Optimal prepayment and default rules for mortgage-backed securities

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October 14, 2009

Abstract

We study the optimal stopping problems embedded in a typical mortgage. Despite a possible non-rational behaviour of the typical borrower of a mortgage, such problems are worth to be solved for the lender to hedge against the prepayment risk, and because many mortgage-backed securities pricing models incorporate this suboptimality via a so-called prepayment function which can depend, at time t, on whether the prepayment is optimal or not. We state the prepayment problem in the context of the optimal stopping theory and present an algorithm to solve the problem via weak convergence of computationally simple trees. Numerical results in the case of the Vasicek model and of the CIR model are also presented. The procedure is extended to the case when both the prepayment as well as the default are possible: in this case, we present a new method of building two-dimensional computationally simple trees and we apply it to the optimal stopping problem.

1 Introduction

The aim of this paper is to study the optimal stopping problems corresponding to the prepayment and default options embedded in a mortgage and their role in pricing mortgage-backed securities (MBS). Although the average borrower of a mortgage

*corresponding author. This work was partly supported by the University of Padova under grant CPDA082105/08.

may not exercise these options in an optimal time for many reasons (among which its non-rationality due to him/her being typically not a professional of financial markets), such problems are worth to be solved for two main reasons. The first is that the lender of a mortgage wants to take into account the risk that many borrowers, though being non-optimal decisors, could exercise their options at the optimal time. The second is that, due to this non-optimality of the exercising times, many pricing models for MBS incorporate this non-optimality via a so-called prepayment function (an analogous of the default intensity in credit risk models), which can explicitly depend on the fact that at time t the prepayment is an optimal decision or not.

We now present the problem in a more formal way. First of all we focus on the typical structure of a single mortgage with its own prepayment option, and to fix the ideas we concentrate on the most simple fixed rate, fixed termination date mortgage (although there are many other kinds of mortgage in the market). Then we present the most common MBS traded in the financial markets, and in particular a class of MBS which are guaranteed against the risk of default of the borrower, so that the only relevant risk is the one relative to prepayment; after this, we present the most common framework of pricing a MBS by use of the prepayment function and how the optimal stopping problem may become relevant. Finally, we extend this approach to the non-guaranteed case, i.e. when a default is also possible.

1.1 Single mortgage

We concentrate on fixed rate, fixed termination date mortgages on an underlying good, which can be a house or a more general physical good. Assume that one (usually called the *borrower*) borrows at time 0 from a so-called *lender* (usually a bank) a capital equal to P at the nominal instantaneous rate $\rho > 0$, and pays it back with a continuous intensity A in the time window $[0, T]$. Then it is easy to see that these quantities are linked by

$$P = \int_0^T e^{-\rho t} A \, dt = A \frac{1 - e^{-\rho T}}{\rho}$$

If we instead assume that the borrower pays the capital back with N equal rates A_N at dates kT/N, $k = 1, \dots, N$ at the nominal rate ρ_N prevailing for each small period $[kT/N, (k+1)T/N]$, then these quantities are linked by

$$P = \sum_{i=1}^{N} \frac{A_N}{(1+\rho_N)^i} = \frac{A_N}{\rho_N} \left(1 - \frac{1}{(1+\rho_N)^N} \right).$$

In both of these payment methods, the borrower pays a total (non-discounted) amount which is greater than P, the difference being related to the fact that the borrower must also pay an interest for his (her) residual debt. With respect to this, it is possible to decompose the (continuous or discrete) rate A (or A_N) in *principal quote* and *interest quote*.

In many mortgages, the borrower has the option to prepay the mortgage at a date $t < T$. The usual convention is that in this case the residual debt (s)he has to repay is given by the remaining rates discounted by the nominal rate. If the rates are paid in continuous time, then the residual debt to be repaid at time t is thus given by

$$F_t := \int_t^T e^{-\rho(u-t)} A \, du = A \frac{1 - e^{-\rho(T-t)}}{\rho} \tag{1}$$

while if the rates are paid in discrete time at time nT/N with $n < N$, then

$$F_n := \sum_{i=n+1}^{N} \frac{A_N}{(1+\rho_N)^{i-n}} = \frac{A_N}{\rho_N} \left(1 - \frac{1}{(1+\rho_N)^{N-n}} \right).$$

Once this option is exercised, the contract terminates. This prepayment option has thus the character of a contingent claim of American type.

From the point of view of the lender, if the borrower exercises his/her prepayment option at time t (in discrete time, nT/N), this means that the lender receives immediately the lump sum F_t (resp. F_n) instead of the future stream of payments with intensity A (resp. A_N), which in continuous time has a market value at time t equal to

$$V_t := \int_t^T B(t,u) A \, du = A \int_t^T B(t,u) \, du \tag{2}$$

while in discrete time its market value at time nT/N is

$$V_n := \sum_{i=n+1}^{N} A_N B\left(n\frac{T}{N}, i\frac{T}{N} \right) = A_N \sum_{i=n+1}^{N} B\left(n\frac{T}{N}, i\frac{T}{N} \right)$$

where in both cases we denote with $B(t,s)$ the price at time t of a zero-coupon bond with maturity $s \geq t$. Thus, the lender is exposed to the risk of early exercise at time $t < T$ of an American option to exchange V_t for F_t. While we have seen that the value of F_t is conventionally fixed and depends only on t and on the deterministic quantities A and ρ, the value of V_t depends on the evolution of the term structure given by $(B(t,s))_{s \in [t,T]}$. The optimal exercise of the prepayment option can thus be triggered by market conditions, usually interest rates falling under a certain level.

Usually, the borrower has another option, called surrender option or default option, which consists in forfeiting the whole contract in exchange for the physical good underlying the mortgage. If the borrower exercises this option, then the lender will no longer receive any stream of payment, but has the right to retain all the payments previously done in addition to the underlying physical good. The options of prepayment and of surrendering are alternative to each other, in the sense that once that the borrower exercises one of the two, the contracts expires and (s)he cannot exercise the other.

1.2 Mortgage-backed securities

Mortgage-backed securities are derivative assets based on the cash flows generated by packages of mortgages (for this brief introduction we follow [11]). The issuer usually aggregates a number of mortgages with approximately equal nominal rate and equal maturity, thus creating a pool of mortgages which is the underlying asset of a MBS. There are different kind of MBS: some examples are

- *pass-through*: the MBS holder receives a fixed fraction of the whole cash flow generated by the mortgage pool;
- *collateralized mortgage obligations (CMOs)*: they work similarly to pass-through, except for the fact that in a CMO there are different tranches, each one with a different priority in receiving the cash flow;
- *stripped interest-only MBS (IOs)*: the MBS holder receives a fixed fraction only of the interest quote generated by the mortgage pool;
- *stripped principal-only MBS (POs)*: the MBS holder receives a fixed fraction only of the principal quote generated by the mortgage pool.

This means that the stream of payments depends on the prepayments of the mortgage which constitutes the pool, this dependence becoming much more dramatic in the latter cases: in fact, in case of a prepayment, an IO-holder no longer receive interest for that single mortgage, while a PO-holder receives the principal of that mortgage before the natural maturity, and a CMO-holder's future cash flow is linked to the priority of his/her CMO. Conversely, for a pass-through holder a prepayment simply means a swap between a lump sum immediately and a fixed stream of payments in the future, which usually have similar market values.

A particular class of MBS consists in those guaranteed by federal and/or US government associations, well known examples being FNMA, FHLMC and GNMA (Federal National Mortgage Association, Federal Home Loan Mortgage Corporation, and Government National Mortgage Association, respectively). These MBS are insured against default in this way: if a mortgage is defaulted by the borrower before its natural maturity, then the corresponding association (FNMA, FHLMC or GNMA) replaces that mortgage in the pool by a lump sum corresponding to a prepayment (and takes in exchange the underlying good), thus bearing the risk of default. For this reason, in order to evaluate a guaranteed MBS, it is not required to model the risk of default (as this is entirely covered by the corresponding association), and the only sources of risk to be modeled are the interest rates dynamics and the prepayment risk. Conversely, if a MBS is not of this kind, the modelling of the default risk is also required: the relevance of this latter problem has been emphasised by the recent crisis of the two associations FNMA and FHLMC in September 2008.

In the sequel, we first concentrate on the problem of pricing a guaranteed MBS. Then we extend our method also to the pricing of non-guaranteed MBS.

1.3 The pricing of guaranteed MBS

Being this the framework, it seems that in order to price a guaranteed MBS one has simply to price, with the aid of the usual no-arbitrage theory, the corresponding stream of payments, which always include an American-style option corresponding to the prepayment option of each borrower. Thus, for a mortgage having rate A and a nominal interest rate ρ and maturity T, the price of a pass-through MBS at time t would simply be

$$V_t - \operatorname*{ess\,sup}_{\tau\in[t,T]} \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^{\tau} r_u\, du}(V_\tau - F_\tau) \mid \mathcal{F}_t] = V_t - \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^{\hat{\tau}} r_u\, du}(V_{\hat{\tau}} - F_{\hat{\tau}}) \mid \mathcal{F}_t] \quad (3)$$

where F and V are respectively given by Equations (1) and (2), $(\mathcal{F}_t)_t$ is a suitable filtration representing the amount of information up to time t (which could be typically the filtration generated by the short rate r), the ess sup is taken over all the $(\mathcal{F}_t)_t$-stopping times τ taking values in the interval $[t, T]$, and $\mathbb{Q}$ is a suitable equivalent martingale measure. In particular, it would happen that every borrower of a given pool finds optimal to prepay at the same stopping time $\hat{\tau}$.

The above argument would be valid provided all the agents in the market, especially the borrowers who control the prepayment options, would act in a rational way. The fact is that in this case, while the lenders are typically professionals of the financial world (banks, insurance companies and so on), generally fully informed about the financial markets, borrowers can be or are typically not acquainted with rational economical reasoning and/or not fully informed about the financial markets; even if they are, there could be other non-optimal reasons for exercising their prepayment options at a non-optimal time (typical reasons could be that the house which is the underlying of the mortgage has been sold, or that the borrower becomes aware of the optimality of prepayment with some delay with respect to the optimal time). Thus, in a given pool, one can observe different prepayment times, optimal as well as non-optimal, for different borrowers.

In order to price a MBS one has thus to model this non-optimal prepayment behaviour and incorporate it in the MBS price. This is usually done by using the so-called *prepayment function.* Let us call τ the (optimal or non-optimal) prepayment time of a "typical" single borrower: this is a $(\mathcal{F}_t)_t$-stopping time, thus a random variable, taking values in the set $[0, T]$; call $F(\cdot|\theta)$ its cumulative distribution function with respect to $\mathbb{Q}$, conditional on a state variable θ, defined by $F(t|\theta) := \mathbb{Q}\{\tau \le t \mid \theta\}$; assume that τ has also a conditional density $f(\cdot|\theta) = F'(\cdot|\theta)$. Then the *prepayment function* (also known as *hazard function* or *risk function* in other areas, and which is the exact analogous of the *default intensity* in credit risk) is defined as

$$\pi(t|\theta) := \lim_{\Delta t\to 0^+} \frac{\mathbb{Q}\{t < \tau \le t + \Delta t \mid \tau > t, \theta\}}{\Delta t} = \frac{f(t|\theta)}{1 - F(t|\theta)}$$

which gives the density of prepayment at time t conditioned on the fact that the borrower has not yet prepaid (and to the state variable θ). It is well known that this is equivalent to saying that $F(t|\theta) = 1 - \exp(-\int_0^t \pi(s|\theta)\, ds)$ or that $f(t|\theta) = \pi(t|\theta)\exp(-\int_0^t \pi(s|\theta)\, ds)$.

Once we specify the hazard function $\pi(\cdot|\theta)$ and the interest rates dynamics, we can obtain the price of a MBS by incorporating $\pi(\cdot|\theta)$ into the pricing formula in a way similar to credit risk. In particular, for a mortgage pool having nominal interest rate ρ and maturity T, the price of a pass-through MBS at time t would be

$$
\begin{aligned}
& V_t - \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^\tau r_u\, du}(V_\tau - F_\tau) \mid \mathcal{F}_t] = \\
= \;& V_t - \mathbb{E}_{\mathbb{Q}}\left[\int_t^T \pi(u|\theta) e^{-\int_t^u (r_v + \pi(v|\theta))dv} (V_u - F_u)\, du \;\middle|\; \mathcal{F}_t \right]
\end{aligned}
$$

where, this time, τ is a prepayment time whose law is characterised by the prepayment function $\pi(\cdot|\theta)$. Notice that this formula is similar to a typical pricing formula for a defaultable security where the recovery value $V_\tau - F_\tau$ is paid at default [16].

In many model specifications, the prepayment function π depends on whether at a given time t the prepayment is or not optimal with respect to the criterion of Equation (3) (a notable exception being [13], where a different criterion is used): if for all $t \in [0, T]$ we define

$$
\theta_t := \begin{cases} 1 & \text{if at time } t \text{ it is optimal to prepay,} \\ 0 & \text{otherwise,} \end{cases}
$$

then, for example, Stanton's model [17] specifies π as $\pi(t|\theta_t) := \lambda + \theta_t \rho$, with λ, ρ being two given positive constants: in this way, an exogenous (i.e. non-optimal but happening due to exogenous reasons) prepayment has a hazard function λ, while if it is optimal to prepay at time t the hazard function is augmented by ρ. In other previous models [8, 9], it was assumed that $\pi(t|\theta_t) := \lambda(t) + \infty \times \theta_t$, i.e. an exogenous prepayment has a hazard function $\lambda(t)$, while all the borrowers always prepay immediately when it is optimal (the prepayment function explodes).

This means that, while in order to properly price a MBS one has to take into account the possibly non-optimal prepayment behaviour of the borrowers, in some models the prepayment function π can explicitly depend on the fact that, for a rational borrower, it would be optimal to prepay or not. This also means that solving the problem (3), while not sufficient to price a MBS, may be necessary for many models.

1.4 The pricing of non-guaranteed MBS

In the case when also a default decision is available, the pricing is furtherly complicated by the fact that the borrower owns two options wich are mutually exclusive, so that the total value of a single mortgage at time t, if the borrower were fully rational, would be $V_t - W_t$, where

$$
W_t = \operatorname*{ess\,sup}_{\tau, \delta \in [t,T]} \mathbb{E}_{\mathbb{Q}}[I_{\{\tau < \delta\}} e^{-\int_t^\tau r_u\, du}(V_\tau - F_\tau) + I_{\{\delta < \tau\}} e^{-\int_t^\tau r_u\, du}(V_\delta - H_\delta) \mid \mathcal{F}_t] \quad (4)
$$

τ and δ indicating the stopping times corresponding respectively to a prepayment decision and to a default decision and the process $H = (H_t)_{t \in [0,T]}$ representing the price of the underlying physical good.

As in the case of prepayment, also the default decision can be triggered by non-rational reasons; for example, a default decision can usually have also legal consequences for the borrower (for example, a bad credit scoring) which go beyond the single transaction $V_\delta - H_\delta$. Conversely, even if the house has a high value at a given time, it is also possible that the borrower is simply not able to repay the mortgage anymore and cannot avoid to forfeit the house. For these reasons, in order to evaluate correctly a MBS, it is important to model this non-optimal behaviour. Analogously to the case when only the prepayment risk is present, the prepayment/default behavior can depend on the fact that at the current time it is optimal or not to prepay and/or to default. For example, in [7] the non-optimal behaviour is modeled as a generalisation of the Stanton model, with the intensities $\lambda_{sp}, \lambda_{sd}$ corresponding to exogenous (i.e. non-optimal) prepayment and default respectively, and the additional intensities λ_p, λ_d corresponding to endogenous (i.e. optimal) prepayment and default.

In the next sections we concentrate on solving the optimal stopping problems (3) and (4), starting from the former. In particular, in Section 2 we present results on optimal prepayment with no default in continuous time: as in this case we do not have closed form solutions, we turn our attention to the discrete time version of this problem with the aim to obtain an approximate solution, discussing various methods which can be used. In Section 3 we focus on an efficient method, introduced in a quite general setting in [14], to build discrete time models based on trees which are computationally simple, and we present two applications to the Vasicek model and to the CIR model, with some examples where we solve the prepayment problem numerically. In Section 4 we show how to apply the solution of the prepayment problem to the pricing of guaranteed MBS where the prepayment function can depend on the optimality of prepayment at the current time. In Section 5 we pass to the more general problem (4) and present numerical results on optimal prepayment and default in discrete time for MBS not of the FNMA/GNMA type by extending the procedure of computationally simple trees to this case, where a 2-dimensional tree for both r and H will be needed, presenting also an application to the CIR model. In doing this, we extend and simplify some aspects of the building of two-dimensional trees with respects to what one can find in the present literature (see [1, 12, 18] and the discussion in Section 5).

2 Optimal rational prepayment

From now on, we make the simplifying assumption that the short rate is a Markov process, so that the entire term structure $(B(t,s))_{0\leq t\leq s\leq T}$ can be obtained from the process r; of course, many of the results can be generalised to the situation where there is a d-dimensional Markov factor process that drives the term structure.

If the evolution of r follows a continuous time dynamics, we assume that its evolution has the stochastic differential

$$dr_t = \mu(r_t)\ dt + \sigma(r_t)\ dW_t \tag{5}$$

where W is a Brownian motion and μ, σ are functions such that Equation (5) has a unique strong solution. In this case, the price of the zero coupon bond $B(t,T)$ is given by $B(t,T) = \tilde{B}(t,T,r_t)$, where $\tilde{B}$ is the solution of the partial differential equation

$$\begin{cases} \tilde{B}_t(t,T,r) + (L\tilde{B})(t,T,r) = r\tilde{B}(t,T,r), & (t,r) \in (0,T) \times \mathbb{R}, \\ \tilde{B}(T,T,r) = 1, & r \in \mathbb{R} \end{cases}$$

and L is the infinitesimal generator of r

$$(Lf)(t,T,r) := \mu(r) f_r(t,T,r) + \frac{1}{2}\sigma^2(r) f_{rr}(t,T,r)$$

It is well known that, if one wants to solve an optimal stopping problem of the kind of (3) in continuous time, its solution in terms of both the optimal value as well as an optimal stopping time is linked to the free boundary value problem

$$\begin{cases} \max(f_t + Lf - rf, \psi - f) = 0, & (t,r) \in (0,T) \times \mathbb{R}, \\ f(T,r) = \psi(T,r) = 0, & r \in \mathbb{R} \end{cases} \tag{6}$$

where $\psi(t,r)$ is given by

$$\begin{aligned} \psi(t,r) &= V(t,r) - F(t) = A \int_t^T B(t,u)\, du - A \frac{1 - e^{-\rho(T-t)}}{\rho} = \\ &= A \left(\int_t^T \tilde{B}(t,u,r)\, du - \frac{1 - e^{-\rho(T-t)}}{\rho} \right) \end{aligned}$$

An optimal stopping time is given by

$$\tau := \inf\{t \le T \mid f(t, r_t) = \psi(t, r_t)\}$$

Unfortunately, Equation (6) admits an explicit solution only in very few cases, and usually one has to implement numerical methods, among which we cite finite differences or finite elements methods to solve Equation (6) or weak convergence of discrete time processes to solve directly problem (3).

It is well known that one-dimensional explicit finite differences schemes are equivalent to trinomial trees [11], thus they can be viewed as a particular case of weak convergence methods, while implicit finite differences methods present better results in terms of stability but not in terms of computational cost; finite elements methods in dimension 1 are more or less equivalent to finite differences methods. In more than 1 dimension, as concerns the finite differences method the same comments of the one-dimensional case can be made, while finite elements methods are more flexible but require the choice of a triangularization of the space domain, which is a nontrivial problem by itself, while not presenting evident advantages over the other computational methods. Moreover, both methods require a truncation of the state space and conditions on the second-order operator which, if not satisfied, lead to a transformation of the state space which can appear artificial (see [7, 17] for examples).

For the reasons above, while not claiming that those analytic methods are worse than those based on weak convergence, we prefer to focus on this latter class. When discretising a diffusion process, one can choose different approximation schemes, the most common being Euler discretisation of the SDE and binomial trees. In both cases, if we consider N subdivisions of the interval $[0,T]$, the original problem (3) becomes

$$\operatorname*{ess\ sup}_{\tau\in[n,N]} \mathbb{E}_{\mathbb{Q}}[e^{-\frac{1}{N}\sum_{i=n}^{\tau} r_i}(V_\tau - F_\tau) \mid \mathcal{F}_n] \tag{7}$$

where the ess sup is now taken over all the $(\mathcal{F}_n)_n$-stopping times τ taking integer values between n and N, and $V-F$ is given by

$$V_n - F_n = A_N \sum_{i=n+1}^{N} B\left(n\frac{T}{N}, i\frac{T}{N}\right) - \frac{A_N}{\rho_N}\left(1 - \frac{1}{(1+\rho_N)^{N-n}}\right).$$

with the zero-coupon values B now given by

$$B\left(n\frac{T}{N}, i\frac{T}{N}\right) = \mathbb{E}_{\mathbb{Q}}[e^{-\frac{1}{N}\sum_{k=n}^{i-1} r_k} \mid \mathcal{F}_n]$$

If the short rate r is Markov (meaning now a Markov chain), then the zero-coupon values $B(\frac{n}{N}T, \frac{i}{N}T)$ are deterministic functions of n and r_n, thus in problem (7) the conditional expectation gives a function which only depends on n and r_n, as well as $V_n - F_n$. In this case, problem (7) can be solved by using the Dynamic Programming principle: define recursively the functions W_n, $n = 0, \dots, N$ in this way:

$$\begin{aligned} W_N(r) &:= V_N(r) - F_N(r) \equiv 0, \\ W_n(r) &:= \max[(V_n - F_n)(r), \mathbb{E}_{\mathbb{Q}}[e^{-\frac{1}{N}r_{n+1}} W_{n+1}(r_{n+1}) \mid r_n = r]] \end{aligned} \tag{8}$$

where an optimal stopping time is given by

$$\hat{\tau} := \inf\{n \le N \mid W_n(r_n) = (V_n - F_n)(r_n)\}$$

Given an initial value r_0, the computational cost of calculating the functions $(W_n)_n$ depends on the model that we choose for the discrete-time evolution of r. The most common choices are:

- Euler scheme discretising a continuous-time diffusion: in this case the conditional distributions of r are Gaussian, so that the state space is an infinite set (typically the whole real line). In this case, if the $(W_n)_n$ cannot be calculated in an analytical way, it is very difficult to evaluate them numerically. One of the most common choices is to perform a quantisation, thus reducing the problem to an optimisation problem on a discrete-space process.

- Binomial trees. In this case the state space for r is a finite set, but its cardinality depends on the type of the tree. If the tree is not recombining, then the state space for r_n can consist of up to 2^n points. If the tree is recombining, then the cardinality of the state space for r_n grows with n at most linearly.

In view of this, the most efficient choice would be to use a recombining binomial tree dynamics for r. In fact assume that, at each time step n, the number of states that r_n can assume is at most Kn (in the common binomial case, $K = 1$). This means that one can use backward induction in order to calculate the functions (8), having to calculate them in every one of the $\sum_{n=1}^{N} Kn = K\frac{N(N+1)}{2}$ nodes of the tree. Conversely, if the tree is not recombining, the nodes can be up to $\sum_{n=1}^{N} 2^n = 2(2^N - 1)$ and calculating the functions (8) is much more time consuming. At last, calculating the functions (8) within the Euler discretisation can be more or less time consuming depending on the kind (recombining or not) of quantisation scheme used.

In the following, we solve numerically the discrete-time problem (7) using a recombining tree technique: in doing this we will follow the Nelson-Rawaswamy approach of "computationally simple trees" [14].

3 Computationally simple trees

Nelson and Rawaswamy define a *computationally simple tree* as a tree where the number of nodes at each time $n \leq N$ grows at most linearly with the number of time intervals.

As noticed before, in this case the total number of nodes up to time N is at most equal to $K\frac{N(N+1)}{2}$. If we use a binomial tree, typically $K = 1$, and in order to calculate the functions (8) one has to calculate at each node of the tree two expectations (of $V_n(r_n)$ and of $W_{n+1}(r_{n+1})$) over two possible future outcomes and then a maximum, ending up with a constant number of operations at each node of the tree.

3.1 Weak convergence

As we build the process $(r_n)_n$ in order to approximate (via weak convergence) the diffusion process of Equation (5), we need sufficient conditions for this weak convergence to take place. We thus present the following theorem from [10], which is also used in [14]: although in that paper the theorem is used in a one-dimensional version, here we state it in its full generality in order to use it again in Section 5, where a 2-dimensional version will be needed.

Theorem 3.1 *Let X a stochastic process with dynamics*

$$dX_t = \bar{\mu}(X_t)\ dt + \Sigma(X_t)\ dZ_t$$

with $X_0 \equiv x \in \mathbb{R}^d$, and $\bar{\mu} : \mathbb{R}^d \to \mathbb{R}^d$ and $a := \Sigma\Sigma^T : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are continuous and such that there exists a pathwise unique solution for all $x \in \mathbb{R}^d$. For $h > 0$, let now $(X_n^h)_n$ be a d-dimensional Markov chain with $X_0^h = x$ and transition kernel $\nu_h(x, \Gamma)$, and define, for all $x \in \mathbb{R}^d$,

$$\begin{aligned}
\bar{\mu}_h(x) &= \frac{1}{h}\int_{|y-x|\leq 1} (y-x)\nu_h(x, \mathrm{d}y), \\
a_h(x) &= \frac{1}{h}\int_{|y-x|\leq 1} (y-x)(y-x)^T \nu_h(x, \mathrm{d}y).
\end{aligned}$$

Assume that for all $R > 0$, $\varepsilon > 0$,

$$\begin{aligned}
\sup_{|x|\le R} |\bar{\mu}_h(x) - \bar{\mu}(x)| &\rightarrow 0, \qquad (9)\\
\sup_{|x|\le R} |a_h(x) - a(x)| &\rightarrow 0, \qquad (10)\\
\sup_{|x|\le R} \frac{1}{h}\nu_h(x, \{y : |y - x| \ge \varepsilon\}) &\rightarrow 0. \qquad (11)
\end{aligned}$$

Define now $X_h(t) := X^h_{[t/h]}$ for all $t \ge 0$. Then, for $h \to 0$, $(X_h)_h$ converges weakly to X.

In order to use this theorem to obtain a weak convergence scheme for a process r solution of Equation (5), we let $\bar{\mu} := \mu$, $\Sigma := \sigma$ and $Z := W$ and assume that σ is non-negative, and in this way we obtain the one-dimensional version used in [14]. It also follows that a weak convergence can be achieved even without building a tree for the evolution of the X^h, $h > 0$: this is the case, for example, of Euler schemes for SDEs, where the transition kernels ν_h are typically Gaussian. Thus, in order to obtain a recombining binomial tree, one has to make further assumptions as is shown in the following subsection.

3.2 Computationally simple trees

In order to obtain Markov chains $(r^h)_h$ which evolve along a tree, we use the construction presented in [14]. Take the interval $[0, T]$ and divide it into N equal subintervals having length $h := T/N$. For each h, consider a process $(r^h_t)_{t\in[0,T]}$, which is constant on the subintervals and, at each time hk, $k = 1, \ldots, N$, jumps upwards or downwards with probability q and $1 - q$, respectively. More precisely, take $q_h, R^+_h, R^-_h : \mathbb{R} \to \mathbb{R}$ such that $0 \le q_h(r) \le 1$ and $-\infty < R^-_h(r) \le R^+_h(r) < \infty$ for all $r \in \mathbb{R}$, $k = 0, 1, \ldots, N$. Now define the process r^h as a Markov chain with $r^h_0 = r_0$ for all $h > 0$ and transition kernel given by

$$\nu_h(r, \cdot) := q_h(r)\delta_{R^+_h(r)}(\cdot) + (1 - q_h(r))\delta_{R^-_h(r)}(\cdot) \qquad (12)$$

where $\delta_x(\Gamma) := \mathbf{1}_\Gamma(x)$, $\Gamma \subseteq \mathbb{R}$ is the Dirac delta centered in x. This means that r^h evolves as a tree with the two possible future outcomes $R^\pm_h(r)$ in each state r, with probability $q_h(r)$ and $1 - q_h(r)$ respectively. In order to establish the weak convergence $r^h \Rightarrow r$, we write explicitly

$$\begin{aligned}
\bar{\mu}_h(r) &= \frac{q_h(r)[R^+_h(r) - r] + (1 - q_h(r))[R^-_h(r) - r]}{h},\\
a_h(r) = \sigma^2_h(r) &= \frac{q_h(r)[R^+_h(r) - r]^2 + (1 - q_h(r))[R^-_h(r) - r]^2}{h},
\end{aligned}$$

and make the following assumption.

Assumption 1 *Assume that the conditions in Equations (9–10) hold and that for all $\delta > 0$, $T > 0$,*

$$\lim_{h\downarrow 0} \sup_{|r|\le\delta, 0\le t\le T} |R^+_h(r, t) - r| = \lim_{h\downarrow 0} \sup_{|r|\le\delta, 0\le t\le T} |R^-_h(r, t) - r| = 0, \qquad (13)$$

Notice that Equation (13) implies Equation (11). We can now state the following result from [14].

Theorem 3.2 *Under Assumption 1, for $h \downarrow 0$ the sequence $(r^h)_h$ converges weakly to r, which is solution of Equation (5).*

In order to obtain a computationally simple tree, besides satisfying Assumption 1, the functions R_h^-, R_h^+ must also satisfy this condition: we must require that the total displacement of a up-movement followed by a down-movement is equal to the analogous displacement when the movements have reverse order. This means that the equality

$$R_h^+(r) - r + R_h^-\left(R_h^+(r)\right) - R_h^+(r) = R_h^-(r) - r + R_h^+\left(R_h^-(r)\right) - R_h^-(r)$$

i.e. $R_h^-(R_h^+(r)) = R_h^+(R_h^-(r))$, must be true for all $r \in \mathbb{R}$, $h > 0$.

We now follow Nelson and Rawaswany and present two models for the interest rate r where we can build computationally simple trees which weakly converge to r.

3.3 Weak convergence to the Vasicek model

Assume that r follows the Vasicek model

$$\mathrm{d}r_t = \beta(\alpha - r_t)\mathrm{d}t + \sigma \mathrm{d}W_t, \tag{14}$$

with $\beta > 0$, and define

$$\begin{aligned} R_h^+(r) &:= r + \sigma\sqrt{h}, \\ R_h^-(r) &:= r - \sigma\sqrt{h}, \\ q_h(r) &:= \max\left(0, \min\left(\frac{1}{2} + \sqrt{h}\frac{\beta}{2\sigma}(\alpha - r), 1\right)\right) \end{aligned}$$

With this choice, the local drift and second moment are respectively

$$\mu_h(r) = \begin{cases} \beta(\alpha - r) & \text{if } 0 < q_h(r) < 1 \\ \sigma/\sqrt{h} & \text{if } q_h(r) = 1 \\ -\sigma/\sqrt{h} & \text{if } q_h(r) = 0, \end{cases}$$

and $\sigma_h^2(r) = \sigma^2$. By Theorem 3.2, with this choice the sequence $(r^h)_h$ converges weakly to r.

3.3.1 Numerical examples

We present a numerical solution of the prepayment problem in Figure 1. We set $r_0 = 0.03$, $T = 2$ years, and the nominal rate $\rho = 0.04$, with model parameters $\beta = 0.02$, $\sigma = 0.1$, $\alpha = 0.15$. We divide the interval $[0, 2]$ into $N = 12$, 24 and 48 subintervals (corresponding to prepayment decisions taken every 2 months, 1 month and 15 days respectively). For each node, we indicate with a red cross an optimal decision to prepay (stop) and with a blue dot an optimal decision to continue.

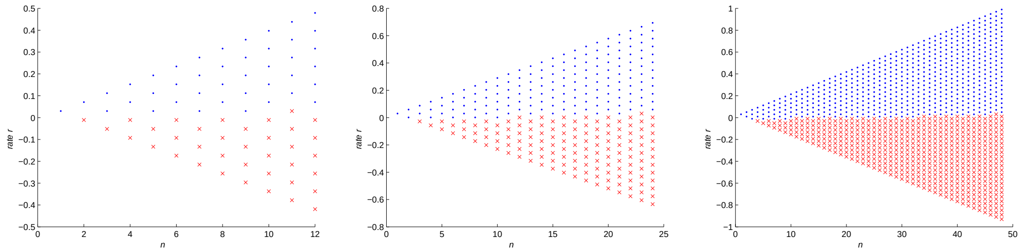

Figure 1: Optimal prepayment with the Vasicek model with $N = 12, 24, 48$ (other parameters: $r_0 = 0.03$, $T = 2$, $\rho = 0.04$, $\beta = 0.02$, $\sigma = 0.1$, $\alpha = 0.15$). For each node, we indicate with a red cross an optimal decision to prepay (stop) and with a blue dot an optimal decision to continue.

The method is efficient enough to be applied also to longer maturities with the same prepayment frequencies: take for example $T = 20$ years and $N = 240$, corresponding to a prepayment decision taken every month, with all the other parameters kept equal ($r_0 = 0.03$, $\rho = 0.04$, $\beta = 0.02$, $\sigma = 0.1$, $\alpha = 0.15$). In order to emphasize the functional form of the optimal prepayment boundary, we omitted the interest rates outside the interval $[-0.2; 0.2]$ because the probability of r going outside this interval is very small: we present the result in Figure 2.

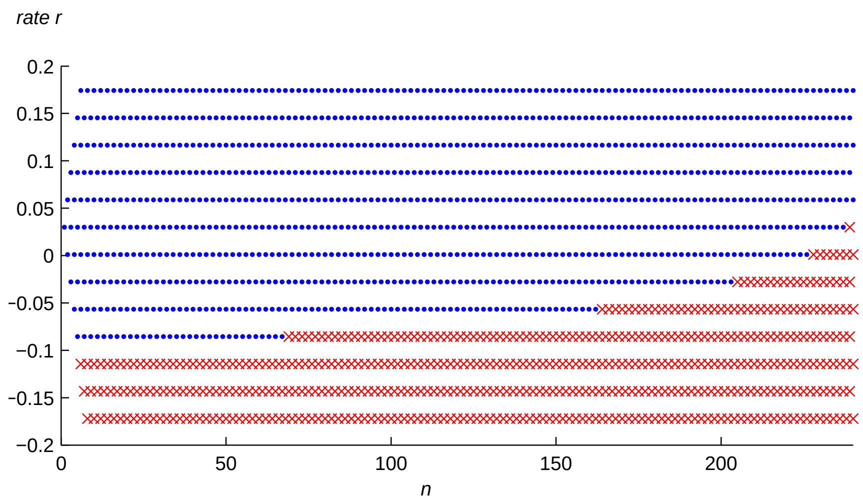

Figure 2: Optimal prepayment with $T = 20$, $N = 240$ (other parameters: $r_0 = 0.03$, $\rho = 0.04$, $\beta = 0.02$, $\sigma = 0.1$, $\alpha = 0.15$). Again, for each node we indicate with a red cross an optimal decision to prepay and with a blue dot an optimal decision to continue. We also omitted the interest rates outside the interval $[-0.2; 0.2]$ because the probability of r going outside this interval is very small.

3.4 Weak convergence to the CIR model

In order to build a computationally simple tree for the CIR model, we first make a transformation of the state variable r_t of the form $X_t := X(r_t)$ with $X \in C^2$. If r satisfies Equation (5), then

$$\mathrm{d}X(r_t) = \left(\mu(r_t)\frac{\partial X(r_t)}{\partial r} + \frac{1}{2}\sigma^2(r_t)\frac{\partial^2 X(r_t)}{\partial r_t^2} + \frac{\partial X(r_t)}{\partial t}\right)\mathrm{d}t + \sigma(r_t)\frac{\partial X(r_t)}{\partial r}\mathrm{d}W_t.$$

Now choose X such that

$$X(r) = \int^r \frac{\mathrm{d}z}{\sigma(z)}, \tag{15}$$

With this choice, we have $\sigma(r)\frac{\partial X(r)}{\partial r} \equiv 1$, so that the diffusion term of $(X(r_t))_t$ is constant and we can again build a computationally simple tree in a similar way as we did for the Vasicek model. If X is invertible (a sufficient condition for this is that $\sigma(r) > 0$ for all r, which we assumed), then we can come back to the process r by applying the inverse transformation.

If we assume that r follows the CIR model

$$\mathrm{d}r_t = k(\mu - r_t)\mathrm{d}t + \sigma\sqrt{r_t}\mathrm{d}W_t, \tag{16}$$

with $k > 0, \mu > 0$, then a suitable transformation X is given by

$$X(r) := \int^r \frac{\mathrm{d}z}{\sigma\sqrt{z}} = \frac{2\sqrt{r}}{\sigma}, \tag{17}$$

with $x_0 = X(r_0)$, the inverse transformation being

$$R(x) := \begin{cases} \frac{\sigma^2 x^2}{4} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

We can easily see that $\sigma(0) = 0 < \mu(0)$, so that $r = 0$ is not an absorbing state. Because of this, Nelson and Rawaswamy present a procedure that requires, when r is near 0, to make an upward jump which is a multiple of the corresponding downward jump. For this reason, we define

$$J_h^+(x) := \begin{cases} \text{the smallest even integer } j \text{ such that} \\ \frac{\sigma^2(x+j\sqrt{h})^2}{4} - \frac{\sigma^2 x^2}{4} \geq k(\mu - \frac{\sigma^2 x^2}{4})h \end{cases} \tag{19}$$

$$J_h^-(x) := \begin{cases} \text{the smallest even integer } j \text{ such that} \\ \text{either } \frac{\sigma^2(x-j\sqrt{h})^2}{4} - \frac{\sigma^2 x^2}{4} \leq k(\mu - \frac{\sigma^2 x^2}{4})h \text{ or } x - j\sqrt{h} \leq 0, \end{cases} \tag{20}$$

$$q_h(x) := \begin{cases} \frac{hk(\mu - R(x)) + R(x) - R_h^-(x)}{R_h^+(x) - R_h^-(x)}, & \text{if } R_h^+(x) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

$$R_h^\pm(x) \equiv R(x \pm J_h^\pm(x)\sqrt{h}), \tag{22}$$

The quantities $J_h^\pm(x)$ are defined in such a way that $0 \leq q_h(x) \leq 1$ in Equation (21), so that the local drift converges to the drift of Equation (16). The following result holds.

Corollary 3.1 *Define the processes $(r^h)_h$ using (12) and (17–21). For $h \downarrow 0$, the sequence $(r^h)_h \Rightarrow r$ weakly, which is the solution of Equation (16).*

3.4.1 Numerical examples

We again present a numerical solution of the prepayment problem in Figure 3. We set $r_0 = 0.03$, $T = 2$ years, and nominal rate $\rho = 0.04$, with model parameters $\beta = 0.02$, $\sigma = 0.1$, $\alpha = 0.15$. As before, we divide the interval $[0, 2]$ into $N = 12$, 24 and 48 subintervals, and for each node we indicate with a red cross an optimal decision to prepay (stop) and with a blue dot an optimal decision to continue.

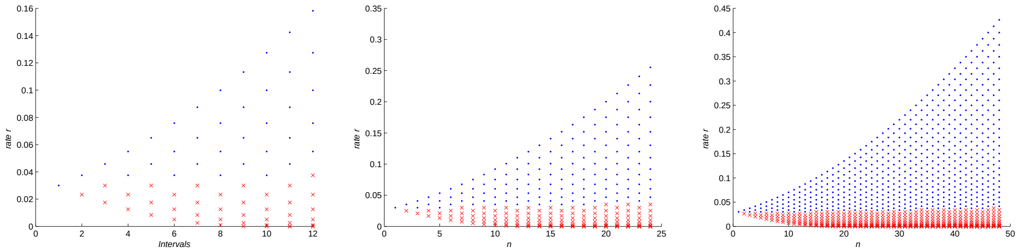

Figure 3: Optimal prepayment with the CIR model with $N = 12, 24, 48$ (other parameters: $r_0 = 0.03$, $T = 2$, $\rho = 0.04$, $\beta = 0.02$, $\sigma = 0.1$, $\alpha = 0.15$). Again, for each node we indicate with a red cross an optimal decision to prepay and with a blue dot an optimal decision to continue.

Also in this case, the method remains efficient enough to be applied to longer maturities with the same prepayment frequencies: in the same "long maturity" example as before ($T = 20$, $N = 240$, $r_0 = 0.03$, $\rho = 0.04$, $\beta = 0.02$, $\sigma = 0.1$, $\alpha = 0.15$) we have the result in Figure 4. As before, in order to emphasize the functional form of the optimal prepayment boundary, we omitted the interest rates outside the interval $[0; 0.2]$ because the probability of r going outside this interval is very small.

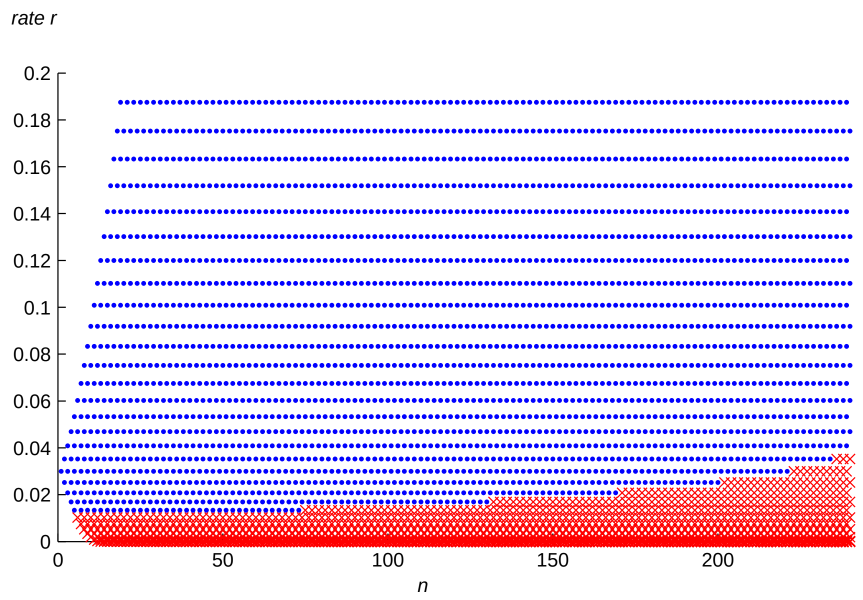

Figure 4: Optimal prepayment with $T = 20$, $N = 240$ (other parameters: $r_0 = 0.03$, $\rho = 0.04$, $\beta = 0.02$, $\sigma = 0.1$, $\alpha = 0.15$). Again, for each node we indicate with a red cross an optimal decision to prepay and with a blue dot an optimal decision to continue, and we omitted the interest rates outside the interval $[-0.2; 0.2]$ because the probability of r going outside this interval is very small.

3.5 Flexibility of the algorithm

The algorithm is flexible enough to incorporate a more complex rate structure than the "classical" constant rate. For example, assume that a mortgage is written with a so-called entrance interest rate ρ_1 up to the time T_1 and an interest rate $\rho_2 > \rho_1$ for the remaining time $[T_1, T]$, while the repayments rates are adjusted in such a way that they are constant in the two periods $[0, T_1]$ and $[T_1, T]$. Then it can be proved that, if we divide the interval $[0, T]$ into N intervals such that $T_1 = N_1 \frac{T}{N}$ for some integer N_1, the repayment rates in the two periods are respectively

$$\begin{aligned} A(n) &= A_1 = \frac{P\rho_1}{1 - (\frac{1}{1+\rho_1})^N}, \quad n \le N_1, \\ A(n) &= A_2 = \frac{P\rho_2}{1 - (\frac{1}{1+\rho_2})^{N-N_1}} \frac{1 - (\frac{1}{1+\rho_1})^{N-N_1}}{1 - (\frac{1}{1+\rho_1})^N}, \quad n > N_1. \end{aligned}$$

In this case, the prepayment option is written on

$$F_t = \int_0^T e^{-\int_t^u \rho(s)\ ds} A(u)\ du, \qquad V_t = \int_0^T B(t,u) A(u)\ du$$

where $\rho(s) := \rho_1$ for $s \leq T_1$ and $\rho(s) := \rho_2$ for $s > T_1$, which can be discretised in a recursive way as

$$F_n := \sum_{i=n+1}^{N} \frac{A(i)}{(1+\rho(i))^{i-n}} = \frac{A(n+1)}{1+\rho(n+1)} + \frac{F_{n+1}}{1+\rho(n+1)}$$

where $\rho(n) := \rho_1$ for $n \leq N_1$ and $\rho(n) := \rho_2$ for $n > N_1$, and

$$\begin{aligned} V_n &:= \sum_{i=n+1}^{N} A(i) B\left(n\frac{T}{N}, i\frac{T}{N}\right) = \\ &= A(n+1) B\left(n\frac{T}{N}, (n+1)\frac{T}{N}\right) + \mathbb{E}[e^{-\frac{1}{N} r_{n+1}} V_{n+1} \mid r_n] \end{aligned}$$

and both the theoretical framework and the construction of the algorithm follow in an analogous way as before.

As an example, we present a numerical result for a mortgage having entrance rate $\rho_1 = 0.03$ for the first $T_1 = 3$ years and rate $\rho_2 = 0.04$ for the remaining 17 years, for a total of $T = 20$ years divided into $N = 120$ subintervals (corresponding to prepayment decisions taken every 2 months) when the short rate follows a CIR model with parameters $r_0 = 0.03$, $\beta = 0.02$, $\sigma = 0.1$, $\alpha = 0.15$. The result is shown in Figure 5.

4 The pricing of mortgage-backed securities

We already saw in Section 1.3 that in order to price a MBS one has to take into account the non-optimal behaviour of the borrowers: we choose to do this via a prepayment function $\pi(t|\theta)$, where θ could be a stochastic process.

Once we specify the hazard function $\pi(\cdot|\theta)$ and the interest rates dynamics, we can obtain the price of a MBS by incorporating $\pi(\cdot|\theta)$ into the pricing formula. In particular, for a mortgage pool having nominal interest rate ρ and maturity T, the price of a pass-through MBS at time t would be $V_t - O_t$, where O_t is the price at time t of the so-called prepayment option, defined by

$$\begin{aligned} O_t &:= \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^\tau r_u\ du}(V_\tau - F_\tau) \mid \mathcal{F}_t] = \\ &= \mathbb{E}_{\mathbb{Q}}\left[\int_t^T \pi(u|\theta) e^{-\int_t^u (r_v + \pi(v|\theta)) dv} (V_u - F_u)\ du \,\middle|\, \mathcal{F}_t\right] \end{aligned}$$

where τ is a prepayment time whose law is characterised by the prepayment function $\pi(\cdot|\theta)$. Notice that in this case, as τ is not typically an optimal prepayment time,

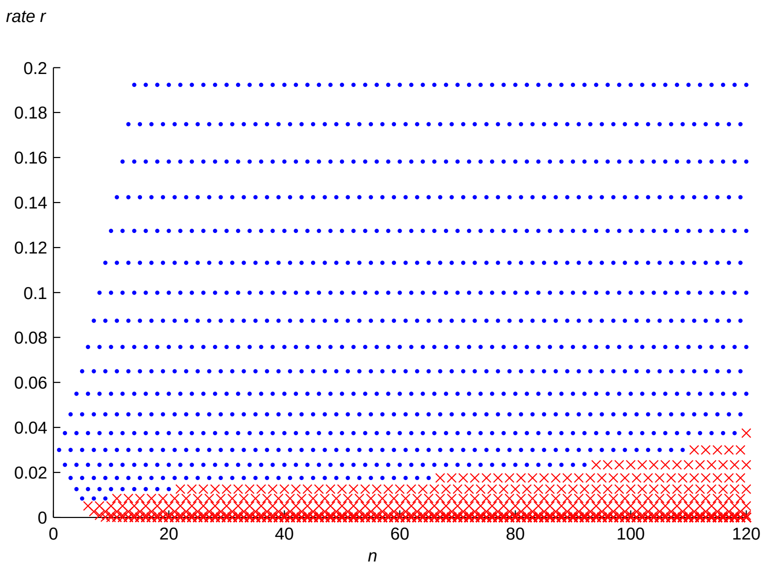

Figure 5: Optimal prepayment of a mortgage with entrance rate $\rho_1 = 0.03$ for the first $T_1 = 3$ years and rate $\rho_2 = 0.04$ for the remaining 17 years, for a total of $T = 20$ years divided into $N = 120$ subintervals, when the short rate follows a CIR model with parameters $r_0 = 0.03$, $\beta = 0.02$, $\sigma = 0.1$, $\alpha = 0.15$.

the value of O_t can also be less than zero. This quantity can be discretised in a recursive way as

$$\begin{aligned} O_n &= \mathbb{E}\left[\sum_{i=n}^{N} \pi(i|\theta_i) e^{-\sum_{j=n}^{i}(r_j+\pi(j|\theta_j))}(V_i - F_i)\middle| \mathcal{F}_n\right] = \\ &= e^{-r_n-\pi(n|\theta_n)}(\pi(n|\theta_n)(V_n - F_n) + \mathbb{E}[O_{n+1}|\mathcal{F}_n]) \end{aligned}$$

In order to investigate the impact of the non-optimal prepayment structure on the value of the MBS, we can easily adapt the computationally simple tree algorithm of the previous section to this framework.

As an example, if we adopt the Stanton model for π, then $\pi(t|\theta_t) := \lambda + \theta_t \bar{\rho}$, where $\lambda, \bar{\rho} > 0$ correspond respectively to the intensity of exogenous payment and to the additional intensity of endogenous payment, and θ_t is a stochastic process adapted to $(\mathcal{F}_t)_t$ equal to 1 in the case when at time t it is optimal to prepay, and 0 otherwise.

In the following numerical example, we assume that the short rate follows the CIR model with $r_0 = 0.03$, $\beta = 0.02$, $\sigma = 0.1$, $\alpha = 0.15$, and also take as parameters of the Stanton prepayment function $\lambda = 0.0338$ and $\bar{\rho} = 0.6452$ on a mortgage with rate $\rho = 0.04$ and $T = 20$. By taking $N = 240$, we can calculate the value of the prepayment option O_n at each node. We can, for example, represent the states where $O_n > 0$ (or $O_n < 0$), i.e. the market value V_n of a typical mortgage is worth more (respectively, less) than the pass-through MBS $V_n - O_n$ (which also contains the prepayment option exercised in a non-optimal way), with a blue dot (respectively, with a red cross) and obtain Figure 6.

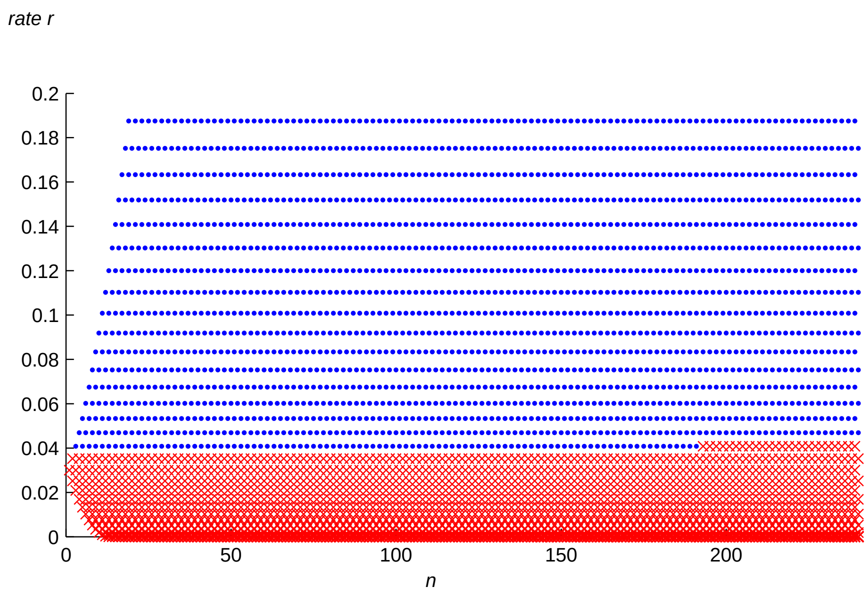

Figure 6: The states where the market value of the prepayment option, obtained with a Stanton intensity with $\lambda = 0.0338$ and $\bar{\rho} = 0.6452$, is positive (blue dot) or negative (red cross), in the case of a mortgage with rate $\rho = 0.04$ and $T = 20$. We assume that the interest rates follow a CIR model with $r_0 = 0.03$, $\beta = 0.02$, $\sigma = 0.1$, $\alpha = 0.15$ and we discretise using $N = 240$ subdivisions.

5 Optimal prepayment/default behavior

In this section, we analyse the situation when the risk of prepayment is not the only risk in the model and the default risk is also present. Usually this is done by identify a stochastic process, called the "price of the house", which triggers the default decision of the borrower: actually, this is not a real price of a real house, as even in a single pool the price of the underlying houses can follow different time evolutions depending on their locations, types and environmental conditions, but must be interpreted as an appropriate proxy (for example, a real estate market index). For example, in the Downing-Stanton-Wallace model [7] it is assumed that the "price of the house" evolves as a geometric Brownian motion

$$dH_t = H_t((r_t - q_H)\ dt + \sigma_H\ dW_t^H) \tag{23}$$

where r is the short rate, q_H is a constant dividend, $\sigma_H > 0$ is the volatility parameter, and W^H is a Brownian motion, not necessarily independent of r. In the case of default, the borrower forfeits the house to the lender; thus, the default option is an option of American type to exchange the market value of the stream of payments V_t for the value of the house H_t.

As said in Section 1, since the option of prepayment and of surrendering are alternative to each other, it is impossible to evaluate them separately. Instead, their optimal combined value is given by Equation (4), provided that the borrower acts in a rational way, which may not happen. For this reason, also the default behaviour at time t is usually represented via a default intensity, which can depend on the fact that at time t is optimal to default or not.

From what is stated before, it is easily seen that finding a solution to the problem (4) in terms of optimal stopping times $(\hat{\tau}, \hat{\delta})$ can be necessary in order to properly price a MBS, in analogy with the case when only the prepayment risk is present. Unfortunately, similarly to problem (3), also this problem has no general solution in closed form when formulated in continuous time. We thus concentrate ourselves on finding a numerical method based on computationally simple trees which solve problem (4).

The first thing to do is to formulate problem (4) in discrete time, which is done in this way:

$$W_n = \operatorname{ess} \sup_{\tau,\delta\in[n,N]} \mathbb{E}[I_{\{\tau<\delta\}} e^{-\sum_{j=n}^{\tau-1} r_j}(V_\tau - F_\tau) + I_{\{\delta<\tau\}} e^{-\sum_{j=n}^{\tau-1} r_j}(V_\delta - H_\delta) \mid \mathcal{F}_n] \tag{24}$$

As in Section 2, if the short rate r and the "price of the house" H are Markov chains, then V_n is a deterministic functions of n and r_n, thus the conditional expectation with respect to $\mathcal{F}_n$ is a deterministic function of r_n and H_n, and the problem can be solved by using the Dynamic Programming principle. Define recursively the functions W_n, $n = 0, \ldots, N$ in this way:

$$\begin{aligned} W_N(r, H) &:= 0, \\ W_n(r, H) &:= \max\Big((V_n - F_n)(r), V_n(r) - H, \\ &\qquad \mathbb{E}_{\mathbb{Q}}[e^{-\frac{1}{N} r_{n+1}} W_{n+1}(r_{n+1}, H_{n+1}) \mid r_n = r, H_n = H]\Big) \end{aligned}$$

Then $W_n = W_n(r_n, H_n)$ for every $n = 0, \ldots, N$, and two optimal stopping times are given by

$$\begin{aligned} \hat{\tau} &:= \inf\{n \le N \mid W_n(r_n, H_n) = (V_n - F_n)(r_n)\}, \\ \hat{\delta} &:= \inf\{n \le N \mid W_n(r_n, H_n) = V_n(r_n) - H_n\} \end{aligned}$$

Given initial values r_0, H_0, the computational cost of calculating the functions $(W_n)_n$ depends again on the model that we choose for the discrete-time evolution of (r, H).

For this reason, we now build a two-dimensional computationally simple tree similar to those of Section 3. In particular, we will build two binomial trees for r and H respectively, and in this way the nodes at each time step n will be n^2. The total number of nodes will thus be $\sum_1^N n^2 = \frac{N(N+1)(2N+1)}{6} = O(N^3)$, which is still an acceptable computational cost for the pricing.

We now illustrate explicitly the construction of the two-dimensional tree, which presents some new aspects with respect to what one can find in literature. In fact, the construction of a two-dimensional tree for state variables which are separate Markov processes driven by Brownian motion which are possibly correlated, first presented in [12], is by now a standard (see for example [1]): but that is not our case. Conversely, the construction of a two-dimensional tree for state variables which are not Markov by their own but form a two-dimensional Markov process, driven by possibly correlated Brownian motions, is less common and was firstly sketched out in [18] along these steps:

1. transform the state variables in the spirit of [14] (as seen in Section 3.4) in order for the diffusion to be constantly 1 for both: however, these new state variables can stilll be driven by correlated Brownian motions;

2. make an affine transformation in order to obtain other new state variables with unitary diffusion term and driven by independent Brownian motions;

3. for each state variable, build a binomial (or trinomial) tree, and combine each node by 4 (9) joint probabilities to the 4 (9) future outcomes;

4. at each node, convert the new variables back to the original ones, and price contingent claims via backward induction.

We now present a new construction which allows to skip Step 2), by calculating in a suitable way the joint probabilities in Step 3) in order to preserve the original correlation structure. Since this construction is new (at least to the authors' knowledge), we present it in some detail. We finally notice that, after this work was done, we were aware of a recent paper [3] where the authors model exactly our state variables with two-dimensional trees, but still using the 4-step procedure of [18].

The first step is to use Theorem 3.1 in our context. Assume that r follows a CIR process as in Equation (16) and that H evolves as in Equation (23), with $W^H = \rho_{rH} W + \sqrt{1-\rho_{rH}^2} W'$, with $\rho_{rH} \in [-1, 1]$ and W' independent of W. Then

$$
\begin{aligned}
\bar{\mu}(r, H, t) &= \begin{pmatrix} k(\mu - r) \\ H(r - q_H) \end{pmatrix}, \\
a(r, H, t) &= \begin{pmatrix} \sigma_r^2 r_t & \sigma_r \sqrt{r_t} \sigma_H \rho_{rH} H \\ \sigma_r \sqrt{r_t} \sigma_H \rho_{rH} H & \sigma_H^2 H^2 \end{pmatrix}
\end{aligned}
$$

Notice that we explicitly generalise [7], where the calculations were carried out with $\rho_{rH} = 0$.

Now we build a 2-dimensional tree for the discretisation of (r, H). For $h > 0$, define (r^h, H^h) as a Markov chain with $(r_0^h, H_0^H) := (r_0, H_0)$ and transition kernel

$$
\begin{aligned}
\nu_h((r,H),\cdot) \quad := \quad & q_1^h(r,h)\delta_{(r_h^+,H_h^+)(r,H)}(\cdot) + q_2^h(r,h)\delta_{(r_h^+,H_h^-)(r,H)}(\cdot) + \\
& + q_3^h(r,h)\delta_{(r_h^-,H_h^+)(r,H)}(\cdot) + q_4^h(r,h)\delta_{(r_h^-,H_h^-)(r,H)}(\cdot)
\end{aligned}
$$

where $\sum_{i=1}^4 q_i^h(r,H) = 1$ for all r, H, $h > 0$, and $r_h^\pm, H_h^\pm$ are still to be specified.

We now make, for the two variables, a state transform in order to have a computationally simple tree in both the dimensions. For r we adopt the usual transformation $r_n = \frac{\sigma_r^2 X_n^2}{4}$, while for H we search for a transformation $S(H)$ such that

$$
S(H) = \int^H \frac{\mathrm{d}z}{\sigma_H z} = \frac{\log H}{\sigma_H}
$$

and $H(S) = e^{\sigma_H S_t}$ is thus a suitable transformation. Define then

$$
H_h^+(S) = e^{(S+\sqrt{h})\sigma_H} = H(S)e^{\sqrt{h}\sigma_H}, \qquad H_h^-(S) = e^{(S-\sqrt{h})\sigma_H} = H(S)e^{-\sqrt{h}\sigma_H}
$$

so $H_h^\pm(H) = He^{\pm\sqrt{h}\sigma_H}$. We then have

$$
\bar{\mu}_h(r,H) \quad = \quad \frac{1}{h}\begin{pmatrix} (r_h^+ - r)q_h^r + (r_h^- - r)(1-q_h^r) \\ (H_h^+ - H)q_h^H + (H_h^- - H)(1-q_h^H) \end{pmatrix},
$$

$$
a_h(r,H) \quad = \quad \frac{1}{h}\begin{pmatrix} (r_h^+ - r)^2 q_h^r + (r_h^- - r)^2(1-q_h^r) & \begin{matrix} (r_h^+ - r)(H_h^+ - H)q_h^1 + \\ +(r_h^+ - r)(H_h^- - H)q_h^2 + \\ +(r_h^- - r)(H_h^+ - H)q_h^3 + \\ +(r_h^- - r)(H_h^- - H)q_h^4 \end{matrix} \\ \begin{matrix} (r_h^+ - r)(H_h^+ - H)q_h^1 + \\ +(r_h^+ - r)(H_h^- - H)q_h^2 + \\ +(r_h^- - r)(H_h^+ - H)q_h^3 + \\ +(r_h^- - r)(H_h^- - H)q_h^4 \end{matrix} & (H_h^+ - H)^2 q_h^H + (H_h^- - H)^2(1-q_h^H) \end{pmatrix}
$$

where $q_h^r = q_h^r(r,H) := q_h^1(r,H) + q_h^2(r,H)$ is the probability of an up-movement for r and $q_h^H = q_h^H(r,H) := q_h^1(r,H) + q_h^3(r,H)$ is the probability of an up-movement for H.

Theorem 3.1 implies that the marginal distributions of r and H converge; thus we impose, as in the one-dimensional case, that $q_h^r \equiv q_h$ as defined in Equation (21) and

$$
\begin{aligned}
q_h^1(r,h) + q_h^3(r,h) = q_h^H(r,H) \quad & = \quad \frac{hH(r - q_H) + H - H_h^-(H)}{H_h^+(H) - H_h^-(H)} = \\
& = \quad \frac{h(r_t - q_H) + 1 - e^{-\sqrt{h}\sigma_H}}{e^{\sqrt{h}\sigma_H} - e^{-\sqrt{h}\sigma_H}}
\end{aligned}
$$

for states r, H such that $q_r, q_c \in [0,1]$. If this does not happen, we force q_r and q_c to be 0 or 1 as suitable, as in the one-dimensional case. Anyway, for $h \to 0$, this happens with smaller and smaller probability, as one can easily verify that

$$
\lim_{h\to 0} q_h^r(r,H) = \lim_{h\to 0} q_h^H(r,H) = \frac{1}{2}.
$$

uniformly on compact sets with respect to r, H. Also, with this definition of q_h^H the local mean and variance of H^h converge to those of H uniformly on compact sets (see [14]). Now we only have to choose a marginal probability among $q_h^1, q_h^2, q_h^3, q_h^4$ in order to accomodate for the convergence of the covariance. We have that

$$\begin{cases} q_h^2 = q_h^r - q_h^1, \\ q_h^3 = q_h^H - q_h^1, \\ q_h^4 = 1 - q_h^r - q_h^H + q_h^1 \end{cases}$$

so we can express everything in terms of q_h^1. We can rewrite $(a_h)_{12}$ as

$$\begin{aligned}(a_h)_{12} \quad &= \quad \frac{1}{h}\big((r^- - r)(H^- - H) + (r^+ - r^-)(H^- - H)q_h^r + (r^- - r)(H^+ - H^-)q_h^H + \\ &\quad + (r^+ - r^-)(H^+ - H^-)q_h^1\big) = (i) + (ii) + (iii) + (iv)\end{aligned}$$

By calculating the limits for $h \to 0$, we have

$$\begin{array}{rcll} \lim\limits_{h\to 0}(i) &=& \sigma\sigma_H\sqrt{r}H & \lim\limits_{h\to 0} J_h^-(r), \\ \lim\limits_{h\to 0}(ii) &=& -\sigma\sigma_H\sqrt{r}H & \lim\limits_{h\to 0}(J_h^+(r) + J_h^-(r))q_h^r(r,H), \\ \lim\limits_{h\to 0}(iii) &=& -2\sigma\sigma_H\sqrt{r}H & \lim\limits_{h\to 0} J_h^-(r)q_h^H(r,H), \\ \lim\limits_{h\to 0}(iv) &=& 2\sigma\sigma_H\sqrt{r}H & \lim\limits_{h\to 0}(J_h^+(r) + J_h^-(r))q_h^1(r,H), \end{array}$$

Imposing that the limit of $(a_h)_{12}$ converges to $\rho_{rH}\sigma\sigma_H\sqrt{r}H$ is equivalent to impose that

$$\rho_{rH} = \lim_{h\to 0}\big(J_h^-(r) - (J_h^+(r) + J_h^-(r))q_h^r(r,H) - 2J_h^-(r)q_h^H(r,H) + 2(J_h^+(r) + J_h^-(r))q_h^1(r,H)\big)$$

We thus let

$$q_h^1(r,H) := \frac{\rho}{2(J_h^+(r) + J_h^-(r))} + \frac{J_h^-(r)}{2(J_h^+(r) + J_h^-(r))} - \frac{1}{2}q_h^r(r,H) - \frac{J_h^-(r)}{J_h^+(r) + J_h^-(r))}q_h^H(r,H)$$

if the right hand side belongs to $[0,1]$, otherwise we let $q_2^h(r,H)$ equal to 0 or 1 suitably.

By using Theorem 3.1 to all of the above, it is possible to prove the following

Theorem 5.1 *Define $r^\pm$, $H^\pm$ and q_i, $i = 1,\dots,4$ as above. Then $(r^h, H^h)_h$ converges weakly to (r,H) solution of Equations (16,23).*

We now present an implementation of this algorithm where we can explicitly see the two optimal boundaries between the continuation, default and prepayment regions. Since a 3-dimensional graphic would have been quite difficult to understand, we choose to present the boundaries at 4 different times, namely $n = N/4, N/2, 3N/4, N$, each node being represented as a red cross in case of default, as a green + in case of prepayment and as a blue dot in case of continuation. We let $r_0 = 0.03$, $H_0 = 110$, $\rho_{rH} = 0.035$, $\beta = 0.02$, $\sigma_r = 0.1$, $\alpha = 0.15$, $\sigma_H = 0.4$, $q_H = 0.3$ and $\rho_{rH} = 0.1$. For this simulation we put $T = 12$ years with $N = 96$ and represent the results for the house price in log scale. Again, the most extreme values for r and H ($r > 0.2$ and $H > 1000$), states which are very rare, have not been drawn. The result is shown in Figure 7.

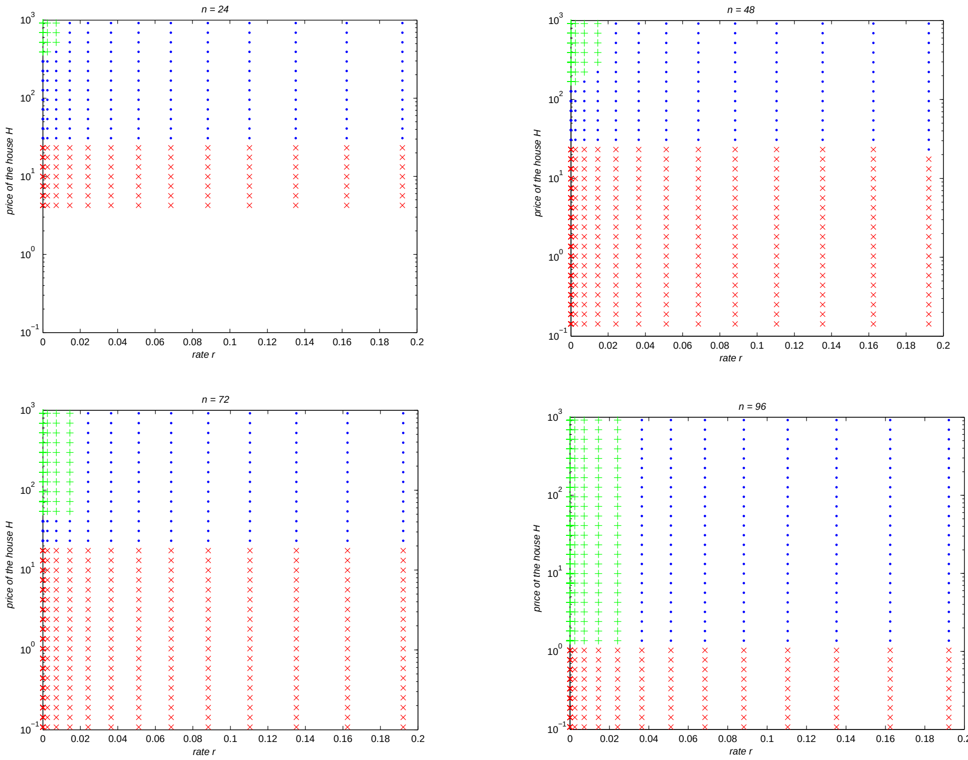

Figure 7: Prepayment and default states at $n = N/4, N/2, 3N/4, N$, with $N = 96$ indicating a total maturity of $T = 12$ years (other parameters $r_0 = 0.03$, $H_0 = 110$, $\rho_{rH} = 0.035$, $\beta = 0.02$, $\sigma_r = 0.1$, $\alpha = 0.15$, $\sigma_H = 0.4$, $q_H = 0.3$ and $\rho_{rH} = 0.1$). Each node indicates a red cross in case of default, a green + in case of prepayment and a blue dot in case of continuation. The results for the house price are in log scale, and the most extreme values for r and H ($r > 0.2$ and $H > 1000$) have not been drawn.

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