Pricing of American Options under the Rough Heston model

Elizabeth Zúñiga

Supervisors:
Etienne Chevalier
Sergio Pulido

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Classical models in finance assume that the log-asset price X_t is given by:

$$dX_t = \mu_t dt + \sigma_t dW_t$$

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- Stochastic volatility models (Heston model): These models reproduce correctly the term structure of ATM skew

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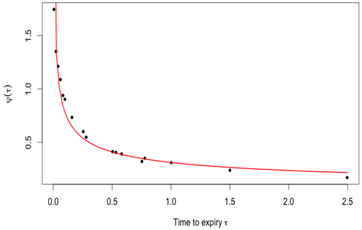
- Empirical studies indicate volatility is rougher than BM [Gatheral et al, 2014].

For small τ, in a model where the volatility is driven by a fractional Brownian motion with Hurst parameter H, $\psi(\tau) = \tau^{H-1/2}$ with $H = 0.1$. (see Figure of S&P ATM volatility skews)



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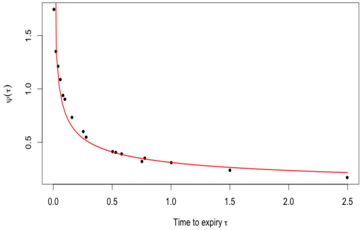

- New class of models: fractional stochastic volatility models (rough volatility models).
  - match roughness of time series data
  - fit implied volatility smiles remarkably well

Drawback: Loss of tractability, neither Markov nor semi-martingales.

The **Heston** model is a stochastic volatility model where:

$$dS_t = S_t \sqrt{V_t} dW_t$$

$$V_t = V_0 + \int_0^t \lambda(\theta - V_s) ds + \int_0^t \lambda \nu \sqrt{V_s} dB_s$$

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Proposition (The characteristic function in the Heston model)

The characteristic function of the log-price $X_t = \log(S_t/S_0)$ satisfies:

$$\mathbb{E}[e^{iaX_t}] = \exp(g(a,t) + V_0 h(a,t)),$$

where h is the solution of the following Riccati equation

$$\partial_t h = \frac{1}{2}(-a^2 - ia) + \lambda(ia\rho\nu - 1)h(a,s) + \frac{(\lambda\nu)^2}{2} h^2(a,s), \quad h(a,0) = 0$$

and

$$g(a,t) = \theta\lambda \int_0^t h(a,s)ds$$

$$dS_t = S_t \sqrt{V_t} dW_t,$$

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Riemann-Liouville fractional integral and differential operators:

$$I^r f(t) = \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}f(s)ds, \quad D^r f(t) = \frac{1}{\Gamma(1-r)}\frac{d}{dt}\int_0^t (t-s)^{-r}f(s)ds$$

We call stochastic Volterra equations to the stochastic convolution equations of the form:

$$X_t = X_0 + \int_0^t K(t-s) b(X_s) ds + \int_0^t K(t-s) \sigma(X_s) dW_s, \tag{1}$$

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- W is a multi-dimensional BM. $K \in L^2_{loc}(\mathbb{R}_+, \mathbb{R}^{d\times d})$ is called kernel.
- b and σ continuous and satisfy for some constant c_{LG} the linear growth conditions:

$$|b(x)| \vee |\sigma(x)| \leq c_{LG}(1+|x|), \quad x \in \mathbb{R}^d \qquad (2)$$

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We refer to the solutions of (1) as **affine Volterra processes** if $a(x) = \sigma(x)\sigma(x)^T$ and $b(x)$ are affine maps given by:

$$a(x) = A^0 + x_1 A^1 + \ldots x_d A^d \qquad b(x) = b^0 + x_1 b1 + \ldots x_d b^d$$

for some $d-$dimensional symmetric matrices A^i and vectors b^i.

Theorem (Bernstein–Widder theorem)

If K is completely monotone, there exists μ positive such that

$$K(t) = \int_0^{+\infty} e^{-t\gamma}\mu(d\gamma) \tag{3}$$

- In the fractional case with: $K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $\mu(d\gamma) = \frac{\gamma^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}d\gamma$, $\alpha \in (1/2, 1)$.

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We approximate $\mu(d\gamma)$ by $\sum_{i=1}^{n} c_i \delta_{\gamma_i}(d\gamma)$ [Carmona et al, 2000], then K by K^n

$$K^n(t) = \sum_{i=1}^{n} c_i e^{-\gamma_i t}.$$

For $n \in \mathbb{N}$, we define the following process:

$$X_t^n = X_0 + \int_0^t K^n(t-s)b(X_s^n)ds + \int_0^t K^n(t-s)\sigma(X_s^n)dW_s, \qquad t \in [0,T] \tag{4}$$

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For $n \in \mathbb{N}$ and $t \geq 0$, we have

$$X_t^n = X_0 + \sum_{i=1}^{n} c_i Y_t^{\gamma_i}$$

where for any $i \in \{1,...,n\}$, Y^{γ_i} is the affine process given by:

$$dY_t^{\gamma_i} = \left(-\gamma_i Y_t^{\gamma_i} + b_0 + b_1 X_0 + b_1 \sum_{j=1}^{n} c_j Y_t^{\gamma_j}\right) dt + \sigma \left(X_0 + \sum_{j=1}^{n} c_j Y_t^{\gamma_j}\right) dW_t \tag{5}$$

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$Y = (Y^{(\gamma_i)})_{i=1}^n$ is affine, Markovian and can be simulated.

Proposition (Convergence in law of X^n to X: Abi Jaber and El Euch (2018))

Let $X^n = (X_t^n)_{t\, leT}$ continuous weak solutions to (4), with the kernel K^n such that there exist $\gamma > 0$ and $C > 0$:

$$\sup_{n\geq 1}\left(\int_0^{T-h}|K^n(u+h)-K^n(u)|^2 du + \int_0^h |K^n(u)|^2 du\right) \leq Ch^{2\gamma}, \tag{6}$$

and

$$\int_0^t |K(u)-K^n(u)|^2 du \to 0 \tag{7}$$

for any $t\in[0,T]$ as n goes to infinity. Then $(X^n)_{n\geq 1}$ is tight for the uniform topology and any point limit X is a solution of (1).

Proposition states the convergence in law of the family X^n to X whenever (1) admits a unique weak solution.

The adjusted forward process $(u_t)_{t\geq 0}$ and its factor-representation $(u_t^n)_{t\geq 0}$ are defined as follows:

$$u_t^n(x) = \mathbb{E}\left[X_{t+x}^n - \int_0^x K^n(x-s) b(X_{t+s}^n) ds \middle| \mathcal{F}_t \right]$$

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Lemma

Assume that K^n *satisfies* (6). *Let* $T > 0$ *and* $p > 2/\gamma$ *be such that* $sup_{n\geq 1, t\leq T}\mathbb{E}[|X_t^n|^p]$ *is finite. Then for every* $s, t \in [0,T]$ *and* $x, y \in [0,M]$

$$\mathbb{E}[|u_t^n(y) - u_s^n(x)|^p] \leq C(\max\,(|t-s|, |y-x|))^{p\gamma}$$

where $C = C(p, K, M, T)$. *As a consequence* u^n *admits a Hölder continuous modification of order* $\alpha \in [0, \gamma - p/2[$. *Denoting this version by* u^n, *we have*

$$\mathbb{E}\left[\left(\max_{(t,y)\neq(s,x)} \frac{|u_t^n(y) - u_s^n(x)|}{|(t-s, y-x)|^\alpha}\right)^p\right] < \infty$$

Proposition (Convergence of (u^n))

u^n *converges in law to* u *under the uniform topology when* n *goes to infinity.*

$$X_t = \ln(S_t) = \sqrt{V_t}dWt - \tfrac{1}{2}V_t dt,$$
$$u_t(x) = V_0 + \int_0^t K(t-s+x)\left(\lambda(\theta - Vs)ds + \eta\sqrt{V_s}dWs\right)$$

The conditional Fourier Laplace transform of the process $(\{u_t^n(.)\},(X_t^n)_{t\geq 0})$ is given by:

$$\mathbb{E}\left[exp\left(wX_T^n + \int_0^\infty h^n(x)u_T^n(x)dx\right)\Big|\mathcal{F}_t\right] = exp\left(wX_t^n + \phi^n(T-t,w) + \int_0^\infty \Psi^n(T-t,x,h^n,w)u_t^n(x)dx\right.$$

fractional Riccati equations:

$$\partial_t\phi^n(t,w) = \mathcal{R}_{\phi^n}\left(\psi^n(t,w)\right), \qquad \phi^n(0)=0$$

$$\Psi^n(t,x,h,w) = h^n(x-t)\mathbf{1}_{\{x\geq t\}} + \Im_{\Psi^n}\left(w,\psi^n(t-x,w)\right)\mathbf{1}_{\{x<t\}}, \quad \Psi^n(0,x)=h^n(x)$$

where

$$\mathcal{R}_{\phi^n}\left(\psi^n(t,w)\right) = \lambda\theta\psi^n(t,w), \quad \Im_{\Psi^n}(w,\psi^n(t)) = \frac{1}{2}(w^2-w) + \left(\rho\eta w - \lambda + \frac{\eta^2}{2}\psi^n(t,w)\right)\psi^n(t,w)$$

$$\psi^n(t) = \int_0^\infty h^n(x)K^n(t+x)dx + K * \Im_{\Psi^n}\left(\psi^n(.,w)\right)(t)$$

We denote the value of the American option at time i by U_i, $i = 0, \ldots .M$.

Theorem

We will suppose that $h^n \to h$ uniformly in n, $K^n \to K$ in $L^2_{loc}(\mathbb{R}+)$, (u^n, X^n) is tight and converges weakly to (u, X). Then,

$$U_0^n \to U_0$$

uniformly in n.

- Idea of the proof:
  - Approximate the American option with Bermuda (uniformly in n)
  - Show the convergence of the Bermuda: We prove the result by induction on the number of dates M of the Bermudan option. The convergence in law of the forward process is used alongside an argument of density (which comes from the Stone-Weierstrass theorem) that allows us to reduce the problem to payoffs of the form $exp(wX_T + \int h(x)u_T(x)dx)$.

Time grid: $t_k = k\Delta t$, $k = 1, \ldots, N$, $\Delta t = T/N$.

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Euler scheme for the log price $X_t = log(S_t)$

$$X_{t_{k+1}} = X_{t_k} + \left(r - \frac{V_{t_k}}{2}\right)\Delta t + \sqrt{(V_{t_k})^+ \Delta t}(\rho\mathcal{N}(0,1) + \sqrt{1-\rho^2}\tilde{\mathcal{N}}(0,1))$$

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Explicit-implicit Euler scheme for the variance process V^n

$$V_{t_k} = u_0^n(t_k) + \sum_{i=1}^{n} c_i^n Y_{t_k}^i, \qquad Y_0^i = 0 \quad \text{for all} \quad i = 1, \ldots, n$$

$$Y_{t_{k+1}}^i = \frac{1}{1+\gamma_i \Delta t}\left(Y_{t_k}^i - \lambda V_{t_k}\Delta t + \eta\sqrt{(V_{t_k})^+ \Delta t}\right)\mathcal{N}(0,1), \quad i = 1, \ldots, n$$

with

$$u_0^n(t_k) = V_0 + \lambda\theta \sum_{i=1}^{n} c_i \left(\frac{1 - e^{-\gamma_i^n t_k}}{\gamma_i^n}\right)$$

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$$u_0^n(t_k) = V_0 + \lambda\theta \sum_{i=1}^{n} c_i \left(\frac{1 - e^{-\gamma_i^n t_k}}{\gamma_i^n}\right)$$

and parameters

$$c_i^n = \frac{(r_n^{1-\alpha} - 1) r_n^{(\alpha-1)(1+n/2)}}{\Gamma(\alpha)\Gamma(2-\alpha)} r_n^{(\alpha-1)i}, \qquad \gamma_i^n = \frac{1-\alpha}{2-\alpha}\frac{r_n^{2-\alpha} - 1}{r_n^{1-\alpha} - 1} r_n^{i-1-n/2}$$

Longstaff-Schwartz methodology.

$T = 1, \quad strike = 100, \quad \rho = -0.7, \quad \theta = 0.02, \quad \lambda = 0.3, \quad \nu = 0.3, \quad V_0 = 0.02 \quad \alpha = 0.6$

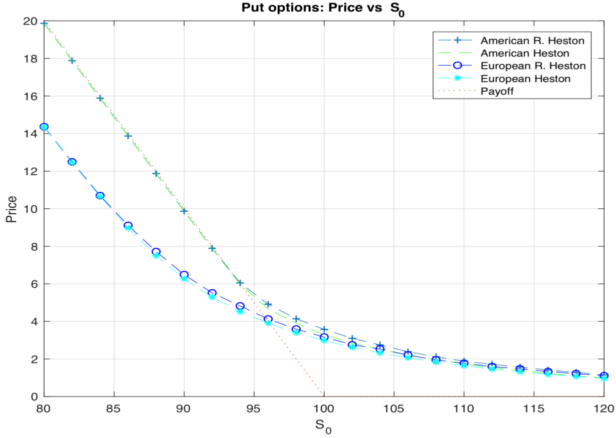
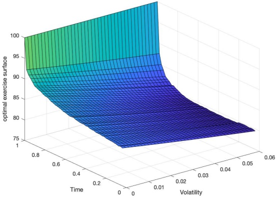
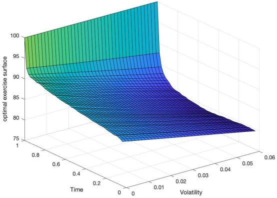


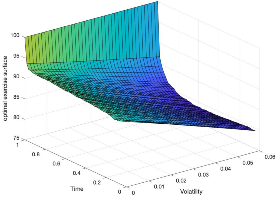
Figure: Put options with 10^5 simulations, 50 time step, 20 factors and $r_{20} = 2.5$



(a) $\alpha = 0.6$



(b) $\alpha = 0.8$



(c) $\alpha = 1$

FIGURE: Optimal exercise surface of American Put options in the Rough Heston

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