Optimal portfolio for HARA utility functions when risky assets are exponential additive processes

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February 8, 2010

Abstract

In this paper we analyse a market where the risky assets follow exponential additive processes, which can be viewed as time-inhomogeneous generalisations of geometric Levy processes. In this market we show that, when an investor wants to maximize a HARA utility function of his/her terminal wealth, his/her optimal strategy consists in keeping proportions of wealth in the risky assets which depend only on time but not on the current wealth level or on the prices of the risky assets. In the time-homogeneous case, the optimal strategy is to keep constant proportions of wealth, a result already found in [11] which extends the classical Merton's result [16] to this market. While the one-dimensional case has been extensively treated (see e.g. [1, 3, 9, 10, 17]) and the multidimensional case has been treated only in the time-homogeneous case [4, 11, 13], to the authors' knowledge this is the first time that such results are obtained for exponential additive processes in the multidimensional case. We use these results to show that the optimal solution in the presence of jumps has the form of the analogous one without jumps but with the asset yields vector reduced by suitable quantities: in the one-dimensional case, we extend a result by [3, 9]. We conclude with four examples.

1 Introduction

In these last years a growing attention has been paid to financial models which incorporate jumps; after the recent financial crisis of 2008, this trend is perhaps

*corresponding author. The two authors wish to thank Claudia Ceci, Jean Jacod, Valentina Prezioso and all the participants to the "Research week on financial mathematics and econometrics" held in Florence, September 9–11, 2009 and to the Workshop on Quantitative Finance held in Palermo, January 28–29, 2010, and two anonymous referees for their useful comments. This work was partly supported by PRIN Research Grant "Probability and finance" under grant 2008YYYBE4 and by the University of Padova under grant CPDA082105/08.

going to increase; due to possible inhomogeneities in the asset prices currently observed in financial markets (see [7, Chapter 14] for a brief discussion), an attention to time-inhomogeneous processes could also be paid. While the classical way of incorporating jumps in the risky assets' dynamics is that of jump-diffusion models (see [19] for a survey), in the recent years the introduction of jumps has been often made with less intuitive instruments, the typical way being that of a more general Levy process with infinite activity in any time interval (see [5, 6] for two of the most used models of this kind): this in principle allows one to neglect or totally eliminate the role of Brownian motion from financial models, even if one can as well work with models where both the Brownian component as well the presence of infinite jumps can be taken into account.

In the study of financial markets the problem of maximising the expected utility of the terminal wealth is perhaps the most studied problem: in this particular setting, this is witnessed by the large amount of literature dedicated to this problem in the presence of jumps (see [1, 2, 3, 4, 9, 10, 11, 13] for some examples): the solution techniques are mainly two, namely the dynamic programming approach, leading to a suitable HJB equation, and the convex duality method. However, these works are mainly confined to the one-dimensional case, with the three notable exceptions of [4, 11, 13]. In the first paper [4], the authors solve the problem for a HARA utility function and a model where the risky assets' dynamics are only driven by Poisson jumps, with a nontrivial correlation structure: the optimal strategy turns out to be the solution of a system of algebraic equations. In the second paper [11], the utility function can be both of the HARA type or of the CARA (i.e., exponential) type and the risky assets' dynamics are driven by general n-dimensional Levy processes: the optimal portfolio is the solution of a system of integral equations. Finally, in the third paper [13], a link is traced between the so-called numeraire portfolio and the solution of this problem in the case of a logarithmic utility: the processes driving the risky assets in this case are jump-diffusions, and the optimal portfolio is again the solution of a system of integral equations. It is however worth noticing that all these models, at least in the multidimensional case, are time-homogeneous, while to the authors' knowledge nothing has been written for the multidimensional time-inhomogeneous case.

This article wants to fill two gaps. The first one is to show that explicit results can be obtained with the dynamic programming approach also in the time-dependent case: while the authors are confident that similar results could be obtained also using convex duality techniques, the proof using dynamic programming is straightforward in the particular case of HARA utility functions. We obtain as sufficient conditions the system of integral equations (15), which is a time-dependent generalisation of the corresponding first-order condition of [11] and generalises analogous conditions found in [1, 2, 3, 4, 9, 10, 13] under more restricting assumptions. The second one is to furnish a bridge between the formulation with ordinary exponentials, typical of many models used in practice (see e.g. [5, 6, 14]), and the formulation with stochastic exponentials, more typical of works dealing with the utility maximisation problem (see e.g. [1, 2, 4, 9, 10, 11, 13]): we thus explicitly provide both the descriptions for three of the most used models driven by jumps, namely the Kou model [14], the Variance Gamma model [5] and the Tempered Stable (or CGMY, for Carr-Geman-Madan-Yor, [6]) model.

The paper is organised as follows. In Section 2 we formulate the model, where the risky assets are driven by additive processes, and the utility maximisation problem. In Section 3, we briefly present the dynamic programming approach and the link between this problem and the solution of a suitable Hamilton-Jacobi-Bellman (HJB) equation via a verification theorem suitable for this case. We then explicitly solve the HJB equation in both the instances of the HARA case, namely with a logarithmic utility or with a power utility. We finally present an analysis of the solution in the case when the optimal portfolio proportions lies in the interior of the admissible space of controls H: first we present a characterisation of these optimal proportions in terms of the first-order sufficient condition in Equation (15) cited above. Then we compare the solution to the analogous solution in the case of no jumps: we show that the optimal solution in the presence of jumps has the form of the analogous one without jumps but with the asset yields vector reduced by suitable quantities: in the one-dimensional case, we extend a result by [3, 9] and prove that the exposure in the risky asset is less in absolute value than the analogous one without jumps, and always with the same sign. This can be interpreted as the effect of an increase of the second moment of the driving process, argument that we show with a formal first order expansion of the jump component. Section 4 contains four examples: the first three are the Kou, Variance Gamma and CGMY models in the one-dimensional case, with the Levy measure reformulated in terms of stochastic exponentials and a numerical solution given for various risk-aversion coefficients, which also include the logarithmic case; the fourth example is the multidimensional model of [4], where we show that the first-order sufficient condition in Equation (15) reduces to the set of algebraic equations already presented in [4].

2 The optimal portfolio problem

We consider a portfolio with a locally riskless asset B which we assume identically equal to 1 (this can be obtained without loss of generality by considering discounted prices) and n risky assets S^i, $i = 1, \dots, n$: since we want these risky assets to be exponential additive processes, possibly correlated both in the diffusion part as well as in the jumps, we formulate their dynamics as

$$dS^i_t = S^i_{t-}\, dR^i_t, \quad i = 1, \dots, n, \tag{1}$$

where $R = (R^1, \dots, R^n)$ is the n-dimensional additive process with dynamics

$$dR_t = \mu(t)dt + \sigma(t)dW_t + \int_{\mathbb{R}^n} x(N(dx, dt) - \nu_t(dx)dt)$$

with $\mu = (\mu_1, \dots, \mu_n) : [0, T] \to \mathbb{R}^n$, $\sigma = (\sigma_{ij})_{ij} : [0, T] \to \mathbb{R}^{n\times d}$ measurable functions, $W = (W^1, \dots, W^d)$ a d-dimensional Brownian motion and $N(dx, dt)$

a Poisson random measure on $\mathbb{R}^n$ with compensating measure $\nu_t(dx)dt$. We can thus rewrite the dynamics of $S_t = (S_t^1, \dots, S_t^n)$ in a more compact vectorial notation as

$$dS_t = \text{diag}(S_{t-})\ dR_t$$

where diag(v) denotes the diagonal matrix in $\mathbb{R}^{n\times n}$ having in the principal diagonal the elements of the n-dimensional vector v.

In order to guarantee that the price of the n assets stays a.s. positive for all $t \in [0,T]$, we assume that

$$\text{supp}(\nu_t) \subseteq X := \{x \in \mathbb{R}^n | x_i \geq -1 \quad \forall i = 1, \dots, n\} \tag{2}$$

and that $\nu_t(\partial X) = 0$ for all $t \in [0,T]$.

We also want the assets to have finite variance: a sufficient condition for this to hold true is to impose that

$$\int_0^T \left(\|\mu(t)\|_n + \|\sigma(t)\|_{n\times d}^2 + \int_{\mathbb{R}^n} \|x\|_n^2\ \nu_t(dx) \right)\ dt < +\infty \tag{3}$$

where $\|x\|_n^2 := \sum_{i=1}^n x_i^2$ and $\|A\|_{n\times d}^2 := \sum_{i=1}^n \sum_{j=1}^d A_{ij}^2$. In fact, in this case the process R has finite variance, and by modifying the proof of [18, Theorem V.67] to this case one obtains that $\mathbb{E}[\|S_t\|_n^2] < +\infty$ for all $t \in [0,T]$, i.e., the risky assets S^i, $i = 1, \dots, n$ have all finite variance.

Remark 2.1. *When dealing with exponential additive processes, it is usual to write them as* e^{L_t}*, with* L *a suitable additive process (see for example [7, Section 14.2]). It is very simple to see that, under the assumption (2), their representation is equivalent to that of Equation (1). In fact, this equation has solution*

$$S_t^i = e^{R_t^i - \frac{1}{2}\int_0^t \sum_{j=1}^n \sigma_{ij}^2(s)\ ds} \prod_{0<s\leq t} (1 + \Delta R_s^i) e^{-\Delta R_s^i}$$

(see [18, Theorem II.37]), where $\Delta R_s^i := R_s^i - R_{s-}^i$*. If, for* $t \in [0,T]$*, we have* supp$(\nu_t) \subseteq X$ *and* $\nu_t(\partial X) = 0$*, as in our case, then* $1 + \Delta R_t^i > 0$ $\mathbb{P}$*-a.s. for* $i = 1, \dots, n$*, so the* S^i*,* $i = 1, \dots, n$*, are strictly positive processes, which can thus be written as* $e^{L_t^i}$*, where*

$$L_t^i := R_t^i - \frac{1}{2}\int_0^t \sum_{j=1}^d \sigma_{ij}^2(s)\ ds + \sum_{0<s\leq t} (\log(1 + \Delta R_s^i) - \Delta R_s^i)$$

are additive processes.

Remark 2.2. *By relaxing the requirement that* $\nu_t(\partial X) = 0$*, it would be possible to take into account the case when some (or all) of the risky assets may default due to a jump to 0. While this in principle could be done with techniques similar to those presented here, we leave this case to a future work.*

Let now $h_t := (h_t^1, \dots, h_t^n)$ be the proportions of the portfolio invested respectively in the assets $(S^1, \dots, S^n)$ at time t; then the dynamics of the portfolio value V^h can be written as

$$dV_t^h = \sum_{i=1}^n \frac{V_{t-}^h h_{t-}^i}{S_{t-}^i} dS_t^i = V_{t-}^h \langle h_{t-}, dR_t \rangle \tag{4}$$

where $\langle x, y \rangle := \sum_{i=1}^n x_i y_i$ denotes the scalar product in $\mathbb{R}^n$, provided that also V^h stays $\mathbb{P}$-a.s. positive. In the spirit of Remark 2.1, we notice that V^h as solution of Equation (4) can be written as

$$V_t^h = e^{\langle h_{t-}, dR_t \rangle - \frac{1}{2}\int_0^t \langle h_{s-}, \Sigma(s) h_{s-} \rangle \, ds} \prod_{0<s\le t} (1 + \langle h_{s-}, \Delta R_s \rangle) e^{-\langle h_{s-}, \Delta R_s \rangle}$$

with $\Sigma(t) = (a_{ij}(t))_{ij} := \sigma(t)\sigma^T(t)$. To require that V stays positive is thus equivalent to requiring that $\langle h_{s-}, \Delta R_s \rangle > -1$ $\mathbb{P}$-a.s. for all $t \in [0, T]$, i.e. that

$$h_t \in H_t := \{ h \in \mathbb{R}^n \mid \langle h, x \rangle > -1 \quad \nu_t(dx)\text{-a.s. } \} \qquad \forall t \in [0, T] \tag{5}$$

Example 2.3. *If the jumps of all the risky assets are unbounded in both directions, i.e.* $\mathrm{supp}(\nu_t) \equiv X = \{x \in \mathbb{R}^n | x_i \ge -1, \ i = 1, \dots, n\}$ *for all* $t \in [0, T]$*, then we get that* $H_t \equiv \{h \in \mathbb{R}^n \mid h_i \ge 0, \sum_{i=1}^n h_i \le 1\}$*, i.e. in order for* V *to stay positive the process* h *can take values in the* n*-dimensional unit simplex in* $\mathbb{R}^n$*.*

Example 2.4. *If* $n = 1$ *and* $\mathrm{supp}(\nu_t) = [-m(t), M(t)]$*, with* $-1 \le -m(t) \le 0 \le M(t) \le +\infty$ *for all* $t \in [0, T]$*, then* $H_t = [-\frac{1}{M(t)}, \frac{1}{m(t)}]$ *for all* $t \in [0, T]$*, which is the generalisation of a result in [15] to this time-dependent case. In this example, we recover also the particular cases when jumps can only be positive, i.e.* $m(t) \equiv 0$*, so that the strategy* h *is unbounded from above, or when jumps can only be negative, i.e.* $M(t) \equiv 0$*, so that* h *is unbounded from below. If* $-m(t) \equiv -1$ *and* $M(t) \equiv +\infty$*, then* $H_t \equiv [0, 1]$*, i.e. the investor will never take a leveraged or short position in the risky asset.*

We now fix a time horizon T in order to maximise, over the strategy h, the expected utility

$$\sup_h \mathbb{E}[U(V_T^h)] \tag{6}$$

where U is a HARA utility function (e.g., $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ with $\gamma > 0$, $\gamma \neq 1$, or $U(x) = \log x$, which can be regarded as the "limiting" case as $\gamma \to 1$). For technical reasons, we choose to work with bounded controls h (as we have seen in the previous two examples, in many relevant cases this is not a restriction). More precisely, we fix a closed bounded convex set $H \subset \mathbb{R}^n$ such that $H \subset \mathrm{int}(\cap_{t\in[0,T]} H_t)$[1], and we call a strategy h *admissible*, and indicate this with

[1] As the n-dimensional simplex of Example 2.3 is always included in each H_t, $t \in [0, T]$, such a H always exists.

$h \in \mathcal{A}[t,T]$, if it is predictable, $h_u \in H$ $\mathbb{P}$-a.s. for all $u \in [t,T]$ and Equation (4) has a unique strong solution $V^{t,v}$ for each initial condition $V_t = v$.

We notice that if Equation (3) holds, then for all $h \in \mathcal{A}[t,T]$ the process $\int_0^{\cdot} \langle h_{s-}, dR_s\rangle$ has finite variance, hence also V^h has. Moreover, as we are dealing with HARA utility functions, the following result is also useful.

Lemma 2.5. *For all $\bar{t} \in [0,T]$ and $h \in \mathcal{A}[\bar{t},T]$ we have $\mathbb{E}[\sup_{t\in[\bar{t},T]}(V_t^h)^{2(1-\gamma)}] \leq C$ for a suitable $C \in \mathbb{R}$.*

Proof. For all $h \in \mathcal{A}[\bar{t},T]$, we have

$$\begin{aligned} d(V_t^h)^{1-\gamma} &= (V_{t-}^h)^{1-\gamma}\Bigg[(1-\gamma)\langle h_t, dR_t\rangle - \frac{1}{2}\gamma(1-\gamma)\|\sigma^T(t)h_t\|_n^2\ dt + \\ &\quad + \int_{\mathbb{R}^n}[(1+\langle h_t,x\rangle)^{1-\gamma} - 1 - (1-\gamma)\langle h_t,x\rangle](N(dt,dx) - \nu_t(dx))\Bigg] \end{aligned} \tag{7}$$

Since the process $\int_{\bar{t}}^{\cdot}(1-\gamma)\langle h_t, dR_t\rangle$ is in L^2 and

$$\mathbb{E}\left[\int_{\bar{t}}^T \frac{1}{2}\gamma(1-\gamma)\|\sigma^T(t)h_t\|_n^2\ dt\right] \leq \frac{1}{2}|\gamma(1-\gamma)| \sup_{h\in H}\|h\|_n \int_{\bar{t}}^T \|\sigma^T(t)\|_{d\times n}^2\ dt < +\infty$$

in order to verify that $(V_t^h)^{1-\gamma} \in L^2$ it is sufficient to check the finiteness of

$$\begin{aligned} &\mathrm{Var}\left[\int_{\bar{t}}^T \int_{\mathbb{R}^n}[(1+\langle h_t,x\rangle)^{1-\gamma} - 1 - (1-\gamma)\langle h_t,x\rangle](N(dt,dx) - \nu_t(dx)\ dt)\right] = \\ &= \int_{\bar{t}}^T \int_{\mathbb{R}^n}[(1+\langle h_t,x\rangle)^{1-\gamma} - 1 - (1-\gamma)\langle h_t,x\rangle]^2\nu_t(dx)\ dt = \\ &= \int_{\bar{t}}^T \int_{\|x\|_n<\epsilon}[(1+\langle h_t,x\rangle)^{1-\gamma} - 1 - (1-\gamma)\langle h_t,x\rangle]^2\nu_t(dx)\ dt + \\ &\quad + \int_{\bar{t}}^T \int_{\|x\|_n\geq\epsilon}[(1+\langle h_t,x\rangle)^{1-\gamma} - 1 - (1-\gamma)\langle h_t,x\rangle]^2\nu_t(dx)\ dt \end{aligned}$$

Now, in a neighbourhood of 0 we have that $(1+\langle h_t,x\rangle)^{1-\gamma} - 1 - (1-\gamma)\langle h_t,x\rangle = O(\langle h_t,x\rangle^2)$, so the first integral is finite (recall that h_t takes values in a bounded H). As concerns the second integral, since h_t takes values in $H \subset \mathrm{int}(\cap_{t\in[0,T]}H_t)$, there exists a $\delta > 0$ such that $\langle h_t,x\rangle \geq -1+\delta$ ν_t-a.s. for all t, so the function $(1+\langle h_t,x\rangle)^{1-\gamma} - 1 - (1-\gamma)\langle h_t,x\rangle$ is bounded from below and with linear growth, so also the second integral converges. It is now sufficient to apply [18, Theorem V.67] on Equation (7) to obtain that $\mathbb{E}[\sup_{t\in[\bar{t},T]}(V_t^h)^{2(1-\gamma)}] \leq C$ for a suitable $C \in \mathbb{R}$. □

3 Solution with dynamic programming

We define $J^h(t,v) := \mathbb{E}[U(V_T^{h;t,v})]$ and the value function as

$$J(t,v) := \sup_{h\in\mathcal{A}[t,T]} J^h(t,v) = \sup_{h\in\mathcal{A}[t,T]} \mathbb{E}[U(V_T^{h;t,v})] \tag{8}$$

where $\{V_s^{h;t,v}, s \geq t\}$ is the solution of Equation (4) with initial condition $V_t := v > 0$. The initial problem is thus equivalent to calculate $J(0, V_0)$.

It is well known that, by the dynamic programming principle and the Markovianity of V^h, we can write

$$\begin{aligned} J(t,v) &= \sup_{h\in\mathcal{A}[t,T]} \mathbb{E}[\mathbb{E}[U(V_T^{h;t,v})|\mathcal{F}_{t+u}]] = \sup_{h\in\mathcal{A}[t,T]} \mathbb{E}\left[U\left(V_T^{h;t+u,V_{t+u}^{h;t,v}}\right)\right] = \\ &= \sup_{h\in\mathcal{A}[t,t+u]} \mathbb{E}[J(t+u, V_{t+u}^{h;t,v})] \end{aligned} \tag{9}$$

for all u such that $t + u \leq T$, and by formal arguments we arrive to the HJB (Hamilton-Jacobi-Bellman) equation

$$0 = \frac{\partial J}{\partial t}(t,v) + \sup_{h\in H} A^h J(t,v) \tag{10}$$

where for all $h \in H$ we define

$$\begin{aligned} A^h J(t,v) &= \frac{\partial J}{\partial v}(t,v)v\langle h, \mu(t)\rangle + \frac{1}{2}v^2\langle\Sigma(t)h,h\rangle\frac{\partial^2 J}{\partial v^2}(t,v) + \\ &+ \int_{\mathbb{R}^n}\left(J(t,v+v\langle h,x\rangle) - J(t,v) - v\langle h,x\rangle\frac{\partial J}{\partial v}(t,v)\right)\nu_t(dx) \end{aligned} \tag{11}$$

with $\Sigma(t) = (a_{ij}(t))_{ij} := \sigma(t)\sigma^T(t)$, is called the *infinitesimal generator* of the controlled process V^h, which is linked to it by the so-called *Dynkyn formula*

$$\mathbb{E}[f(T, V_T^h)] - \mathbb{E}[f(t, V_t^h)] = \mathbb{E}\left[\int_t^T A^{h_u} f(u, V_u^h)\, du\right] \tag{12}$$

for f good enough. In order for the differential problem of HJB equation to be well defined, we also impose the terminal condition

$$J(T,v) = U(v). \tag{13}$$

The next theorem links formally the HJB equation with our problem. This theorem belongs to the class of the so-called *verification theorems* for stochastic control. While theorems like this can be explicitly found in literature for Levy processes (see [17] for several cases), the authors did not find an analogous result for additive processes. We thus proceed to state and prove a verification theorem, following results of [7] and [8].

Define

$$\mathcal{D} := \{f \in C^{1,2}([0,T]\times\mathbb{R}) \mid \forall t \in [0,T],\ \text{the Dynkyn formula (12) holds } \forall h \in \mathcal{A}[t,T]\}$$

The usual choice for possibly discontinuous Markov processes is $\mathcal{D} := C_0^2([0,T]\times\mathbb{R})$, the C^2 functions vanishing at infinity: in fact for this space it is always possible to prove that the Dynkyn formula holds. However, this space is too small for our purposes, as HARA utility functions are unbounded, so we have to define $\mathcal{D}$ more generally as above.

We can now state the following verification theorem, which is a particular case of [8, Theorem III.8.1].

Theorem 3.1 (verification theorem)**.** *Let $K \in \mathcal{D}$ be a classical solution to (10) with terminal condition (13). Then, for all $(t, v) \in [0, T] \times \mathbb{R}$,*

(a) $K(t, v) \geq J^h(t, v)$ for every admissible control $h \in \mathcal{A}[t, T]$;

(b) if there exists an admissible control $h^* \in \mathcal{A}[t, T]$ such that

$$h^*_s \in \arg\max_h A^h K(s, V^h_s) \qquad \mathbb{P}\text{-a.s. for all } s \in [t, T],$$

then $K(t, v) = J^{h^*}(t, v) = J(t, v)$.

Thus, the utility maximisation problem boils down to find a regular solution of the HJB equation. We see that this is the case for the particular situation that we have.

3.1 HARA utility

We consider the general case of a HARA utility function $U(v) = \frac{v^{1-\gamma}}{1-\gamma}$ for $\gamma > 0$, $\gamma \neq 1$ or $U(v) = \log v$ (which we conventionally indicate as "the case $\gamma = 1$") and, in analogy with the diffusion case, search for a solution of the kind $J(t, v) := U(e^{\phi(t)}v)$, where $\phi(t)$ is a C^1 deterministic function of time such that $\phi(T) = 0$. We then have, for all $\gamma > 0$ (remember that $\gamma = 1$ corresponds to $U(v) = \log v$),

$$\begin{aligned} \frac{\partial J}{\partial t}(t, v) &= U'(e^{\phi(t)}v)ve^{\phi(t)}\phi'(t) = \phi'(t)(ve^{\phi(t)})^{1-\gamma}, \\ v\frac{\partial J}{\partial v}(t, v) &= U'(e^{\phi(t)}v)e^{\phi(t)} = (ve^{\phi(t)})^{1-\gamma}, \\ v^2\frac{\partial^2 J}{\partial v^2}(t, v) &= U''(e^{\phi(t)}v)e^{2\phi(t)} = -\gamma(ve^{\phi(t)})^{1-\gamma} \end{aligned}$$

Then the HJB equation for $\gamma \neq 1$ becomes

$$\begin{aligned} 0 &= \phi'(t)(ve^{\phi(t)})^{1-\gamma} + (ve^{\phi(t)})^{1-\gamma} \sup_{h \in H}\Bigg[\langle h, \mu(t)\rangle - \frac{1}{2}\gamma\langle\Sigma(t)h, h\rangle + \\ &\quad + \int_{\mathbb{R}^n}\left[\frac{1}{1-\gamma}\Big((1 + \langle h, x\rangle)^{1-\gamma} - 1\Big) - \langle h, x\rangle\right]\nu_t(dx)\Bigg] \end{aligned}$$

and for $\gamma = 1$ becomes

$$0 = \phi'(t) + \sup_{h \in H}\left[\langle h, \mu(t)\rangle - \frac{1}{2}\gamma\langle\Sigma(t)h, h\rangle + \int_{\mathbb{R}^n}\Big[\log(1 + \langle h, x\rangle) - \langle h, x\rangle\Big]\nu_t(dx)\right]$$

By dividing the HJB equation for $(ve^{\phi(t)})^{1-\gamma}$, for all $\gamma > 0$ we obtain

$$0 = \phi'(t) + \sup_{h \in H} F(t, h)$$

with terminal condition $\phi(T) = 0$, where for $\gamma \neq 1$ the function F is defined as

$$\begin{aligned} F(t,h) &:= \langle h, \mu(t) \rangle - \frac{1}{2}\gamma \langle \Sigma(t) h, h \rangle + \\ &+ \int_{\mathbb{R}^n} \left[\frac{1}{1-\gamma} \Big((1 + \langle h, x \rangle)^{1-\gamma} - 1 \Big) - \langle h, x \rangle \right] \nu_t(dx) \end{aligned}$$

and for $\gamma = 1$ as

$$F(t,h) := \langle h, \mu(t) \rangle - \frac{1}{2} \langle \Sigma(t) h, h \rangle + \int_{\mathbb{R}^n} \Big(\log(1 + \langle h, x \rangle) - \langle h, x \rangle \Big) \nu_t(dx)$$

For all $t \in [0,T]$ and $\gamma > 0$, the function $F(t,\cdot)$ is strictly concave in h, as the sum of a linear function, a nonpositive quadratic form and a strictly concave function. Thus, being $F(t,\cdot)$ strictly concave on the convex bounded set H, there exists a unique solution $h^*(t) := \arg\max_{h \in H} F(t,h)$.

Remark 3.2. *As the function F does not explicitly depend on the capital v or on other state variables, we can deduce that the optimal strategy $h^*(t)$ consists in investing wealth proportions in each risky asset not depending on the current level of wealth, but only depending on the quantities $\mu(t)$, $\sigma(t)$ and on the measure ν_t. The optimal strategy thus results to be totally myopic, in the sense that it does not depend neither on any value of the state variables V or S^i, $i = 1, \ldots, n$, nor on the time to maturity $T - t$ (but we remark that this is quite typical of HARA utility functions with models driven by additive processes, see [1, 2, 4, 9, 16, 10, 11, 13] for some particular cases). In the time-homogeneous case, i.e. when $\mu(t) \equiv \mu$, $\sigma(t) \equiv \sigma$ and $\nu_t \equiv \nu$, the function F does not depend on time, so the optimal strategy h^* consists in investing wealth proportions in each risky asset which are constant in time.*

Define now $\lambda(t) := F(t, h^*(t))$; then the HJB equation becomes

$$0 = \phi'(t) + \lambda(t)$$

with terminal condition $\phi(T) = 0$, and we have $\phi(t) = \int_t^T \lambda(u) du$, hence a candidate value function is

$$J(t,v) = U\left(v e^{\int_t^T \lambda(u) du} \right)$$

and a candidate Markov optimal control is $h_t \equiv h^*(t)$.

In order to apply the verification theorem, we only need to prove that $J \in \mathcal{D}$. For all $\bar{t} \in [0,T]$, $h \in \mathcal{A}[\bar{t},T]$, by applying the Itô formula we have

$$dJ(t, V_t^h) = A^h J(t, V_t^h)\, dt + dM_t$$

with

$$\begin{aligned} dM_t &= e^{(1-\gamma)\int_t^T \lambda(u) du} (V_t^h)^{1-\gamma} \langle h_t, \sigma(t)\, dW_t \rangle + \\ &\quad + e^{(1-\gamma)\int_t^T \lambda(u) du} (V_{t-}^h)^{1-\gamma} \int_{\mathbb{R}^n} [(1 + \langle h_t, x \rangle)^{1-\gamma} - 1](N(dt,dx) - \nu_t(dx)) \end{aligned}$$

for the case $\gamma \neq 1$, and

$$dM_t = \langle h_t, \sigma(t)\ dW_t\rangle + \int_{\mathbb{R}^n} \log(1 + \langle h_t, x\rangle)(N(dt, dx) - \nu_t(dx))$$

for the case $\gamma = 1$. Now, the Dynkyn formula holds if M is a martingale: in order to prove this, it is sufficient to check that, for all $\gamma > 0$,

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{\bar{t}}^{T} e^{(1-\gamma)\int_t^T \lambda(u)du}(V_t^h)^{1-\gamma}\langle h_t, \sigma(t)\ dW_t\rangle\right)^2\right] = \\
&= \quad \mathbb{E}\left[\int_{\bar{t}}^{T} e^{2(1-\gamma)\int_t^T \lambda(u)du}(V_t^h)^{2(1-\gamma)}\|\sigma^T(t)h_t\|^2\ dt\right] \leq \\
&\leq \quad \sup_{h\in H}\|h\|_n^2 \int_{\bar{t}}^{T} e^{2(1-\gamma)\int_t^T \lambda(u)du}\mathbb{E}\left[(V_t^h)^{2(1-\gamma)}\right]\|\sigma^T(t)\|^2\ dt \leq \\
&\leq \quad \sup_{h\in H}\|h\|_n^2 \int_{\bar{t}}^{T} e^{2(1-\gamma)\int_t^T \lambda(u)du} C\|\sigma^T(t)\|^2\ dt < +\infty
\end{aligned}$$

where C is the constant of Lemma 2.5, and an analogous condition for the pure jump stochastic integral. To this concern, we know from the proof of Lemma 2.5 that $1 + \langle h_t, x\rangle \geq \delta$ for a suitable $\delta > 0$. Now, for the case $\gamma \neq 1$, the function $x \to (1 + \langle h_t, x\rangle)^{1-\gamma} - 1$ is bounded from below and with linear growth, so it is possible to find a constant D such that $|1 + \langle h_t, x\rangle)^{1-\gamma} - 1| \leq D\sup_{h\in H}\|h\|_n^2\|x\|_n$, thus we have

$$\begin{aligned}
&\mathbb{E}\left[\int_{\bar{t}}^{T}\int_{\mathbb{R}^n}\left|e^{(1-\gamma)\int_t^T \lambda(u)du}(V_{t-}^h)^{1-\gamma}[(1 + \langle h_t, x\rangle)^{1-\gamma} - 1]\right|\ \nu_t(dx)\ dt\right] \leq \\
&\leq \quad \int_{\bar{t}}^{T} e^{(1-\gamma)\int_t^T \lambda(u)du}\mathbb{E}\left[|(V_{t-}^h)^{1-\gamma}|\right]\int_{\mathbb{R}^n}|(1 + \langle h_t, x\rangle)^{1-\gamma} - 1|\ \nu_t(dx)\ dt \leq \\
&\leq \quad \int_{\bar{t}}^{T} e^{(1-\gamma)\int_t^T \lambda(u)du}\mathbb{E}\left[|(V_{t-}^h)^{1-\gamma}|\right] D\sup_{h\in H}\|h\|_n^2\int_{\mathbb{R}^n}\|x\|_n\ \nu_t(dx)\ dt \leq \\
&\leq \quad CD\sup_{h\in H}\|h\|_n^2\int_{\bar{t}}^{T} e^{(1-\gamma)\int_t^T \lambda(u)du}\int_{\mathbb{R}^n}\|x\|_n\ \nu_t(dx)\ dt < +\infty
\end{aligned}$$

where C is the constant of Lemma 2.5. Thus we can conclude that, for the case $\gamma \neq 1$, M is a martingale. For the case $\gamma = 1$, we notice that also the function $x \to \log(1 + \langle h_t, x\rangle)$ is bounded from below and with linear growth, so it is possible to find a constant D such that $|\log(1+\langle h_t, x\rangle)| \leq D\sup_{h\in H}\|h\|_n^2\|x\|_n$, thus we have

$$\begin{aligned}\mathbb{E}\left[\int_{\bar t}^{T}\int_{\mathbb{R}^n}|\log(1+\langle h_t,x\rangle)|\ \nu_t(dx)\ dt\right]\leq\\ \leq\ &\int_{\bar t}^{T} D\sup_{h\in H}\|h\|_n^2\int_{\mathbb{R}^n}\|x\|_n\ \nu_t(dx)\ dt\leq\\ \leq\ &D\sup_{h\in H}\|h\|_n^2\int_{\bar t}^{T}\int_{\mathbb{R}^n}\|x\|_n\ \nu_t(dx)\ dt<+\infty\end{aligned}$$

so M is a martingale also in the case $\gamma = 1$. We conclude that it is possible to apply the Dynkyn formula on J, so $J \in \mathcal{D}$ and we can apply the verification theorem.

Remark 3.3. *In the time-homogeneous case, under the assumption that the first-order condition (15) below holds, we find the same optimal portfolio as in [11] and in [13]: in this latter paper (which however treated only the logarithmic case), the portfolio was called the numeraire portfolio and the jump part of the Levy processes was given by a compound Poisson process, i.e. the Levy measure* ν *was of the form* $\nu(x) := \lambda\mu_Z(x)$, *with* μ_Z *being the probability distribution function of the n-dimensional jump and* λ *the intensity of the Poisson process counting the jumps.*

3.2 Analysis of the solution

As to now, we only obtained an existence result for the optimal strategy h^* in terms of the maximisation of a convex function $F(t, h)$ over the compact and convex set H. It is well known that this maximisation problem can have an internal solution or a solution on the boundary of H. In order to obtain more analytic results (and in line with most of the literature on this topic, see e.g. [1, 2, 4, 9, 10, 11, 13]), we now assume to have an internal solution: as $F(t, \cdot) \in C^1(H)$ for all $t \in [0, T]$, this corresponds to say that, for all $t \in [0, T]$, $h^*(t)$ satisfies the first-order conditions

$$0=\mu_i(t)-\gamma\sum_{j=1}^{n}a_{ij}(t)h_j+\int_{\mathbb{R}}\left(\frac{x_i}{1+\langle h,x\rangle}-x_i\right)\nu_t(dx)\quad\forall i=1,\dots,n.\tag{14}$$

which can be written in a more compact vectorial notation as

$$0=\mu(t)-\gamma\Sigma(t)h+\int_{\mathbb{R}^n}x((1+\langle h,x\rangle)^{-\gamma}-1)\ \nu_t(dx)\tag{15}$$

These conditions usually do not admit an explicit solution, but a numerical solution can be easily found.

Remark 3.4. *The fact that the first-order condition (15) has a solution* $h^*(t) \in H$ *has to be verified case by case (see the first three examples in Section 4 for* *numerical cases when this does not happen). In general, even proving that for all $t \in [0,T]$ the first-order condition (15) has a solution in the natural domain H_t, possibly outside of H, does not seem an easy task: we in fact notice that even in the most general time-homogeneous case in [11] the existence of an optimal h^* satisfying the first-order conditions is assumed and not proved. Despite this open situation in the general case, in the particular case of a compensating measure with finite support the existence of an optimal $h^*(t) \in H$ can be proved for a suitable choice of H: see for example [13] for the logarithmic case and [4] for the general case.*

Now we compare our solution with the solution in the case when there are no jumps, i.e. $\nu_t \equiv 0$ for all $t \in [0,T]$, the other parameters $\mu(\cdot)$ and $\sigma(\cdot)$ being unchanged. It is well known that, if the matrices $\Sigma(t)$ are positive definite for all $t \in [0,T]$ (i.e., $d \geq n$ and the $\sigma(t)$, $t \in [0,T]$ have all full rank n), then the optimal portfolio without jumps is given by

$$h_t^c := \frac{1}{\gamma}\Sigma^{-1}(t)\mu(t) \tag{16}$$

If the optimal portfolio proportions h^* satisfy the first-order condition (15) and the matrices $\Sigma(t)$ are positive definite for all $t \in [0,T]$, then we find out that the optimal portfolio in the presence of jumps is equal to the optimal portfolio without jumps but with the assets having a different yield. In fact, by the non-degeneracy of $\Sigma(t)$ the condition (15) can be written as

$$h_t^* = \frac{1}{\gamma}\Sigma^{-1}(t)(\mu(t) - \mu_t^J(h_t^*)) = h_t^c - \frac{1}{\gamma}\Sigma^{-1}(t)\mu_t^J(h_t^*) \tag{17}$$

where, for all $h \in H$, $t \in [0,T]$, $\mu_t^J(h)$ is the vector defined by

$$\mu_t^J(h) := \int_{\mathbb{R}^n} x(1 - (1 + \langle h, x\rangle)^{-\gamma})\ \nu_t(dx)$$

Thus, the optimal portfolio h_t^* can be represented as the algebraic sum of the optimal portfolio without jumps

$$h_t^c := \frac{1}{\gamma}\Sigma^{-1}(t)\mu(t) \tag{18}$$

and the term $\frac{1}{\gamma}\Sigma^{-1}(t)\mu_t^J(h_t^*)$, where $\mu_t^J(h_t^*)$ and can be interpreted as a "jump dividend", i.e. a term which subtracts (if positive) something from the yield of the risky assets. In the n-dimensional case, however, one cannot say in general if $\mu_t^J(h_t^*)$ has all positive components or not, and even in this case one has to take into account the fact that $\Sigma(t)$ can possibly be non-diagonal and transform positive vectors in vectors with some negative components, and one has to assess the sign of the difference between the optimal portfolio without jumps and the optimal portfolio with jumps (i.e., the term $\frac{1}{\gamma}\Sigma^{-1}(t)\mu_t^J(h_t^*)$) case by case.

The situation is different when $n = 1$: in this case we generalise a result of [3, 9] and find out that the fraction of optimal portfolio invested in the risky asset in the presence of jumps is always less in absolute value than the corresponding fraction without jumps, and always with the same sign.

Lemma 3.5. *If $n = 1$ and h^* satisfies the first-order condition (15) with $\sigma(t) > 0$ for all $t \in [0,T]$, then for all $t \in [0,T]$ one of the two following chains of inequalities holds:*

$$0 \le h_t^* \le \frac{\mu(t)}{\gamma\sigma^2(t)} \qquad \text{or} \qquad \frac{\mu(t)}{\gamma\sigma^2(t)} \le h_t^* \le 0$$

Proof. In fact, in this case μ^J becomes

$$\mu_t^J(h) := \int_{\mathbb{R}} x(1 - (1 + hx)^{-\gamma})\ \nu_t(dx)$$

It is easy to see that $x(1 - (1 + \langle h, x\rangle)^{-\gamma})$ has ν_t-a.s. the same sign as h for all $t \in [0,T]$, $h \in H$. This means that $\mu_t^J(h_t^*) \ge 0$ if $h_t^* \ge 0$ and $\mu_t^J(h_t^*) \le 0$ if $h_t^* \le 0$. Now, write Equation (17) as

$$h_t^* + \frac{\mu_t^J(h_t^*)}{\gamma\sigma^2(t)} = \frac{\mu(t)}{\gamma\sigma^2(t)}$$

It is easy to see that all the addends in the left- and right-hand side must have the same sign, and this concludes the proof. □

We can give another interpretation of this result, interpretation which is also supported by strong numerical evidence, by making a first-order approximation of the function $x \to (1 + x)^{-\gamma}$ in Equation (15): in this way we write

$$0 = \mu(t) - \gamma\Sigma(t)\tilde{h} + \int_{\mathbb{R}^n} x(1 - \gamma\langle \tilde{h}, x\rangle + o(\gamma\langle \tilde{h}, x\rangle) - 1)\ \nu_t(dx)$$

By neglecting the term $o(\gamma\langle \tilde{h}, x\rangle)$, one arrives at

$$\mu(t) - \gamma\left[\sum_{j=1}^{n} a_{ij}(t)\tilde{h}_j + \sum_{j=1}^{n}\int_{\mathbb{R}^n} x_i x_j \tilde{h}_j \nu_t(dx)\right] = 0$$

By collecting the vector $\tilde{h}$, which now appears linearly, we obtain the approximation

$$\mu(t) - \gamma[\Sigma_t + C_t]\tilde{h} = 0 \tag{19}$$

where $C_t = (C_{ij}(t))_{ij}$ is the second moment matrix of the Levy measure ν_t, defined as

$$C_{ij}(t) := \int_{\mathbb{R}^n} x_i x_j \nu_t(dx)$$

and finally one can think to approximate the optimal portfolio proportions with

$$h_t^* \simeq \tilde{h}_t := \frac{1}{\gamma}[\Sigma_t + C_t]^{-1}\mu(t) \tag{20}$$

i.e., the optimal portfolio proportions, at the first order, are the same as the corresponding ones in the no-jump case when we substitute the volatility matrix Σ_t with the total covariance matrix $\Sigma_t + C_t$. In the one-dimensional case this has the effect of increasing the total variance, thus this approximation gives immediately the result of Lemma 3.5, a thing also remarked (without an explicative argument) in [9]. Of course the goodness of this first-order approximation depends on the particular numerical case that one has, but we remark that in the first three examples of Section 4 we found out that this approximation is rather good.

4 Examples

We now analyse three well-known models in dimension one, namely the Kou model, the Variance Gamma model and the CGMY model, and as the fourth example a pure jump multidimensional model proposed in [4]. All these four models are incidentally time-homogeneous, thus based on Levy processes. The first three models are usually presented in literature using the ordinary exponential notation presented in Remark 2.1, i.e. assuming that the price of the risky asset is of the form $S_t = e^{L_t}$, with L suitable Levy process. This allows to be able to put any Levy measure on L without restrictions (but does not allow to use stochastic calculus), while the formulation that we used with stochastic exponentials forced us to impose that jumps should not be less than or equal to -1 for the risky assets not to default. There is however a standard way to pass from one formulation to another, thanks to the following result [7, Proposition 8.22], which we partially rewrite here for the reader's convenience.

Proposition 4.1. *Let $(L_t)_{t\geq 0}$ a real Levy process with characteristic triplet $(\sigma_L^2, \nu_L, \gamma_L)$, and let $S_t := e^{L_t}$. Then we also have $S = \mathcal{E}(R)$, where R is the Levy process*

$$R_t = L_t + \frac{\sigma^2 t}{2} + \int_0^t \int_{\mathbb{R}} (1 + x - e^x) N_L(ds, dx)$$

with N_L jump measure of L. In particular, the characteristic triplet (σ^2, ν, γ) of R is

$$\begin{aligned} \sigma &= \sigma_L; \\ \nu(A) &= \nu_L\left(\{x | e^x - 1 \in A\}\right) = \int_{\mathbb{R}} 1_{\{e^x - 1 \in A\}} \nu_L(dx); \\ \gamma &= \gamma_L + \frac{\sigma_L^2}{2} + \int_{\mathbb{R}} \left((e^x - 1) - x\right) \nu_L(dx). \end{aligned}$$

In the first three examples that follow, jumps are not constrained, i.e. they can assume any value in $(-1, +\infty)$, thus we have the same situation as in Example 2.4, with $H_t \equiv [0, 1]$ for all $t \in [0, T]$. Hence, we can take H to be any closed interval strictly contained in $(0, 1)$ and search for an optimal solution $h^* \in H$ which solves Equation (15). Notice that, if we do not specify H a priori and find a solution $h^* \in (0, 1)$ of Equation (15), we can specify H such that $h^* \in H$ and apply the verification theorem. If Equation (15) has not a solution in $(0,1)$, then we have a solution on the boundary, so we indicate as the solution the point $h^* = \max H$ or $h^* = \min H$, according to the case. For each of the three examples, we take γ from 0.2 to 2 (thus, for $\gamma = 1$ we have the particular case of $U(v) = \log v$) and denote with h^* the optimal solution of the our general problem with jumps. We also indicate with h^c the solution in the case of no jumps in Equation (18), where applicable (i.e. when $\sigma > 0$), and with h^T the solution calculated with the second-order approximation in Equation (20), i.e. in a purely diffusive case but with local second moment equal to the total local second moment of the current Levy process (Brownian + jump parts). All these numerical examples have been calculated using Octave.

4.1 The Kou model

The Kou model [14] in ordinary exponential form (i.e. with $S_t = e^{L_t}$) is characterised by a Levy measure of the kind

$$\nu_L(dx) = \lambda(1-p)\eta_+ e^{-\eta_+ x} 1_{\{x>0\}}\, dx + \lambda p \eta_- e^{\eta_- x} 1_{\{x<0\}} dx$$

with $\eta_+ > 1$, $\eta_- > 0$ and $p \in [0,1]$. By using Proposition 4.1, we have $S_t = \mathcal{E}(R)_t$, where the characteristic triplet of R is given by

$$\begin{aligned}
\sigma &= \sigma_L, \\
\nu(dx) &= \lambda(1-p)\eta_+(1+x)^{-\eta_+-1} 1_{\{x>0\}}\, dx + \lambda p \eta_- (1+x)^{\eta_- -1} 1_{\{-1<x<0\}} dx, \\
\mu &= \mu_L + \frac{\sigma_L^2}{2} + \frac{\lambda(1-p)}{(1-\eta_+)\eta_+} + \frac{\lambda p}{(1+\eta_-)\eta_-}.
\end{aligned}$$

The total second moment of R is then given by

$$\begin{aligned}
\mathrm{Var}[R_t] &= \sigma^2 t + t \int_{-1}^{+\infty} x^2 \nu(dx) = \\
&= \sigma^2 t + \frac{2\lambda p}{(\eta_- + 1)(\eta_- + 2)} t + \frac{2\lambda(1-p)}{(\eta_+ - 1)(\eta_+ - 2)} t
\end{aligned}$$

Example 4.2. *Consider, as in [14], the parameters $\eta_+ = 10$, $\eta_- = 5$, $\lambda = 1$, $p = 0.4$, $\sigma_L = 0.16$, and assume also that $\mu_L = 0$: this gives us $\mu = 0.0328$. We now plug these values in the first-order condition in Equation (15) for different values of γ from 0.2 to 2 and compare the resulting h^* with the optimal portfolio in the purely diffusive case, first simply with $\nu \equiv 0$ (call this optimal portfolio h^c) and then with the same second moment (call this h^T). We obtain the results in the following table.*

γ	h^*	h^c	h^T
0.2	$\max H$	*6.40625*	*2.67474*
0.4	$\max H$	*3.20313*	*1.33737*
0.6	*0.83690*	*2.13542*	*0.89158*
0.8	*0.64066*	*1.60156*	*0.66869*
1	*0.51773*	*1.28125*	*0.53495*
1.2	*0.43401*	*1.06771*	*0.44579*
1.4	*0.37345*	*0.91518*	*0.38211*
1.6	*0.32765*	*0.80078*	*0.33434*
1.8	*0.29183*	*0.71181*	*0.29719*
2	*0.26304*	*0.64063*	*0.26747*

As we can see, in the first two rows (which however correspond to a very low risk aversion) the optimal h^* is equal to the maximum over H: this means that the first order condition (15) does not hold, but still we can apply the verification theorem. In all the other cases, the optimal h^* in the presence of jumps is a reasonable value between 0 and 1. These values are all less than the analogous proportions without jumps, presented in the third column, and this confirms numerically Lemma 3.4. Notice that, if we had assumed $\mu_L = 0.04$, then also in the logarithmic case (i.e when $\gamma = 1$) we would have obtained $h^* = \max H$. Finally, it is worth noticing that the variance of the driving process seems to explain almost entirely the optimal portfolios, at least for risk-aversion coefficients γ which are not too low: in fact, columns 2 and 4 are obtained with the driving process of the risky asset having the same local variance but different paths (diffusion + jumps in the first case, pure diffusion in the second case) and the resulting optimal portfolios come nearer and nearer as γ increases: thus the approximation in Equation (20) is numerically rather good in this case for $\gamma > 1$.

4.2 The Variance Gamma model

The Variance Gamma model [5] in ordinary exponential form (i.e. with $S_t = e^{L_t}$) is characterised by a Levy measure of the kind

$$\nu_L(dx) = \frac{ce^{-\lambda_+ x}}{x} 1_{\{x>0\}} dx + \frac{ce^{-\lambda_- |x|}}{|x|} 1_{\{x<0\}} dx$$

with $\lambda_+ > 1$ and $\lambda_-, c > 0$. Its most common interpretation is that L is a Brownian motion with drift subordinated to a Gamma process with unit mean rate (see [5] for details). If θ is the drift of the Brownian motion, σ its standard deviation and the Gamma process has variance ν, then the coefficients c, λ_+ and λ_- are given by

$$c = \frac{1}{\nu}, \qquad \lambda_\pm = \sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}} \mp \frac{\theta}{\sigma^2}$$

and we have that $\mathbb{E}[L_t] = \theta t$, $\mathrm{Var}[L_t] = (\theta^2\nu + \sigma^2)t = c(\lambda_+^{-2} + \lambda_-^{-2})t$. Since this process has infinite variation, its use in literature is alternative to the classic diffusion models, without a further Brownian component. We can however make a comparison between this model and a geometric Brownian motion with the same first two moments.

By using Proposition 4.1, we can rewrite $S_t = \mathcal{E}(R)_t$, where the characteristic triplet of R this time is given by

$$\begin{aligned}
\sigma &= \sigma_L, \\
\nu(dx) &= \frac{c(1+x)^{-\lambda_+-1}}{\log(1+x)} 1_{\{x>0\}} dx - \frac{c(1+x)^{\lambda_- -1}}{\log(1+x)} 1_{\{-1<x<0\}} dx, \\
\mu &= \mu_L + \frac{\sigma_L^2}{2} + c \log\left(\frac{\lambda_+}{\lambda_+ - 1}\right) - \frac{c}{\lambda_+} + c\log\left(\frac{\lambda_-}{\lambda_- + 1}\right) + \frac{c}{\lambda_-}.
\end{aligned}$$

In this case, the total second moment of R_t is now equal to

$$\mathrm{Var}[R_t] = t \int_{-1}^{+\infty} x^2 \nu(dx) = c(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2})t$$

Example 4.3. *Consider, as in [5], the parameters $\sigma = 0.1213$, $\nu = 0.1686$ and $\theta = -0.1436$, which as a consequence give, in our parameterisation, $\lambda_+ = 39.78$, $\lambda_- = 20.26$ and $c = 5.93$, and assume also that $\mu_L = 0.005$. As before, we plug these values in the first-order condition in Equation (15) for different values of γ from 0.2 to 2 and compare the resulting h^* only with the optimal portfolio h^T in a purely diffusive case calculated with analogous first two moments. We obtain the results in the following table.*

γ	h^*	h^T
0.2	$\max H$	*3.81986*
0.4	$\max H$	*1.90993*
0.6	$\max H$	*1.27329*
0.8	*0.96305*	*0.95496*
1	*0.77569*	*0.76397*
1.2	*0.64927*	*0.63664*
1.4	*0.55825*	*0.54569*
1.6	*0.48960*	*0.47748*
1.8	*0.43597*	*0.42443*
2	*0.39293*	*0.38199*

As we can see, in the first three cases (which again correspond to very low risk aversions) the optimal h^* is equal to the maximum over H, i.e., the first order condition (15) does not hold, but still we can apply the verification theorem. In all the other cases, the optimal h^* in the presence of jumps is a reasonable value between 0 and 1. Notice that, if we had assumed $\mu_L = 0.01$, then also in the logarithmic case (i.e when $\gamma = 1$) we would have obtained $h^* = \max H$. Finally, by comparing columns 2 and 3, we notice that also in this case the variance seems to explain almost entirely the optimal portfolios: in these two cases, the driving process of the risky asset would have the same local variance but very different paths (discontinuous Levy process in the first case, pure diffusion in *the second case), but the resulting optimal portfolios are all equal up to the 2nd significant digit (provided that they are in the interval* $[0,1]$*). Thus, also in this case the approximation in Equation (20) is numerically rather good, this time without apparent limitations on* γ*.*

4.3 The tempered stable, or CGMY, model

The tempered stable model, also called Carr-Geman-Madan-Yor (CGMY) from the authors who first proposed it in [6], is a generalisation of the Variance Gamma model, and in ordinary exponential form (i.e. with $S_t = e^{L_t}$) is characterised by a Levy measure of the kind

$$\nu_L(dx) = \frac{ce^{-\lambda_+ x}}{x^{\alpha+1}} 1_{\{x>0\}} dx + \frac{ce^{-\lambda_- |x|}}{|x|^{\alpha+1}} 1_{\{x<0\}}$$

with $\lambda_+ > 1$, $\lambda_-, c > 0$ and $0 \leq \alpha < 1$. In fact, if we put $\alpha = 0$, then we are led back to a Variance Gamma model. By using Proposition 4.1, we have $S_t = \mathcal{E}(R)_t$, where the characteristic triplet of R this time is given by

$$\begin{aligned}
\sigma &= \sigma_L, \\
\nu(dx) &= \frac{c(1+x)^{-\lambda_+ - 1}}{(\log(1+x))^{\alpha+1}} 1_{\{x>0\}} dx + \frac{c(1+x)^{\lambda_- - 1}}{(-\log(1+x))^{\alpha+1}} 1_{\{-1<x<0\}} dx, \\
\mu &= \mu_L + \frac{\sigma_L^2}{2} - \frac{c}{\alpha} \Gamma(1-\alpha) \times \\
&\quad \times \left((\lambda_+ - 1)^\alpha + \alpha \lambda_+^{\alpha-1} - \lambda_+^\alpha + (\lambda_- + 1)^\alpha - \alpha \lambda_-^{\alpha-1} - \lambda_-^\alpha \right)
\end{aligned}$$

Even if also the CGMY process is typically a process with infinite activity, in [6] the authors propose to consider an "extended" CGMY process, i.e. to allow also the presence of a Brownian motion with drift in the driving Levy process. In this case, the total variance of R_t is now equal to

$$\mathrm{Var}[R_t] = \sigma^2 t + t \int_{-1}^{+\infty} x^2 \nu(dx) = \sigma^2 t + c\Gamma(2-\alpha)(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}) t$$

thus, depending on the size of the two addends, the jump component can explain much of the variance, besides taking care of the deviation from Gaussianity of all the higher moments.

Example 4.4. *Consider, as in [6, p. 322, HWP asset], the parameters* $\lambda_+ = 31.72$*,* $\lambda_- = 32.36$*,* $c = 25.72$*,* $\alpha = 0.0931$ *and* $\sigma_L = 0.0981$*, and assume also that* $\mu_L = 0.02$*. As before, we plug these values in the first-order condition in Equation (15) for different values of* γ *from 0.2 to 2 and compare the resulting* h^* *with the optimal portfolio in a purely diffusive case, first when we take as local variance only* σ^2 *(call this* h^c*), then when we take as local variance* $\sigma^2 + c\Gamma(2-\alpha)(\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2})$*, i.e. the same variance that we would have including the jump component (call this* h^T*). We obtain the results in the following table.*

γ	h^*	h^c	h^T
0.2	$\max H$	*25.04988*	*3.15704*
0.4	$\max H$	*12.52494*	*1.57852*
0.6	$\max H$	*8.34996*	*1.05235*
0.8	*0.78876*	*6.26247*	*0.78926*
1	*0.63095*	*5.00998*	*0.63141*
1.2	*0.52574*	*4.17498*	*0.52617*
1.4	*0.45059*	*3.57855*	*0.45101*
1.6	*0.39423*	*3.13123*	*0.39463*
1.8	*0.35041*	*2.78332*	*0.35078*
2	*0.31535*	*2.50499*	*0.31570*

As we can see, in the first three cases (which again correspond to very low risk aversions) the optimal h^* is equal to the maximum over H, i.e., the first order condition (15) does not hold, but still we can apply the verification theorem. In all the other cases, the optimal h^* in the presence of jumps is a reasonable value between 0 and 1. These values are all again less than the analogous proportions without jumps in the third column. Notice that, if we had assumed $\mu_L = 0.04$, then also in the logarithmic case (i.e when $\gamma = 1$) we would have obtained $h^* = \max H$. Again, by comparing columns 2 and 4, we notice that again the variance of the driving process seems to explain almost entirely the optimal portfolios: here the driving process of the risky asset have again the same local variance but different paths (diffusion + discontinuous Levy process in the first case, pure diffusion in the second case) and the resulting optimal portfolios are all equal up to the 3rd significant digit (provided that they are in the interval $[0,1]$), so even closer than in the case of the Variance Gamma model. Once again in this case we have a confirmation of the goodness of the approximation in Equation (20).

4.4 A simple multidimensional model

Finally, we present a multidimensional model, taken from [4], where the risky assets evolve with dynamics which presents only jumps: this can be a suitable model for market microstructure, which can also take into account nontrivial correlation structures between the risky assets. In this model, it is assumed that the dynamics of the discounted prices of the risky assets are

$$dS_t^i = S_{t-}^i \left[-r dt + \sum_{j=1}^{k} c_{ij} dN_t^j \right]$$

where N^j, $j = 1, \dots, k$ are independent Poisson processes with intensities λ_j, $j = 1, \dots, k$, respectively, r is the risk-free interest rate and the matrix $(c_{ij})_{i=1,\dots,n, j=1,\dots,k}$ has maximum rank. This can be written in our framework by choosing the Levy measure of the kind $\nu := \sum_{j=1}^k \lambda^j \delta_{c^j}$, with $c^j := (c_1^j, \dots, c_n^j) \in X$, where δ_x is the Dirac delta centered in x, i.e. the measure such that $\delta_x(B) = \mathbf{1}_B(x)$. This corresponds to $N(dx, dt)$ being the random Poisson measure corresponding to a multivariate Poisson process.

In this case, for all $\gamma > 0$ the first order conditions read

$$
\begin{aligned}
0 &= -r + \int_{\mathbb{R}^n} \left[\Big(1 + \langle h, x \rangle \Big)^{-\gamma} x_i - x_i \right] \nu(dx) = \\
&= -r + \sum_{i=1}^{m} \lambda_j c_i^j \left[\Big(1 + \langle h, c^j \rangle \Big)^{-\gamma} - 1 \right] \quad \forall i = 1, \ldots, n
\end{aligned}
$$

both for the log-case ($\gamma = 1$) as for the power case ($\gamma \neq 1$). These conditions can be shown to have a unique solution $h^* \in \mathrm{int}(H)$, and are equivalent to the conditions already present in [4].

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