American Real Option Pricing with Stochastic Volatility and Multiple Priors

Ankush Agarwal, Christian Ewald & Yihan Zou

University of Glasgow

12th European Summer School in Financial Mathematics

September 2019

The Problem

- Real life investment decision: irreversible projects (sunk cost) to decide an optimal time for investment; future market conditions are uncertain (ambiguous); the agent faces a set of equivalent probability measures rather than a certain probability measure.
- Financial application: American option with drift parameter uncertainty/ambiguity.
- Worst-case evaluation: the agent computes the expected profit (or option value) by using the "worst" probability measure and choose its strategy to maximize it. (Axiomatized by Gilboa and Schmeidler (1989))
- Then the evaluation becomes a minimax problem.

Previous Papers

- Optimal consumption problem with κ-ignorance (the drift rate of Brownian motion in stock price varies in $[-\kappa, \kappa]$): Chen and Epstein (2002).
- Evaluation of irreversible investments in finite and infinite horizons: Nishimura and Ozaki (2007).
- Barrier, American straddle, etc.: Cheng and Riedel (2013) and Vorbrink (2011).
- European option under Heston's model and drift uncertainty by numerically solving BSDE: Cohen and Tegnér (2018).

Our Approach

- Extend American/real option prices under drift uncertainty to stochastic volatility case
- Heston's Model: solve Reflected BSDE (RBSDE) numerically by stratification; stratified one-step forward dynamic programming without using RBSDE.

Heston's Model

Financial market model with stock price S_t and volatility $\sqrt{V_t}$ under risk-neutral (Q) measure:

$$\begin{aligned} \mathrm{d}S_t/S_t &= r\mathrm{d}t + \sqrt{V_t}\big(\rho \mathrm{d}\tilde{W}_t^1 + \sqrt{1-\rho^2}\mathrm{d}\tilde{W}_t^2\big), \\ \mathrm{d}V_t &= \alpha(\beta - V_t)\mathrm{d}t + \sigma\sqrt{V_t}\mathrm{d}\tilde{W}_t^1, \end{aligned}$$

with a money market account:

$$\mathrm{d}\gamma_t = r\gamma_t \mathrm{d}t,\ \gamma_0 = 1,$$

where $\alpha, \beta, \sigma, \rho, r$ are parameters. Denote τ as the exercise time, then the American option price at time t is,

$$J_t := J(t, X_t) = \operatorname*{ess\,sup}_{\tau \in T_t} \mathbb{E}^Q[H_\tau \gamma_{\tau - t}^{-1} | \mathcal{F}_t],$$

where $H_t := H(t, X_t) = \Phi(X_t)$ is the payoff function.

Heston's Model with Drift uncertainty

The agent knows the volatility parameter σ, but not the drift of underlying Brownian motions. Setting Q as our refernce measure here, the agent may consider all priors within a set of equivalent probability measures P^{Θ}.

- Conditional Radon-Nikodym derivative and stochastic exponential:

$$\frac{\mathrm{d}Q^{\theta}|_{\mathcal{F}_t}}{\mathrm{d}Q|_{\mathcal{F}_t}} = M_t^{\theta} = \mathcal{E}\big(-\int_0^{\cdot} \theta_s^* \mathrm{d}\tilde{W}_s\big)_t,$$

$$\mathcal{E}\big(-\int_0^{\cdot} \theta_s^* \mathrm{d}W_s\big)_t = \exp\left(-\int_0^t \theta_s^* \mathrm{d}W_s - \frac{1}{2}\int_0^t \theta_s^* \theta_s \mathrm{d}s\right).$$

- The set of multiple priors:

$$P^{\Theta} := \{Q^{\theta} : \theta_t \in \Theta \text{ and } Q^{\theta}\},$$

where the density generator $\theta_t : [0, T] \times \Omega \to \mathbb{R}^2$, and $\theta_t = (\theta_t^1, \theta_t^2)^*$.

Heston's Model with Drift uncertainty

Under Q^{θ}, the dynamics of state variables become:

$$\frac{\mathrm{d}S_t}{S_t} = r\mathrm{d}t - \sqrt{V_t}\big(\rho\theta_t^1 + \sqrt{1-\rho^2}\theta_t^2\big)\mathrm{d}t + \sqrt{V_t}\big(\rho\mathrm{d}\tilde{W}_t^{\theta 1} + \sqrt{1-\rho^2}\mathrm{d}\tilde{W}_t^{\theta 2}\big),$$

$$\mathrm{d}V_t = \alpha(\beta - V_t)\mathrm{d}t - \sigma\theta_t^1\sqrt{V_t}\mathrm{d}t + \sigma\sqrt{V_t}\mathrm{d}\tilde{W}_t^{\theta 1}.$$

Then the worst case evaluation implies that the value function will be,

$$v_t := v(t, X_t) = \operatorname*{ess\,sup}_{\tau \in T_t} \operatorname*{ess\,inf}_{\theta \in \Theta} \mathbb{E}^{Q^{\theta}}[H_\tau \gamma_{\tau - t}^{-1} | \mathcal{F}_t],\ t \in [0, T].$$

Relation with Non-linear RBSDE

Especially, the supremum and infimum are interchangeable (proof is mostly related to El Karoui, Kapoudjian, Pardoux, Peng and Quenez (1997)),

$$\operatorname*{ess\,sup}_{\tau\in T_t}\operatorname*{ess\,inf}_{\theta\in\Theta}\mathbb{E}^{Q^\theta}[H_\tau\gamma_{\tau-t}^{-1}|\mathcal{F}_t]=\operatorname*{ess\,inf}_{\theta\in\Theta}\operatorname*{ess\,sup}_{\tau\in T_t}\mathbb{E}^{Q^\theta}[H_\tau\gamma_{\tau-t}^{-1}|\mathcal{F}_t],$$

The value function v_t can be represented by a solution of RBSDE,

$$-\mathrm{d}Y_t=f(t,Y_t,Z_t)\mathrm{d}t+\mathrm{d}K_t-Z_t^*\mathrm{d}\tilde{W}_t,\ Y_T=H_T,$$

here $f(t,Y_t,Z_t)=\operatorname*{ess\,inf}_{\theta\in\Theta}(-rY_t-\theta_t^*Z_t)$ specifically, and $v_t=Y_t$ for every t.

Setting of Drift Uncertainty

Consider an elliptical uncertainty set:

$$\Theta = \{\theta : \ \theta\Sigma^{-1}\theta^* \leq \chi\},$$

then the generator in RBSDE will become:

$$f(t, Y_t, Z_t) = \min_{\theta \in \Theta} (-rY_t - \theta_t^* Z_t),$$
$$\text{subject to } \theta\Sigma^{-1}\theta^* = \chi,$$

with solution,

$$f(t, Y_t, Z_t) = -rY_t - \sqrt{Z_t^* \Sigma^* Z_t \chi},$$

under κ-ignorance, in which the uncertainty set is a rectangular (compact), we have similar results.

Numerical Schemes

- One-step forward dynamic programming scheme (ODP) for RBSDE:

$$
\begin{aligned}
Z^{\pi}_{t_{i-1}} &= \frac{1}{\Delta_i}\mathbb{E}^{Q}\big(Y^{\pi}_{t_i}\Delta\tilde{W}_{t_i}|X^{\pi}_{t_{i-1}}\big),\\
\tilde{Y}^{\pi}_{t_{i-1}} &= \mathbb{E}^{Q}\big(Y^{\pi}_{t_i} + f(Y^{\pi}_{t_i}, Z^{\pi}_{t_{i-1}})\Delta_i|X^{\pi}_{t_{i-1}}\big),\ Y^{\pi}_{t_n} = H(X^{\pi}_{t_n}),\\
Y^{\pi}_{t_{i-1}} &= \tilde{Y}^{\pi}_{t_{i-1}} \vee H(X^{\pi}_{t_{i-1}}),
\end{aligned}
$$

  for a time grid π: $0 = t_0 < ... < t_n = T$ (convergence proved by Ma and Zhang (2005)). Conditional expectation can be approximated by least square Monte Carlo method.
- ODP with stratification: stratify the simulation paths for X_t in a set of hypercubes, and locally approximate the solution. Gobet, López-Salas, Turkedjiev and Vzquez (2016) uses stratified MDP for European option, but MDP doesn't work in the American option case.

Numerical Schemes

- Stratified ODP without using RBSDE:
  First split the set of density generator into k discrete points, take the stochastic exponential M_t^θ as an extra dimension of state variables, so we still simulate the forward process under Q measure $\Rightarrow$ save huge memory than directly simulating under Q^θ measure, since θ can be time-varying. Then we have,

$$h(S_{t_i}, M_{t_i}^\theta) = \mathbb{E}^Q\left(\frac{M_{t_{i+1}}^\theta Y_{t_{i+1}}^{\tilde{\theta}} \gamma_{\Delta t}^{-1}}{M_{t_i}^\theta} | \mathcal{F}_{t_i}\right) = \mathbb{E}^{Q^\theta}\left(Y_{t_{i+1}}^{\tilde{\theta}} \gamma_{\Delta t}^{-1} | \mathcal{F}_{t_i}\right),$$

$$h(S_{t_i}, M_{t_i}^{\tilde{\theta}}) = \inf_{\theta \in \Theta} \mathbb{E}^Q\left(\frac{M_{t_{i+1}}^\theta Y_{t_{i+1}}^{\tilde{\theta}} \gamma_{\Delta t}^{-1}}{M_{t_i}^\theta} | \mathcal{F}_{t_i}\right),$$

$$Y_{t_i}^{\tilde{\theta}} = h(S_{t_i}, M_{t_i}^{\tilde{\theta}}) \vee H(S_{t_i}),\ Y_{t_n}^{\tilde{\theta}} = H(S_{t_n}).$$

Initial Results

- In one dimensional case, $\theta \in [-\kappa, \kappa]$, the optimal θ equals to $-\kappa$.

Table: The strike price is 40. We use 1000 hypercubes and 2000 paths for each hypercube. Standard deviations are in parentheses. The American option prices without uncertainty are 4.3930, 3.1904, 2.2719, 1.5939 and 1.0862 correspondingly by LSM method (Longstaff and Schwartz (2001)).

S0	LSM (10K Paths)	SRBSDE	SODP	95%CI	
36	3.8132 (0.0278)	3.8509 (0.0537)	3.8247 (0.0290)	3.8167	3.8327
38	2.6089 (0.0297)	2.6262 (0.0394)	2.6102 (0.0299)	2.6019	2.6185
40	1.7253 (0.0207)	1.7339 (0.0321)	1.7195 (0.0388)	1.7088	1.7303
42	1.1182 (0.0197)	1.1199 (0.0219)	1.1168 (0.0413)	1.1053	1.1282
44	0.7186 (0.0155)	0.7163 (0.0184)	0.7143 (0.0311)	0.7057	0.7229

Initial Results

- In Heston's case, θ varies in a elliptical uncertainty set.

Table: The strike price is 10. We use 30 hypercubes and 2000 paths for each hypercube. Standard deviations are in parentheses. The American option prices without uncertainty are 2, 1.1076, 0.52, 0.2137 and 0.082 correspondingly by solving PDE numerically (Ikonen and Toivanen (2008)).

S0	SRBSDE	95%CI		SODP	95%CI	
8	1.8689 (0.0358)	1.8599	1.8798	1.8305 (0.0116)	1.8273	1.8337
9	0.9729 (0.0176)	0.968	0.978	0.959 (0.0099)	0.9562	0.9617
10	0.4080 (0.0283)	0.4002	0.4159	0.4143 (0.0107)	0.4114	0.4173
11	0.1491 (0.0181)	0.1441	0.1541	0.1573 (0.0088)	0.1549	0.1598
12	0.0437 (0.0111)	0.0406	0.0468	0.0551 (0.0075)	0.053	0.0572

Concluding Remarks

- We obtain the American/real option prices with drift uncertainty under stochastic volatility framework by stratified RBSDE scheme, the approach depends on the compactness of uncertainty set.
- We use another scheme with stratification without using RBSDE, and results are close.
- One drawback: stratification depends on the Markovian setting of state variables.
- Further works: extend to multi-asset and multi-factor cases.

Chen, Zengjing and Larry Epstein, "Ambiguity, risk, and asset returns in continuous time," *Econometrica*, 2002, *70* (4), 1403–1443.

Cheng, Xue and Frank Riedel, "Optimal stopping under ambiguity in continuous time," *Mathematics and Financial Economics*, 2013, *7* (1), 29–68.

Cohen, Samuel N and Martin Tegnér, "European Option Pricing with Stochastic Volatility models under Parameter Uncertainty," *Working Papers*, 2018.

Gilboa, Itzhak and David Schmeidler, "Maxmin expected utility with non-unique prior," *Journal of mathematical economics*, 1989, *18* (2), 141–153.

Gobet, Emmanuel, José G López-Salas, Plamen Turkedjiev, and Carlos Vzquez, "Stratified regression Monte-Carlo scheme for semilinear PDEs and BSDEs with large scale parallelization on GPUs," *SIAM Journal on Scientific Computing*, 2016, *38* (6), C652–C677.

Ikonen, Samuli and Jari Toivanen, "Efficient numerical methods for pricing American options under stochastic volatility," *Numerical Methods for Partial Differential Equations: An International Journal*, 2008, *24* (1), 104–126.

Karoui, Nicole El, Christophe Kapoudjian, Étienne Pardoux, Shige Peng, and Marie-Claire Quenez, "Reflected solutions of backward SDE's, and related obstacle problems for PDE's," *the Annals of Probability*, 1997, *25* (2), 702–737.

Longstaff, Francis A and Eduardo S Schwartz, "Valuing American options by simulation: a simple least-squares approach," *The review of financial studies*, 2001, *14* (1), 113–147.

Ma, Jin and Jianfeng Zhang, "Representations and regularities for solutions to BSDEs with reflections," *Stochastic processes and their applications*, 2005, *115* (4), 539–569.

Nishimura, Kiyohiko G and Hiroyuki Ozaki, "Irreversible investment and Knightian uncertainty," *Journal of Economic Theory*, 2007, *136* (1), 668–694.

Vorbrink, Jörg, "American options with multiple priors in continuous time," Technical Report, Working Papers 2011.