta^{\alpha-1}}{\cos(\pi\alpha/2)} \quad \text{and} \quad \theta > \eta \quad \Rightarrow \quad \mathbb{E}^{\mathbb{Q}}\left[e^{Y_t^{\delta_i}}\right] < +\infty \tag{10}$$

- 0 is inaccessible boundary if $2a(\delta_i) \geq \sigma^2$;

Compare the above consequences with Jiao et al.(2017).

CBI-driven Multi-curve pricing

Non-linear products

Exponentially affine forms for the following:

- OIS zero-coupon bonds:

$$B_t(T) = \exp\left(A_0(t,T) + B_0(t,T)^\top Y_t\right), \qquad (11)$$

- Forward multiplicative spreads:

$$S_t^{\delta_i}(T) = \exp\left(A_i(t,T) + B_i(t,T)^\top Y_t\right), \qquad (12)$$

where $S_t^{\delta_i}(T) = \frac{1+\delta_i L_t(T,T+\delta_i)}{1+\delta_i L_t^{OIS}(T,T+\delta_i)}$.

CBI-driven Multi-curve pricing

Non-linear products

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$\Longrightarrow$ Efficient pricing of linear fixed income products: Forward rate agreements, Interest rate swaps...

CBI-driven Multi-curve pricing

Caplet pricing

Knowledge of the characteristic function of the flow of CBI processes:

CBI-driven Multi-curve pricing

Caplet pricing

Knowledge of the characteristic function of the flow of CBI processes: $\Longrightarrow$ Efficient pricing of non-linear fixed income derivatives via Fourier techniques.

Time-0 price of a caplet with strike K delivered at time $T+\delta_i$:

$$P^{Cplt}(T,\delta_i,K) = B_0(T+\delta_i)\mathbb{E}^{\mathbb{Q}^{T+\delta_i}}\left[\left(\exp(X_T^i) - \exp(k_i)\right)^+\right], \tag{13}$$

where $X_T^i = \log\left(\frac{S^{\delta_i}(T)}{B_T(T+\delta_i)}\right)$ and $k_i = \log(1+\delta_i K)$,

such that the modified characteristic function of X_T^i is known in closed form: $\Pi_T^i(z) = B_0(T+\delta_i)\mathbb{E}^{\mathbb{Q}^{T+\delta_i}}\left[e^{izX_T^i}\right]$.

CBI-driven Multi-curve pricing

Caplet pricing

The Fourier inversion technique (see Lee (2004)) then provides:

$$P^{Cplt}(T, \delta_i, K) = R_T^i(k_i) + \frac{1}{\pi} \int_{0-i\epsilon}^{+\infty-i\epsilon} \Re\left(\exp(-izk_i)\frac{\Pi_T^i(z-i)}{-z(z-i)}\right) dz, \tag{14}$$

where $R_T^i(k_i)$ is a residue term depending on ϵ.

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where $R^i_T(k_i)$ is a residue term depending on ϵ.
$\Longrightarrow$ The above integral can be efficiently computed through the FFT (see Carr & Madan (1999)), thus providing fast pricing for calibration.

Calibration of the CBI-driven multi-curve model

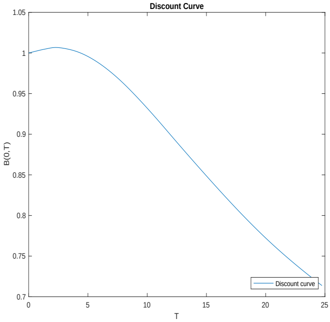
Market data

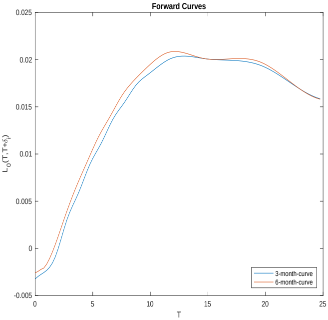
Market data from 25 June 2018, set of tenor $\{3M, 6M\}$:

- Initial term structures of OIS bonds $T \mapsto B_t(T)$ and Ibor rates $T \mapsto L_0(T, T+\delta)$, $\delta \in \{3M, 6M\}$ bootstrapped from linear products (FRAs, swaps);
- Concerning non-linear products, caplet volatility surface for $K \in [-0.13\%, 2\%]$ and $T \in [6M, 6Y]$:
  - Given in terms of normal (Bachelier) implied volatilities;
  - Caplets with maturity larger than two years are indexed to the 6-month forward rate, the others to the 3-month curve.

Calibration of the CBI-driven multi-curve model

Market data





Discount and forward curves, compare with Cuchiero et al. (2018)

Calibration of the CBI-driven multi-curve model

Implementation details

The aim of the calibration procedure was to solve:

$$\min_{p \in \mathcal{P}} \sum_j \sum_i \left(\sigma^{imp}_{mkt}(K_i, T_j) - \sigma^{imp}_{model}(K_i, T_j, p) \right)^2 , \qquad (15)$$

- $\sigma^{imp}_{model}(K_i, T_j, p)$ computed via FFT with 32768 points and 0.05 integration mesh size;
- Multi-threaded Levenberg-Marquardt optimizer with 8 threads.

Calibration of the CBI-driven multi-curve model

Calibration results

b	0.05353549346164644	α	1.317535272783081 4
σ	0.005827989181896429	y_0	$(0.004953850642168643, 0.005076590407389615)^\top$
η	0.04070169217539017	β	$(9.999999554946787E-4, 0.0034047019048037384)^\top$
θ	0.050701692175390174	μ	$(1.4999999999998428, 1.0000000348864304)^\top$

Table: Calibrated parameters.

Constraints satisfied: $\beta(1) \leq \beta(2)$, $\eta > 0$, $\theta > \eta$, $\alpha \in (1, 2)$, $y_0^1, y_0^2 \in \mathbb{R}_+$.

Calibration of the CBI-driven multi-curve model

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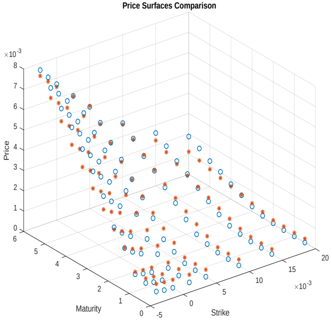

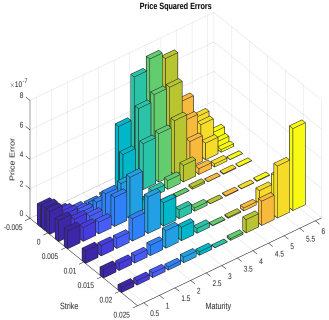

Model prices against market prices: Market prices are represented by blue circles while model prices by red stars.

Conclusions

A multiple yield curve model based on CBI processes:

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- Reproduces easily the typical scenarios of the spreads on the post-crisis interest rate market by means of the self-exciting feature along with the concept of flow of CBI processes;
- Allows for an exact fit to the initially observed term structures as well as immediate pricing of all linear fixed income products thanks to the affine property;
- Prices efficiently of non-linear interest rate derivatives via Fourier techniques and provides tractable calibration to market data with satisfactory results.

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Work in progress: Caplet pricing and calibration of the model via quantization techniques (CF of the model known in closed form).

Thank you for your attention!