Optimal design of derivatives in illiquid market: an alternative approach*

Grazia De Silvestro
Tiziano Vargiolu
Università degli Studi di Padova
Dipartimento di Matematica Pura ed Applicata
via Belzoni 7, 35131 Padova – Italy
gdesil@libero.it, vargiolu@math.unipd.it

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Abstract

In this paper we determine the optimal structure of derivatives written on an illiquid asset, such as an energy event or a weather event. This transaction involves two agents: a "bank" which wants to create and sell a claim contingent to the illiquid asset and aims minimise its shortfall risk, and an "investor" which may buy this claim and aims at maximise its expected utility, possibly investing his residual wealth in a financial market. In the case when the utility and the loss functions are exponential, we characterise the structure and the price of the optimal derivative asset.

1 Introduction

This paper follows the new approach introduced by Barrieu and El Karoui in [1, 2] for the design of derivatives in illiquid markets. Its motivation is the recent developement in the markets of claim contingent to non-financial assets, such as cat-bonds, credit derivatives, energy and weather options. These

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new instruments are usually not quoted on standardised markets; moreover, as they are usually written on underlying assets which are not traded in the markets, they are very illiquid and the standard risk-neutral techniques (perfect replication, super-replication and so on) are not applicable to valuate them.

The new approach introduced in [1, 2] is based on maximising the utility functions of two agents in the market: agent B (the “bank”), which could be for example a bank or an insurance company, but also another kind of firm such as an electricity or gas producer, and agent I (the “investor”). Agent B is exposed to a non-financial risk Θ (for example temperature, or electricity demand, or its policy portfolio in the case of an insurance company), which results in a financial income $X(\Theta)$ at time T. As we do not impose constraints on the sign of X, this quantity can result in possible gains or losses. In order to hedge against this position, agent B wants to create a claim F contingent to Θ and sell it at the price π to a generic “investor”, which is represented by agent I. While agent B has the power to design the claim F, i.e. to specify its functional dependence on Θ, agent I can only choose between accepting or refusing this transaction, and possibly can also invest in a financial market (we will treat both the cases when this happens or not).

In order to specify the problem, one thing is missing, that is the criterion that the two agents use to decide their actions. In [1, 2] the criterion used by both agents was the maximisation of their expected utility functions, which could be calculated under probability measures that in general were different for the two agents. By specifying that both agents used an exponential utility function, the authors obtained that the optimal claim F turns out to be an affine function of the financial income X, with slope equal to the risk-aversion coefficient of the bank divided by the sum of the risk-aversion coefficients of both the agents, and the intercept which depended on the different situations studied in [2], namely, when only the risk Θ is present and the two agents have the same reference probability measure, when the two agents have different probability measures, and when a financial market is introduced where the two agents can invest independently of Θ. In [1], also the case when the investment in the financial market takes into account the risk Θ is present, and the optimal claim F turns out to be the solution of a nonlinear equation involving a double conditional expectation.

In our paper, we show that similar results hold when we substitute agent B's criterion with minimising the expected shortfall, with the variant that only agent I is allowed to operate in the financial market, and agent B is not. This can happen for various reasons: for example, agent B could be a bank's compartment which is not allowed to operate in the stock market because of internal regulations. With these assumptions, in the three situation studied in [2] we obtain an explicit solution for the optimal claim F, which in this case it is a continuous, pathwise affine function with slope equal to that of the utility maximisation case or to 1, and with intercept depending on the different cases. In the case when the investment in the financial market takes into account the risk Θ, as in [1] we obtain that the optimal claim F turns out to be the solution of a nonlinear equation involving a double conditional expectation.

The paper is organised as follows: in Section 2 we present the model and the general problem. In Section 3 we solve the problem in the case where only a non-financial risk is present. In Section 4 we introduce a financial market in our model, and solve the problem in the case when the optimal investments in the market can be considered independent of the non-financial risk, while in the case when these two quantities are related we characterise the solution via a nonlinear equation involving a double conditional expectation.

2 The model

We work on the probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$, where $\mathbb{P}$ is the real world probability measure. The time t flows in $[0, T]$, where T is the maturity time and the capitalization factor from 0 to T is taken to be 1 without loss of generality, as we will deal with derivative assets of European type. In this space we consider a non-financial risk Θ, which can be, for instance, a catastrophic or a weather event or, as we saw before, the non-payment of a loan. The main assumption on Θ is that there is not a financial market where it could be hedged. We suppose that Θ assumes values in $D(\Theta)$ which could be a subset of $\mathbb{R}^n$ but also a part of $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^n)$. Hence the risk Θ can be simply a random variable or a stochastic process. The realisation of the random variable Θ is known at time T or, if a process, it is $(\mathcal{F}_t)_t$-adapted. Then a financial market, whose assets may be correlated to Θ, is introduced.

We have two different agents: a "bank", denoted by agent B, and an "investor", denoted by agent I. The investor has an initial wealth $x > 0$ at time $t = 0$, while the bank has an initial exposure towards the risk Θ, which we denote by $X(\Theta)$; we do not assume that $X(\Theta)$ has constraint on the sign, and we also incorporate in $X(\Theta)$ the initial wealth of agent B. Agent B wants to hedge against $X(\Theta)$ by selling a financial contract whose flow depends on Θ, the possible buyer being agent I. The final payoff of this contract is denoted by $F(\Theta)$. At the time $t = 0$, the investor pays the price π of the contract to agent B. In exchange, it receives $F(\theta)$ at time T if the realisation $\Theta = \theta$ occurs. Thus, the question is to characterise the optimal structure of the contract F and its price π. We also define $\overline{F}(\Theta) := F(\Theta) - \pi$, which represent the difference between the final payoff for the investor and the price paid at time 0.

We assume that the two agents have in general different beliefs on the probability measure $\mathbb{P}$: in particular, we assume that the agents can use the probability measures $\mathbb{P}_i$, $i = B, I$, which we assume to be equivalent to $\mathbb{P}$. We also assume that the two agents have two different ways of hedging away risk: in fact, we assume that agent I adopts the criterion of utility maximisation, i.e. he wants to find the investment that maximises his expected utility, defined by

$$\mathbb{E}_{\mathbb{P}_I}[U_I(W_T^I)]$$

where W_T^I is his final wealth at time T (which will be defined in different ways according to the different possibility of investment treated in this paper). Throughout all this paper, we will assume that the utility function U_I is exponential:

$$U_I(z) = -\exp(-\gamma_I z)$$

Conversely, we assume that agent B minimises the expected shortfall of its final wealth, weighted by a suitable loss function ℓ

$$\min_F \mathbb{E}_{\mathbb{P}_B}[\ell(W_T^B)^+]$$

Throughout all this paper, we will assume that the loss function ℓ is

$$\ell(x) = e^{\gamma_B x} - 1$$

We notice that we can write

$$\ell(x^+) = (e^{\gamma_B x} - 1)^+ = e^{(\gamma_B x)^+} - 1$$

Hence, the choice criterion for a given agent is the maximisation of its expected utility, or the minimisation of its shortfall risk. There is another peculiarity of the relation between the bank and the investor. In fact, the bank determines the optimal structure of the contract, i.e. the functional dependence of F on Θ, whereas the investor can only accept or refuse to buy this contract. In doing this, agent I only compares its expected utility in the two different final situations of buying the contract or doing nothing.

We will analyse this problem in two different situations. In Section 3, we suppose that both the bank and the investor can only operate on the risk Θ, so their wealth position will be determined only in terms of $X(\Theta)$, $F(\Theta)$, π and the initial wealth of the investor x. In Section 4, we assume that a financial market exists where only agent I can operate, while agent B is not allowed to do so (this can happen for various reasons: agent B

could be a big insurance company and there could not exist a market big enough to influence its wealth in a significant way; or agent B could be a bank's compartment which is not allowed to operate in the stock market for internal regulations).

3 The case when Θ is the only source of risk

First we focus on the case when Θ as the only source of risk. We divide this first case in two subcases: in the first one we analyse the problem when both the agents have the same view of the distribution of Θ (i.e. we have $\mathbb{P}_B = \mathbb{P}_I = \mathbb{P}$); in the second one we analyse the case when the agents adopt the different probability measures $\mathbb{P}_B \neq \mathbb{P}_I$.

Generally agent B has to solve the problem

$$\min_{\overline{F}} \mathbb{E}_{\mathbb{P}_B}[\exp \gamma_B(\overline{F}(\Theta) - X(\Theta))^+ - 1] \tag{1}$$

where $\overline{F}$ is a Θ-measurable random variable. The bank is not free in choosing $\overline{F}$, because (if he wants to sell it to agent I) $\overline{F}$ has to satisfy agent I's budget constraint, which in this case results from the utility in the case when agent I buys F being greater than in the case when agent I does not. This is written as

$$\mathbb{E}_{\mathbb{P}_I}[\exp(-\gamma_I(\overline{F}(\Theta) + x))] \leq \exp(-\gamma_I x)$$

We can easily see that this constraint can be rewritten as

$$\mathbb{E}_{\mathbb{P}_I}[\exp(-\gamma_I \overline{F}(\Theta))] \leq 1 \tag{2}$$

3.1 Same probability measure

We first consider the case when the two agents use the same probability measure $\mathbb{P}$.

Theorem 1. *If Θ is the only source of risk and both the agents adopt the same probability measure $\mathbb{P}$, then the optimal structure of the contract is given by*

$$\begin{aligned} \overline{F}^*_{\lambda_1^*}(\Theta) &= \frac{\gamma_B}{\gamma_B + \gamma_I}\left(X(\Theta) + \frac{1}{\gamma_B}\log\left(\frac{\lambda_1^*\gamma_I}{\gamma_B}\right)\right)\mathbb{I}_{\{X(\Theta) < \frac{1}{\gamma_I}\log\left(\frac{\lambda_1^*\gamma_I}{\gamma_B}\right)\}} + \\ &\quad + X(\Theta)\mathbb{I}_{\{X(\Theta) \geq \frac{1}{\gamma_I}\log\left(\frac{\lambda_1^*\gamma_I}{\gamma_B}\right)\}} \end{aligned} \tag{3}$$

where λ_1^* is a real positive constant such that

$$\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I \overline{F}^*_{\lambda_1^*}(\Theta))] = 1 \tag{4}$$

provided $\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I X(\Theta))] > 1$. Conversely, if $\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I X(\Theta))] \leq 1$, then the optimal structure of the contract is given by $\overline{F}^*(\Theta) = X(\Theta)$.

Proof. As in [2], by introducing the Lagrange multiplier $\lambda_1 > 0$ we rewrite the problem as

$$\inf_{\lambda_1} \min_{\overline{F}(\cdot)} \mathbb{E}_{\mathbb{P}}[\exp \gamma_B(\overline{F}(\Theta) - X(\Theta))^+ - 1 + \lambda_1(\exp(-\gamma_I \overline{F}(\Theta)) - 1)]$$

and use variational control techniques, converting the problem in a pathwise optimization problem in this way:

$$\begin{aligned}
J(\lambda_1) \quad &:= \quad \min_{\overline{F}(\cdot)} \mathbb{E}_{\mathbb{P}}[\exp \gamma_B(\overline{F}(\Theta) - X(\Theta))^+ - 1 + \lambda_1(\exp(-\gamma_I \overline{F}(\Theta)) - 1)] = \\
&\geq \quad \mathbb{E}_{\mathbb{P}}[\min_{\varphi} \exp \gamma_B(\varphi - X(\Theta))^+ - 1 + \lambda_1(\exp(-\gamma_I \varphi) - 1)] = \\
&= \quad \mathbb{E}_{\mathbb{P}}[\exp \gamma_B(\varphi^* - X(\Theta))^+ - 1 + \lambda_1(\exp(-\gamma_I \varphi^*) - 1)] \geq \\
&\geq \quad J(\lambda_1)
\end{aligned}$$

where

$$\varphi^* = \mathrm{argmin}\, (\exp \gamma_B(\varphi - X(\Theta))^+ - 1 + \lambda_1(\exp(-\gamma_I \varphi) - 1)) \tag{5}$$

provided the minimizer φ^* exists. If so, then φ^* defines a random variable $\overline{F}^*$ (which depends on λ_1) which is a candidate for the solution to the problem.

To solve the problem (1) under the constraint (2), we have to solve (5). We divide this problem in two cases:

1. $\varphi^* - X(\Theta) > 0$
2. $\varphi^* - X(\Theta) \leq 0$

We discuss what happen in case 1. We have the minimum for the problem (5) when the derivative is zero, hence the optimal φ^* solve

$$\gamma_B \exp \gamma_B(\varphi^* - X(\Theta)) - \lambda_1 \gamma_I \exp(-\gamma_I \varphi^*) = 0$$

After some algebra we arrive to

$$\varphi^* = \frac{\gamma_B}{\gamma_B + \gamma_I} X(\Theta) + \frac{1}{\gamma_B + \gamma_I} \log\left(\frac{\lambda_1 \gamma_I}{\gamma_B}\right)$$

This is compatible with the constraint of the case 1. if and only if

$$\frac{\gamma_B}{\gamma_B+\gamma_I}X(\Theta)+\frac{1}{\gamma_B+\gamma_I}\log\left(\frac{\lambda_1\gamma_I}{\gamma_B}\right)-X(\Theta)>0$$

that is

$$X(\Theta)<\frac{1}{\gamma_I}\log\left(\frac{\lambda_1\gamma_I}{\gamma_B}\right)$$

If this does not happen, case 2. applies.

Now we study what happens in the case 2. The problem becomes

$$\lambda_1(\exp(-\gamma_I\overline{F}(\Theta))-1)$$

Being $\varphi\to\exp(-\gamma_I\varphi)$ strictly decreasing, we have the minimum when φ reaches the maximum value allowed by the constraint 2., that is when $\varphi^*=X(\Theta)$.

To conclude we have that the optimal claim $\overline{F}^*$ is

$$\begin{aligned}\overline{F}^*(\Theta) &= \frac{\gamma_B}{\gamma_B+\gamma_I}\left(X(\Theta)+\frac{1}{\gamma_B}\log\left(\frac{\lambda_1\gamma_I}{\gamma_B}\right)\right)\mathbb{I}_{\{X(\Theta)<\frac{1}{\gamma_I}\log\left(\frac{\lambda_1\gamma_I}{\gamma_B}\right)\}}+\\ &\quad +X(\Theta)\mathbb{I}_{\{X(\Theta)\geq\frac{1}{\gamma_I}\log\left(\frac{\lambda_1\gamma_I}{\gamma_B}\right)\}}=\\ &= \frac{\gamma_B}{\gamma_B+\gamma_I}\left(X(\Theta)+\frac{1}{\gamma_B}\log\left(\frac{\lambda_1\gamma_I}{\gamma_B}\right)\right)\bigvee X(\Theta)\end{aligned}$$

where $A\vee B:=\max(A,B)$. To find λ_1^*, we substitute this optimal policy in the constraint (2): to this purpose, we notice that for each fixed value of Θ, the function $\lambda\to\overline{F}^*_\lambda(\Theta)$ is continuous and increasing, so the function $\lambda\to\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I\overline{F}^*_\lambda(\Theta))]$ is decreasing, and by two easy applications of the Monotone Convergence Theorem (see [5]), it is continuous. Since

$$\begin{aligned}\lim_{\lambda_1\to 0}\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I\overline{F}^*_{\lambda_1}(\Theta))] &= \mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I X(\Theta))]>1,\\ \lim_{\lambda_1\to+\infty}\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I\overline{F}^*_{\lambda_1}(\Theta))] &= \mathbb{E}_{\mathbb{P}}[\exp(-\infty)]=0,\end{aligned}$$

then there exists a unique λ_1^* satisfying the budget constraint (2).

Let us now consider the case when $\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I X(\Theta))]\leq 1$. In this case, there does not exist a λ_1^* such that the candidate $\overline{F}^*_{\lambda_1}$ meets the sharp budget constraint (4). Instead, we easily check that $\overline{F}^*(\Theta)=X(\Theta)$ satisfies the original budget constraint (2) and is optimal for (1). □

So we found an optimal policy $\overline{F}^*$ continuous and pathwise linear in $X(\Theta)$, where the slope changes as

$$X(\Theta)<\frac{1}{\gamma_I}\log\left(\frac{\lambda_1\gamma_I}{\gamma_B}\right)$$

or

$$X(\Theta) \geq \frac{1}{\gamma_I} \log\left(\frac{\lambda_1 \gamma_I}{\gamma_B}\right)$$

3.2 Different probability measures

When each agent has a personal view of the distribution of Θ, they adopt two different probability measures. To reduce to the previous situation, we introduce a Radon-Nykodym's derivative to rewrite it under a unique probability measure. As $\mathbb{P}_B$ and $\mathbb{P}_I$ are both equivalent to $\mathbb{P}$, we can write

$$\frac{d\mathbb{P}_B}{d\mathbb{P}} = \exp(Y_B(\Theta)), \quad \frac{d\mathbb{P}_I}{d\mathbb{P}} = \exp(Y_I(\Theta)) \tag{6}$$

and we have

$$\frac{d\mathbb{P}_I}{d\mathbb{P}_B} = \exp(Y_I(\Theta) - Y_B(\Theta))$$

Hence we suppose that $\frac{d\mathbb{P}_B}{d\mathbb{P}}$ and $\frac{d\mathbb{P}_I}{d\mathbb{P}}$ are Θ-measurable. Hence, introducing the change of measure $\frac{d\mathbb{P}_I}{d\mathbb{P}_B}$, the problem (1) for the bank does not change, while the constraint (2) becomes

$$\mathbb{E}_{\mathbb{P}_B}[\exp(-\gamma_I \overline{F}(\Theta) + Y_I(\Theta) - Y_B(\Theta))] \leq 1 \tag{7}$$

Theorem 2. *Let Θ be the only source of risk and the agents have a different view of its distribution in the sense of (6). Then the structure of the optimal contract is given by*

$$\begin{aligned} \overline{F}^*_{\lambda_2^*}(\Theta) &= \frac{\gamma_B}{\gamma_B + \gamma_I}\left(X(\Theta) + \frac{1}{\gamma_B} Z^{B,I}\right) \mathbb{I}_{\{X(\Theta) < \frac{1}{\gamma_I} Z^{B,I}\}} + \\ &\quad + X(\Theta) \mathbb{I}_{\{X(\Theta) \geq Z^{B,I}\}} \end{aligned} \tag{8}$$

where

$$Z^{B,I} = \log\left(\frac{\lambda_2 \gamma_I}{\gamma_B}\right) - (Y_B(\Theta) - Y_I(\Theta)) \tag{9}$$

and λ_2^* is a real positive constant such that Equation (4) is satisfied, provided $\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I X(\Theta))] > 1$. Conversely, if $\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I X(\Theta))] \leq 1$, then the optimal structure of the contract is given by $\overline{F}^*(\Theta) = X(\Theta)$.

Proof. As in the proof of Theorem 1, we introduce the lagrangian with parameter λ_2 and we have to solve

$$\begin{gathered} \min_{\overline{F}} \mathbb{E}_{\mathbb{P}_I}[\exp[\gamma_B(\overline{F}(\Theta) - X(\Theta))^+)] - \\ -1 + \lambda_2(\exp(-\gamma_I \overline{F}(\Theta) + (Y_I(\Theta) - Y_B(\Theta)) - 1)] \end{gathered}$$

As before, we solve

$$\mathbb{E}_{\mathbb{P}_I}[\min_{\varphi^*}[\exp[\gamma_B(\varphi^* - X(\Theta))^+] - 1 + \\ +\lambda_2(\exp(-\gamma_I\varphi^* + (Y_I(\Theta) - Y_B(\Theta)) - 1))]] \tag{10}$$

As before, we divide the two cases:

1. $\varphi^* - X(\Theta) > 0$
2. $\varphi^* - X(\Theta) \leq 0$

We study what happens in case 1. The minimum for the function to be minimised in (10) is reached when the derivative is zero, that is when

$$\gamma_B \exp[\gamma_B(\overline{F}(\Theta) - X(\Theta)))] - \gamma_I\lambda_2 \exp(-\gamma_I\overline{F}(\Theta) + (Y_I(\Theta) - Y_B(\Theta)) = 0$$

After some calculations, we arrive to

$$\varphi^* = \frac{\gamma_B}{\gamma_B + \gamma_I}\left(X(\Theta) - \frac{1}{\gamma_B}(Y_B(\Theta) - Y_I(\Theta))\right) + \frac{1}{\gamma_I + \gamma_B}\log\left(\frac{\gamma_I\lambda_2}{\gamma_B}\right)$$

If we define $Z^{B,I}$ as in Equation (9), we can write

$$\overline{F}^*(\Theta) = \frac{\gamma_B}{\gamma_B + \gamma_I}\left(X^{B,I}(\Theta) - \frac{1}{\gamma_B}Z^{B,I}\right)$$

This is analogous to what we find in the case when the two agents have the same view of the distribution of Θ. As in that case, this satifies the constraint 1. when

$$\frac{\gamma_B}{\gamma_B + \gamma_I}X^{B,I}(\Theta) + \frac{1}{\gamma_B}Z^{B,I} - X(\Theta) > 0$$

that means

$$X(\Theta) < \frac{1}{\gamma_I}Z^{B,I}$$

We now analyze the case 2. The problem to study is

$$\min_{\overline{F}} \mathbb{E}_{\mathbb{P}_I}[\lambda_2(\exp(-\gamma_I\overline{F}(\Theta)) - 1)]$$

We have the minimum when $\overline{F}$ reaches its maximum value, that is when $\overline{F}^*(\Theta) = X(\Theta)$.

To conclude, we have

$$\begin{aligned}
\overline{F}^{*}(\Theta) &= \frac{\gamma_B}{\gamma_B+\gamma_I}\left(X(\Theta)-\frac{1}{\gamma_B}Z^{B,I}\right)\mathbb{I}_{\{X(\Theta)<\frac{1}{\gamma_I}Z^{B,I}\}}+ \\
&\quad +X(\Theta)\mathbb{I}_{\{X(\Theta)\geq\frac{1}{\gamma_I}Z^{B,I}\}}= \\
&= \frac{\gamma_B}{\gamma_B+\gamma_I}\left(X(\Theta)-\frac{1}{\gamma_I}Z^{B,I}\right)\bigvee X(\Theta)
\end{aligned}$$

Using arguments similar to those of Theorem 1, we can prove that there exists a unique λ_2^* such that Equation (4) is satisfied, provided $\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I X(\Theta))] > 1$. Conversely, if this condition does not hold, the second part follows easily by an argument similar to that of Theorem 1. □

We have seen that we find an optimal policy again linear in $X(\Theta)$, where the slope changes as

$$X(\Theta) < \frac{1}{\gamma_I}Z^{B,I}$$

or

$$X(\Theta) \geq \frac{1}{\gamma_I}Z^{B,I}$$

4 Introduction of a financial market

We introduce a financial market where only agent I can invest. We suppose that the financial market is complete, so there exists a unique equivalent martingale measure $\mathbb{Q}_T$. We define

$$H_T = \frac{d\mathbb{Q}_T}{d\mathbb{P}}$$

As Θ is not the only source of risk, the agent I has to choose how to invest in the financial market in both the cases when he faces the risk Θ and when he does not. He can build his optimal market portfolio with respect to his utility. The value of such portfolio at the time T is indicated with $V_I(z)$ when he invests an amount z at the initial time $t = 0$. If the investor buys the contract, he has an initial wealth $z = x - \pi$, that he will invest in the financial market. If he decides not to buy the contract, he will invest all the amount $z = x$ in the market. The capital flows up the time T can be written as $X(\Theta) - F(\Theta)$ for the bank and $F(\Theta) + V_I(x - \pi)$ for the investor if he does the transaction, $V_I(x)$ if he does not.

We assume that both the agents have the same view of the distribution of Θ and we write the problem under the real probability $\mathbb{P}$. As before, the bank wants to minimise the shortfall using a loss function $\ell(x) = \exp \gamma_B x - 1$ so we have

$$\min_{F,\pi,V_B} \mathbb{E}_{\mathbb{P}}[\exp \gamma_B (\overline{F}(\Theta) - X(\Theta))^+ - 1] \tag{11}$$

under the constraint

$$\max_{V_I} \mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I(F(\Theta) + V_I(x-\pi)))] \leq \max_{V_I} \mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I V_I(x))] \tag{12}$$

We will analyse two possible cases:

1. To calculate the optimal portfolios in (12) we do not consider the position of the investor to the risk Θ. That is, we assume that the investor has previously determined the structure of his optimal portfolios, and that he will not change its structure when taking into account the source of risk Θ. This can happen for example when the position $F(\Theta)$ is negligeable with respect to the optimal portfolio $V_I^*(x)$. In this case the constraint becomes

$$\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I(F(\Theta) + V_I^*(x-\pi)))] \leq \mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I V_I^*(x))] \tag{13}$$

   where $V_I^*(z)$ is the optimal portfolio with initial wealth z which solves the problem

$$\max_{V_I} \mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I V_I(z))]$$

2. The risk of the market is as the same order as the one we have to cover and so we cannot neglect $F(\Theta)$.

4.1 Optimal portfolio independent on the agents' position towards the risk Θ

In this case, the choice of the market portfolio on the investor's side does not depend of his position to Θ, so to determine it, we consider only the initial wealth that he invests in it.

We now obtain the optimal portfolio for the investor, and we find the structure of the optimal contract.

A classical result of portfolio theory (see for example [3]) is the characterisation of the optimal portfolio for an utility exponential function with risk-adversion coefficient γ_I, with initial capital z:

$$V_I^*(z) = z + \frac{1}{\gamma_I}\mathbb{E}_{\mathbb{Q}_T}[\log H_T] - \frac{1}{\gamma_I}\log H_T + z \quad \mathbb{P}\text{-a.s.} \tag{14}$$

To come back to the previous analysis, the investor's constraint has to be re-examinated and it is suitable to introduce the quantity $\overline{F}$. Replacing the investor's optimal portfolio in (13), we obtain:

$$\mathbb{E}_{\mathbb{Q}_T}\left[\exp\left(-\gamma_I(F(\Theta)+\tfrac{1}{\gamma_I}\mathbb{E}_{\mathbb{Q}_T}[\log H_T]+x-\pi)\right)\right]\leq$$
$$\leq \mathbb{E}_{\mathbb{Q}_T}\left[\exp\left(-\gamma_I\left(\tfrac{1}{\gamma_I}\mathbb{E}_{\mathbb{Q}_T}[\log H_T]+x\right)\right)\right]$$

By letting $\overline{F}=F-\pi$ we easily obtain

$$\mathbb{E}_{\mathbb{Q}_T}[\exp(-\gamma_I\overline{F}(\Theta))]\leq 1 \tag{15}$$

In order to reduce to the case of Section 3.2, we rewrite this constraint as

$$1\geq \mathbb{E}_{\mathbb{P}}[H_T\exp(-\gamma_I\overline{F}(\Theta))]=\mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[H_T|\Theta]\exp(-\gamma_I\overline{F}(\Theta))]$$

Now we are in the same framework of Section 3.2, with $\log\mathbb{E}_{\mathbb{P}}[H_T|\Theta]$ playing the same role as $Y_B(\Theta)-Y_I(\Theta)$. In fact, this is equivalent to introduce a new change of measure: we define the probability measure $\mathbb{Q}_\Theta$ such that

$$\frac{d\mathbb{Q}_\Theta}{d\mathbb{P}}=\frac{\mathbb{E}_{\mathbb{P}}[H_T|\Theta]}{\mathbb{E}_{\mathbb{P}}[H_T]}=\mathbb{E}_{\mathbb{P}}[H_T|\Theta]$$

Theorem 3. *When a financial market with state price density H_T is introduced the structure of the optimal contract is given by*

$$\begin{aligned}\overline{F}^*_{\lambda_3^*}(\Theta) &= \frac{\gamma_B}{\gamma_B+\gamma_I}\left(X(\Theta)-\frac{1}{\gamma_B}Z^{B,I}\right)\mathbb{I}_{\{X(\Theta)<\frac{1}{\gamma_I}Z^{B,I}\}}+\\ &\quad +X(\Theta)\mathbb{I}_{\{X(\Theta)\geq\frac{1}{\gamma_I}Z^{B,I}\}}\end{aligned} \tag{16}$$

where

$$Z^{B,I}=\log\left(\frac{\lambda_3^*\gamma_I}{\gamma_B}\right)-\log\mathbb{E}_{\mathbb{P}}[H_T|\Theta] \tag{17}$$

and λ_3^* is a real positive constant such that Equation (15) is satisfied, provided $\mathbb{E}_{\mathbb{Q}_T}[\exp(-\gamma_I X(\Theta))]>1$. Conversely, if $\mathbb{E}_{\mathbb{Q}_T}[\exp(-\gamma_I X(\Theta))]\leq 1$, then the optimal structure of the contract is given by $\overline{F}^*(\Theta)=X(\Theta)$.

Proof. The proof proceeds along the lines of that of Theorem 2, with $\log\mathbb{E}_{\mathbb{P}}[H_T|\Theta]$ playing the same role as $Y_B(\Theta)-Y_I(\Theta)$. □

4.2 Optimal portfolio depending on agents' position towards Θ

A more general way to treat the problem is to consider the agents' global position, that is the position towards the market and the risk Θ. So each of them will determine its portfolio in the optimal way with respect to the utility but also considering his view of Θ. We denote with $V_I(z)$ the value of the portfolios at the final time T for the investor when he invests at time 0 an amount z in the financial market. As before, the flows capitalized till time T can be written as follows:

- $X(\Theta) - F(\Theta)$ for the bank;
- $F(\Theta) + V_I(x - \pi)$ for agent I, if the transaction occurs,
- $V_I(x)$ for agent I, if not.

We always write the optimisation problem under the real world probability $\mathbb{P}$. The optimisation program become

$$\min_{F,\pi,V_B} \mathbb{E}_{\mathbb{P}}[\exp \gamma_B (F(\Theta) - X(\Theta))^+ - 1]$$

under the constraint

$$\max_{V_I} \mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I(F(\Theta) + V_I(x-\pi)))] \leq \max_{V_I} \mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I V_I(x))]$$

We now proceed by steps: in the first one we find the optimal portfolio for the investor depending on $F(\Theta)$ and π; then we find a condition on the maximum price π that the investor would accept for the claim $F(\Theta)$; finally, we find the structure of the optimal contract.

4.2.1 Search of the optimal portfolio for the investor

Proposition 1. *The value of the optimal portfolio for the investor when he consider his view of the risk Θ is given by*

$$\begin{aligned} V_I(x-\pi) &= -\frac{1}{\gamma_I}(\log H_T - \mathbb{E}_{\mathbb{Q}_T}[\log H_T]) + x - \pi + \\ &\quad + \frac{1}{\gamma_I}\left(\log \mathbb{E}_{\mathbb{P}}\left[e^{-\gamma_I F(\Theta)}|H_T\right] - \mathbb{E}_{\mathbb{Q}_T} \log \mathbb{E}_{\mathbb{P}}\left[e^{-\gamma_I F(\Theta)}|H_T\right]\right) \end{aligned} \tag{18}$$

Proof. The investor has to solve the problem

$$\max_{V_I} \mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I(F(\Theta) + V_I(x-\pi)))]$$

under the constraint

$$\mathbb{E}_{\mathbb{P}}[H_T V_I(x-\pi)] \leq x-\pi$$

Since V_I has to be H_T-measurable, we rewrite the function to be maximised as

$$\mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I F(\Theta))|H_T]\exp(-\gamma_I V_I(x-\pi))]$$

As in the proof of Theorem 1 we introduce the lagrangian with parameter λ_I and we have to solve

$$\max_{V_I} \mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I F(\Theta)))|H_T]\exp(-\gamma_I V_I(x-\pi))+ \\ +\lambda_I(H_T V_I(x-\pi)-(x-\pi))]$$

We can derive in V_I and see when this derivative is zero

$$-\gamma_I \mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I(F(\Theta)))|H_T]\exp(-\gamma_I V_I(x-\pi))+\lambda_I H_T = 0$$

which gives

$$V_I(x-\pi) = -\frac{1}{\gamma_I}\log\lambda_I - \frac{1}{\gamma_I}\log\frac{H_T}{\gamma_I} + \frac{1}{\gamma_I}\log\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I F(\Theta))|H_T] \quad (19)$$

Now we draw λ_I from the constraint

$$\mathbb{E}_{\mathbb{P}}[H_T V_I(x-\pi)] \leq x-\pi$$

So we have

$$\mathbb{E}_{\mathbb{Q}_T}\left[-\frac{1}{\gamma_I}\log\lambda_I - \frac{1}{\gamma_I}\log\frac{H_T}{\gamma_I} + \frac{1}{\gamma_I}\log\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I(F(\Theta)))|H_T]\right] \leq x-\pi$$

and we easily get

$$-\frac{1}{\gamma_I}\log\lambda_I \leq \frac{1}{\gamma_I}\mathbb{E}_{\mathbb{Q}_T}\log\frac{H_T}{\gamma_I} - \frac{1}{\gamma_I}\mathbb{E}_{\mathbb{Q}_T}\log\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I F(\Theta))|H_T] + x - \pi$$

Substituing what we found in Eq. (19), we easily obtain Eq. (18). □

Remark 1. In this case, the optimal portfolio is equal to the optimal portfolio in the case when $F(\Theta)$ is negligeable (see Section 4.1) plus a second term which depends on $F(\Theta)$.

4.2.2 Search for the maximum price π

By substituting Eq. (18) and Eq. (14) in the investor's constraint, we obtain

$$\exp(\mathbb{E}_{\mathbb{Q}_T}[\log(\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I \overline{F}(\Theta))|H_T])])\mathbb{E}_{\mathbb{Q}_T}\left[\frac{\exp(-\gamma_I F(\Theta))}{\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I F(\Theta))|H_T]}\right] \leq 1 \tag{20}$$

We can see that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_T}\left[\frac{\exp(-\gamma_I F(\Theta))}{\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I F(\Theta))|H_T]}\right] &= \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left[\frac{\exp(-\gamma_I F(\Theta))H_T}{\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I F(\Theta))|H_T]}|H_T\right]\right] = \\
&= \mathbb{E}_{\mathbb{P}}\left[\frac{H_T \mathbb{E}_{\mathbb{P}}\left[\exp(-\gamma_I F(\Theta))|H_T\right]}{\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I F(\Theta))|H_T]}\right] = \\
&= \mathbb{E}_{\mathbb{P}}[H_T] = 1
\end{aligned}$$

So we finally obtain

$$\mathbb{E}_{\mathbb{Q}_T}[\log(\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I \overline{F}(\Theta))|H_T])] \leq 0$$

which translates into

$$\pi \leq -\frac{1}{\gamma_I}\mathbb{E}_{\mathbb{Q}_T}[\log(\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I F(\Theta))|H_T])]$$

which is a condition on the maximum price that the investor would accept to pay for the claim $F(\Theta)$.

4.2.3 The optimal contract

In order to solve the problem, we transform the constraint in Eq. (20) in a form similar to the case when the portfolio is independent of the claim $F(\Theta)$: we rewrite Eq. (20) as

$$\mathbb{E}_{\mathbb{P}}\left[\exp(-\gamma_I \overline{F}(\Theta))H_{\overline{F}}\right] \leq \frac{1}{\mathbb{E}_{\mathbb{P}}[Z_{\overline{F}}]}$$

where $H_{\overline{F}^*}$ is the density of the probability measure $\mathbb{Q}_{\overline{F}^*}$ defined by

$$\frac{d\mathbb{Q}_{\overline{F}}}{d\mathbb{P}} = H_{\overline{F}} = \frac{Z_{\overline{F}}}{\mathbb{E}_{\mathbb{P}}[Z_{\overline{F}}]}$$

and $Z_{\overline{F}^*}$ is defined as

$$Z_{\overline{F}} = \mathbb{E}_{\mathbb{P}}\left[H_T\frac{\exp(\mathbb{E}_{\mathbb{Q}_T}[\log(\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I \overline{F}(\Theta))|H_T])])}{\mathbb{E}_{\mathbb{P}}[\exp(-\gamma_I \overline{F}(\Theta))|H_T]} \mid \Theta\right]$$

Notice that if Θ and H_T are independent under $\mathbb{P}$, we obtain $Z_{\overline{F}^*} = \mathbb{E}_{\mathbb{P}}[H_T|\Theta]$, and $\mathbb{E}_{\mathbb{P}}[Z_{\overline{F}^*}] = 1$; that is, we reobtain the case when the investment is independent of $\overline{F}$. In the general case that we are treating here, the optimal claim $\overline{F}^*$ is the solution of a nonlinear equation which involves a double conditional expectation, as in [1].

Theorem 4. *When a financial market with state price density H_T is introduced, the structure of the optimal contract is given by*

$$\begin{aligned}\overline{F}^*_{\lambda_4^*}(\Theta) &= \frac{\gamma_B}{\gamma_B+\gamma_I}\left(X(\Theta) - \frac{1}{\gamma_B} Z^{B,I}\right)\mathbb{I}_{\{X(\Theta)<\frac{1}{\gamma_I}Z^{B,I}\}} + \\ &\quad + X(\Theta)\mathbb{I}_{\{X(\Theta)\geq\frac{1}{\gamma_I}Z^{B,I}\}} \end{aligned} \quad (21)$$

where

$$Z^{B,I} = \log\left(\frac{\lambda_4^*\gamma_I}{\gamma_B}\right) - \log Z_{\overline{F}^*} \quad (22)$$

and λ_4^* is a real positive constant such that Equation (20) is satisfied, provided $\mathbb{E}_{\mathbb{Q}_T}[\exp(-\gamma_I X(\Theta))] > 1$. Conversely, if $\mathbb{E}_{\mathbb{Q}_T}[\exp(-\gamma_I X(\Theta))] \leq 1$, then the optimal structure of the contract is given by $\overline{F}^*(\Theta) = X(\Theta)$.

Proof. The proof proceeds along the lines of that of Theorem 2, with $\log Z_{\overline{F}^*}$ playing the same role as $Y_B(\Theta) - Y_I(\Theta)$. $\square$

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